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THE EXISTENCE OF CLOSE GEODESICS ON A COMPLETE RIEM ANNIAN MANIFOLD

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Abstract

This paper studies the existence of closed geodesics in the homotopy class of a give closed curve. Let M be a complete Riemannian manifold without boundary, $f\in C^1(S^1, M)$ Look at S^1 as $[0, 2\pi]/\{0, 2\pi\}$. The following results are proved:

A. The initial value problem of heat equation $\partial_t f_t=\tau(f_t)$, $f_0=f$ always admits a globa solution.

B. (Existence of closed geodesics). If there exists a compact set $K\subset M$ such tha $f(S^1)\cap K\neq\varnothing$ and

$$E(f)\leqslant\frac{1}{\pi}\, i\,(\partial K)^2,$$

then there exists a closed geodesic homotopic to f. If

$$E(f)\leqslant\frac{1}{\pi}\, i\,(M\backslash K)^2,$$

then the closed geodesic is minimal.

C. Some estimates about injective radius are obtained.

Some example is found showing that the inequalities in B are sharp.

§1. Introduction and Notations

In the study of closed geodesics great progress has been made since Hilb pioneering works. In 1978 G. Thorbergsson[5] proved several existence theorem 1980 V. Bangert[6] supplemented Thorbergsson's results. In this paper we study existence of closed geodesics in a given homotopy class of a closed curve. The m heorem is as follows:

Let M be a complete Riemannian manifold without boundary. $f\in C^1(S^1, M$ there exists a compact set $k\subset M$ such that $k\cap f(S^1)\neq\varnothing$ and

$$i(\partial k)\geqslant\sqrt{\pi E(f)},$$

then there exists a closed geodesic homotopic to f.

Here $L(f)$ is the length of f, $i(\partial k)$ the injective radius of the boundary of k particular, if

$$i(M/K))\geqslant\sqrt{\pi E(f)},$$

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n the closed geodesic exists. This means that if M is very nice at infinity, then closed geodesics will exist as if on a compact manifold. Note that we allow a closed desic to be a single point. In this paper we also study injective radii and obtain eral local theorems. Now we define some notations. We regard S^1 as $[0, 2\pi]/\{0,$ with local coordinate θ. Let $f: S^1 \to (M, g)$. The energy density

$$e(f)(\theta) = \frac{1}{2}|\partial_\theta f|^2.$$

e energy

$$E(f) = \int_0^{2\pi} e(f)(\theta)\,d\theta.$$

e length

$$L(f) = \int_0^{2\pi} |\partial_\theta f|\,d\theta.$$

e tension tensor $\tau(f)(\theta) = \nabla_{\partial_\theta}\partial_\theta f$, B_r denotes the open ball in R^n centered at the gin with the radius r. For a subset S of a Riemannian manifold, the ε-neighborhood $= \{x \mid d(x, S) < \varepsilon\}$. The injective radius

$$i(S) = \inf_{P\in S} i(P).$$

e Gaussian curvature

$$\mathrm{Riem}(S) = \sup_{P\in S} \{\text{all the sectional curvatures at } P\}.$$

§2. Injective Radii

In this section M is assumed to be an arbitrary Riemannian manifold without undary, but may not be complete. Hence the exponantial map $\exp_p: T_pM \to M$ may t be well-defined on the whole tangent space $T_p\ M$. We first establish a lemma iich is the local version of Lemma 1. 32 in [7], p. 35.

Lemma 1. *Let M', M beRiemannian manifolds. $\varphi \in C^{(X)}(M', M)$, S is a nected open subset of M'. Assume that there is $\varepsilon > 0$ such that $\exp_P$ can be defined on $\subset T_PM'$ for any $p \in S$, and $\varphi(S_\varepsilon/S) \cap \varphi(S) = \varnothing$. If the restricted map $\varphi|S$ is a al isometry, then it is a covering map. Hence $\varphi|S$ is a diffeomorphism if $\varphi(S)$ is nply connected.*

Proof Clearly $\varphi(S)$ is open in M. Fix $q \in \varphi(S)$. Let $\varphi^{-1}(q) \cap S = \{q_\alpha\}$. Assume $(0 < r < \varepsilon)$ is so small that

$$U = \exp_q B_r \subset \varphi(S)$$

contained in a normal coordinate neighborhood of q. Let

$$U_\alpha = \{x \in S \mid d(x, q_\alpha) < r\}.$$

Then $(\varphi|S)^{-1}(U)$ is the disjoint union $\bigcup_\alpha U_\alpha$, and the commutability of the following diagram

$$\begin{array}{ccc} B_r(\subset T_{q_\alpha}M') & \xrightarrow{d\varphi} & B_r(\subset T_qM) \\ \Big\downarrow{\scriptstyle \exp_{q_\alpha}} & & \Big\downarrow{\scriptstyle \exp_q} \\ U_\alpha & \xrightarrow{\varphi|U_\alpha} & U \end{array}$$

implies that $\varphi|U_\alpha \to U$ is a diffeomorphism.

Theorem 1. *Let* $p\in M$. $r_1>r>0$. $\exp_p$ *is defined on* $B_{r_1}\subset T_pM$. $A=\exp_p B_r$ *is simply connected.* p *has no conjugate point in* $\exp_p B_{r_1}$. *Assume that there is* $s>0$ *that* $\exp_p(B_{r+s}\backslash B_r)\cap \exp_p B_r=\varnothing$. *Then*

$$i(p)\geqslant r.$$

Proof Let $M'=B_{r_1}\subset T_pM$, g the Riemaenian metric on M. $g'=\exp_p^* g$. Si has no conjugate point in $\exp_p B_{r_1}$, (M', g') becomes a Riemannian manifold $\varphi=\exp_p$: $M'\to M$ is a local isometry. Let $S=B_r$. Then Lemma 1 shows that $\varphi|$ diffeomorphism and therefore $i(p)\geqslant r$.

Theorem 2. *Let* $p\in M$. $r_1>r>0$. $\exp_p$ *is defined on* $B_{r_1}\subset T_pM$. $A=\exp_p$ *simply connected.*

$$Rime(A)<\left(\frac{\pi}{r}\right)^2.$$

Assume that there is $s>0$ *such that* $\exp_p(B_{r+s}\backslash B_r)\cap \exp_p B_r=\varnothing$. *Then*

$$i(p)\geqslant r.$$

Proof Pick $r_1>r$ so close to r that

$$\text{Riem }(A)<\left(\frac{\pi}{r_1}\right)^2.$$

Compare $\exp_p B_{r_1}$ with the sphere of dimension $n(n=\dim M)$ of radius $\frac{r_1}{\pi}$. I Rauch's comparison theorem p has no conjugate point in $\exp_p B_{r_1}$. Then Theor implies Theorem 2.

Remark. If $\exp_p(B_{r+s}\backslash B_r)\cap \exp_p B_r\neq\varnothing$, for all $s>0$, then there are at two geodesics in $\exp_p B_{r+s}$ from p to x for $x\in\exp_p(B_{r+s}\backslash B_r)\cap\exp_p B_r$. Hence $i($ in this case.

§ 3. Closed Geodesics

In this section we always assume that M is a complete C^∞ Riemannian man without boundary. Given $f\in C^1(S^1, M)$, we shall study the existence of c geodesics in the homotopy class of f. We regard S^1 as $[0, 2\pi]/\{0, 2\pi\}$. Conside initial value problem of heat equation

$$\begin{cases} \dfrac{\partial f_t}{\partial t}=\tau(f_t), \\ f_t|_{t=0}=f \end{cases} \tag{1}$$

which the solution f_t exists at least locally. From now on f_t will always denote the lution of problem (1).

Lemma 2[2]. *Let M' be a compact subset o M, c a constant. Then there exists* $=t_1(M', c)$, *such that, for any $f\in C^1(S^1, M)$, if $f(S^1)\subset M'$ and $e(f)\leqslant c$, then the ution of problem* (1) *exists for $t\in[0, t_1)$.*

Lemma 3[3]. *Iet $f\in C^2(S^1\times[0, t_1))$. If $(\partial_\theta^2-\partial_t)f(\theta, t)\geqslant 0$, then* $\sup\{f(\theta, t)\mid$ $S^1\}$ *is decreasing for $t\in[0, t_1)$.*

Theorem 3. $e(f_t)\leqslant\sup\{e(f_0)(\theta)\mid\theta\in S^1\}$, *for all $t\geqslant 0$.*

Proof Since

$$\partial_t\partial_\theta f_t(\theta)=\partial_\theta\partial_t f_t=\partial_\theta\tau(f_t)=\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_t,$$

have

$$\begin{aligned}&(\partial_\theta^2-\partial_t)e(f_t)(\theta)\\&=(\partial_\theta^2-\partial_t)\frac{1}{2}\langle\partial_\theta f_t, \partial_\theta f_t\rangle\\&=\partial_\theta\langle\nabla_{\partial_\theta}\partial_\theta f_t, \partial_\theta f_t\rangle-\langle\partial_t\partial_\theta f_t, \partial_\theta f_t\rangle\\&=\langle\nabla_{\partial_\theta}\nabla_{\partial_\theta}\partial_\theta f_t, \partial_\theta f_t\rangle+|\nabla_{\partial_\theta}\partial_\theta f_t|^2-\langle\partial_t\partial_\theta f_t, \partial_\theta f_t\rangle\\&=\nabla_{\partial_\theta}\partial_\theta f_t|^2\geqslant 0.\end{aligned}$$

en Lemma 3 implies Theorem 3.

Lemma[2] **4.** $\partial_t E(f_t)=-\int_0^{2\pi}|\tau(f_t)|^2 de$ $(t>0)$. *Hence $E(f_t)$ is a decreasing ction of t.*

Lemma 5[2]. *Let f_t, g_t be solutions of problem* (1). *Then $f_t=g_t$ (for $t\geqslant 0$).*

Theorem 4. *Let M be any complete Riemannian manifold. Given any initial dition $f\in C^1(S^1, M)$, the solution of problem* (1) *exists uniquely for $t\in[0, +\infty)$.*

Proof Let $[0, b)$ be the largest existence interval of the solution of problem). By Lemma 2, $b>0$. we now show that $b=+\infty$. Notice that every $\varphi\in C^1(S^1, M)$ isfies

$$\operatorname{diam}\varphi(S^1)\leqslant\frac{1}{2}L(\varphi(S^1))\leqslant\sqrt{\pi E(\varphi)}.$$

ppose $b<+\infty$. Then for each $\theta\in S^1$ we have

$$\begin{aligned}d(f_t(\theta), f(\theta))&\leqslant d(f_t(\theta), f_t(\theta_0))+d(f_t(\theta_0), f(\theta_0))+d(f(\theta_0), f(\theta))\\&\leqslant\operatorname{diam}f_t(S^1)+\operatorname{diam}f(S^1)+\int_0^t|\partial_t f_t(\theta_0)|dt\\&\leqslant\sqrt{\pi E(f_t)}+\sqrt{\pi E(f)}+\int_0^t|\partial_t f_t(\theta_0)|dt\\&\leqslant 2\sqrt{\pi E(f)}+\int_0^t|\partial_t f_t(\theta_0)|dt.\end{aligned}$$

Chose θ_0 such that

$$\int_0^t|\partial_t f_t(\theta_0)|dt=\min_{\theta\in S^1}\int_0^t|\partial_t f_t(\theta)|dt.$$

Then we obtain

$$\sup_{\theta\in S^1} d(f_t(\theta), f(\theta)) \leqslant 2\sqrt{\pi E(f)} + \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^t |\partial_t f_t(\theta)| dt$$
$$\leqslant 2\sqrt{\pi E(f)} + \frac{1}{2\pi}\sqrt{2\pi b}\left(\int_0^{2\pi}\int_0^t |\partial_t f_t(\theta)|^2 d\theta\, dt\right)^{1/2}$$
$$\leqslant 2\sqrt{\pi E(f)} + \left(\frac{b}{2\pi}\int_0^b (-\partial_t E(f_t))\, dt\right)^{1/2} \quad \text{(Lemma 4)}$$
$$\leqslant 2\sqrt{\pi E(f)} + \left(\frac{b}{2\pi} E(f)\right)^{1/2} < +\infty.$$

From Theorem 4 and Lemma 2 we see that f_t can be extended to $t\in[0, b+\varepsilon$ some $\varepsilon>0$. This contradicts our definition of b. Hence $b=+\infty$, that is, f_t exist all $t>0$. The uniqueness of the solution is from Lemma 5.

In [1], Ottarsson proved a global existence theorem of f_t, but in his pro assumed that M satisfies some boundedness condition which is removed here.

Lemma 6[1]. *If f_t is a bounded solution of problem* (1), *then there exists a geodesic homotopic to f_0.*

If Riem$(M)\leqslant 0$, then the boundedness of f_t is equivalent to the existence closed geodesic in its homotopy class[4]. But in general this is not true. Here example.

Example 1. Consider the surface of revolution in R^3:

$$x^2+y^2=r(z)^2.$$

It can be expressed by the local coordinates (φ, z):

$$x=r(z)\cos\varphi,\ y=r(z)\sin\varphi,\ z=z.$$

The heat equation (1) is

$$\begin{cases} \dfrac{\partial\varphi}{\partial t}=\dfrac{\partial^2\varphi}{\partial\theta^2}+\dfrac{2r'}{r}\dfrac{\partial\varphi}{\partial\theta}\dfrac{\partial z}{\partial\theta}, \\ \dfrac{\partial z}{\partial t}=\dfrac{\partial^2 z}{\partial\theta^2}-\dfrac{rr'}{1+r'^2}\left(\dfrac{\partial\varphi}{\partial\theta}\right)^2+\dfrac{r'r''}{1+r'^2}\left(\dfrac{\partial z}{\partial\theta}\right)^2. \end{cases}$$

If we seek the solution of (2) with the initial condition of the parallel

$$\varphi(\theta, 0)=\theta,\ z(\theta, 0)=z_0$$

then the solution is

$$\varphi(\theta, t; z_0)=\theta,\ z(\theta, t; z_0)=z(t; z_0),$$

where $z(t; z_0)$ satisfies

$$\frac{dz}{dt}=-\frac{r(z)r'(z)}{1+r'(z)^2},$$
$$z(0)=z_0.$$

Hence one can see that every isochrone $t=t_0$ of the solution is the parallel $z=z$ And when t increases, the isochrone goes in the direction along which the ra $r(z(i))$ is decreasing and it will never stop moving unless a geodesic parallel strikes him.

Now take

$$r(z)=e^{-\mathrm{ch} z},$$

$$z_0=1.$$

en as we indicated above, the solution (3) has the property:

$$z(t)\geqslant 1, \text{ for all } t\geqslant 0.$$

e equation (4) is

$$\frac{dz}{dt}=\frac{e^{-2\mathrm{ch} z}\mathrm{sh} z}{1+e^{-2\mathrm{ch} z}\mathrm{sh}^2 z}\geqslant\frac{e^{-2\mathrm{ch} z}\mathrm{sh}\, 1}{1+e^{-2\mathrm{ch} z}\mathrm{sh}^2 z}. \tag{5}$$

w we show that the solution (3) is unbounded. If not, then from (5), we see that re exists a constant $c>0$, such that

$$\frac{dz}{dt}\geqslant c \text{ for all } t\geqslant 0.$$

en $z(t)\geqslant z_0+ct$ is unbounded. On the other hand, the parallel $z=0$ is a closed desic. This shows that the inverse of Lemma 6 does not hold.

Theorem 5. *For any $f\in C^1(S^1, M)$, if there is a compact set K, such that* $^1)\cap k\neq\varnothing$ *and*

$$E(f)\leqslant\frac{1}{\pi}i(\partial k)^2,$$

n there exists a closed geodesic homotopic to f.

Proof By Lemma 6, we only need to prove that if f is not null homotopic, then s bounded. Suppose this is not true. First there is $t_0\geqslant 0$ such that $f_{t_0}(S^1)\cap\partial k\neq\varnothing$. k $p\in f_{t_0}(S^1)\cap\partial k$. There exists $q\in f_{t_0}(S^1)$ such thst $d(p, q)=\max\{d(p, x)\,|\,x\in$ $S^1)\}$. $d(p, q)\geqslant i(p)$ since f is not null homotopic. Then

$$i(\partial k)\leqslant i(p)\leqslant d(p, q)\leqslant\frac{1}{2}L(f_{t_0}(S^1))\leqslant\sqrt{\pi E(f_{t_0})}\leqslant\sqrt{\pi E(f)}\leqslant i(\partial k).$$

is forces

$$d(p, q)=\frac{1}{2}L(f_{t_0}(S^1)).$$

nce $f_{t_0}(S^1)\setminus\{p, q\}$ are two minimal geodesics. Then f_{t_0} must be a closed geodesic ce it is a C^1 curve. Thus $f_t=f_{t_0}$ (for all $t\geqslant t_0$) is again bounded.

Denote $B_r(p)=\{q\,|\,d(p, q)<r\}$. From Theorem 5 we know that if $i(M\setminus B_r(p))\to$ $(r\to\infty)$, then every homotopy class of a closed curve contains a closed geodesic. is tells us that if a manifold behaves elegantly when in remote places, then closed desics will come into life.

Theorem 6. *For any $f\in C^1(S^1,M)$, if there is a compact set K such that $f(S^1)\cap$ $\neq\varnothing$ and $L(f)\leqslant 2i(\partial K)$, then there exists a closed geodesic homotopic to f.*

Proof Take

$$\theta=\frac{2\pi}{L(f)}s$$

as the new parametrization of f, where s is the arc-length. Then

$$E(f)=\frac{L(f)^2}{4\pi}\leqslant\frac{1}{\pi}i(\partial K)^2.$$

Hence Theorem 5 implies Theorem 6.

Now we prove a nonexistence proposition.

Proposition. *Assume $f\in C^2(R)$, $f>0$. The first derivative f' has no zero point. Then there do not exist any nonconstant closed geodesics on the surface M: $x^2+x^2=f(z)$ in R^3.*

Remark. This is also true for the surface

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=f(z)$$

in R^3.

Proof Assume $f'>0$ everywhere. Note that each parallel is not a ge
Suppose $\gamma(s)=(x(s),y(s),z(s))$ is a nonconstant closed geodesic on M. Let

$$z(s_0)=\max_s z(s).$$

There exist a, b, such that

$$a<s_0<b,$$

$$z(a)=z(b)\leqslant z(s)\ \text{ for all } s\in[a,b],$$

and $\gamma|[a,b]$ is a minimal geodesic.

Now consider the orthogonal projection onto the plane $z=z(a)$.

$$\begin{aligned}L(\gamma|[a,b])&\geqslant L(\text{its projection})\\&\geqslant L(\widehat{\gamma(a)\gamma(b)}\text{ on the parallel})\\&>d_M(\gamma(a),\gamma(b)).\end{aligned}$$

This contradicts our choice that $\gamma|[a,b]$ is a minimal geodesic.

Now we present an example to explain that the inequalities in Theorem 1
Theorem 6 are sharp.

Example 2. Let M be the surface of revolution in R^3: $x^2+x^2=r(s)^2$ e
$r(z)=(1+e^{-z})^{1/2}$. $p=(x,x,z)\in M$. Then $i(p)>\pi$ since M encloses the c r
$x^2+y^2=1$. On the other hand

$i(p)<$half-perimeter of the parallel through $p=\pi r(z)$. We therefore ob

$$\pi<i(x,y,z)<\pi r(z).$$

Hence for any compact subset K we have

$$i(\partial K)\geqslant i(M\backslash K)=\pi.$$

Claim: For any $\varepsilon>0$, there exist $f\in C^\infty(S^1,M)$ and a compact set k, suc

$$f(S^1)\cap K\neq\varnothing, E(f)<\left(\frac{1}{\pi}+\varepsilon\right)i(\partial K)^2,\ \text{and}$$

$$L(f)<(2+\varepsilon)i(\partial K),$$

but there do not exist any closed geodesics in the homotopy class of f.

Proof Let f be the parallel: $z=t$.

$$f(\theta)=(r(t)\cos\theta,\ r(t)\sin\theta,\ t),$$
$$\partial_\theta f(\theta)=(-r(t)\sin\theta,\ r(t)\cos\theta,\ 0)$$
$$L(f)=2\pi r(t),$$
$$E(f)=\pi r(t)^2=\pi(1+e^{-t}).$$

anygiven $\varepsilon>0$, we chose $t>-\ln\varepsilon$. Then for any compact set K containing $f(S^1)$, have

$$f(S^1)\cap K\neq\varnothing,$$
$$E(f)<\left(\frac{1}{\pi}+\varepsilon\right)i(M\backslash K)^2\leqslant\left(\frac{1}{\pi}+\varepsilon\right)i(\partial K)^2,$$
$$L(f)<(2+\varepsilon)i(M\backslash K)\leqslant(2+\varepsilon)i(\partial K).$$

the other hand f is not null homotopic. The nonexistence proposition implies t there is no closed geodesic in the homotopy class of f.

Here is another example which shows that the closed geodesics in Theorem 5 and orem 6 may not be the shortest.

Example 3. Let M be the surface of revolution in R^3: $x^2+y^2=r(z)^2$, where) $\in C^\infty(R)$. We choose the $r(z)$ such that

$$r(z)>0 \text{ for all } z,$$
$$r(z)=e^z \text{ for } z<0,$$
$$r(z)=1 \text{ for } z>1.$$

en

$$i(x,\ y,\ z)=\pi \text{ for } z>1+\pi.$$

f be the parallel $z=2\pi$. $K=f(S^1)$. Then

$$f(S^1)\cap K\neq\varnothing,\ E(f)\leqslant\frac{1}{\pi}i(\partial K)^2, \text{ and } L(f)\leqslant 2i(\partial K),$$

t there do not exist any minimal closed geodesics homotopic to f.

Theorem 7. *Let $f\in C^1(S^1,\ M)$. If there is a compact set $K\subset M$, such that*

$$L(f)\leqslant 2i(M\backslash K),$$

n the homotopy class $[f]$ contains a minimal closed geodesic.

Proof Let

$$L_0=\inf\{L(g)\,|\,g\in[f]\}.$$

k a C^1 minimal sequence $\{\gamma_n\}$ in $[f]$. If $\{\gamma_n\}$ is unbounded, then for any N there sts $n>N$ such that $\gamma_n(S^1)\cap(M\backslash K)\neq\varnothing$. If $L(f)=L_0$, then f is a minimal closed desic. If $L(f)>L_0$, then for some sufficiently big n

$$L(\gamma_n)<L(f)\leqslant 2i(M/K)\leqslant 2i(p),$$

ere $p\in\gamma_n(S^1)\cap(M/K)$. This implies that γ_n lies entirely in a normal coordinate ghborhood of p. Hence γ_n is null homotopic, and so is f as well. In this case, $[f]$ ...tains the constant closed geodesic.

Now we assume that $\{\gamma_n\}$ is bounded, namely, $\gamma_n(S^1)$ all falls in a compact subset of M which is independent of n. Let M' be the universal Riemannian covering of M. Let α_n be the lift of γ_n such that every $\alpha_n(S^1)$ falls in a compact subset of M' which is independent of n. Pick aminimal geodesic β_n in M' connecting the endpoints of α_n. The projection of β_n in M compose a minimal sequence for L_0 in $[f]$. Since the initial tangent vector $\dot{\beta}_n(0)$ falls in a compact subset of TM', it must have a subsequence converging to some $x \in T_pM'$. Then the geodesic starting from the projection of p with the initial tangent vector——the projection of x——must be a minimal closed geodesic in $[f]$.

The inequality in Theorem 7 is sharp as we indicated in Example 2. thoerem also can be stated as follows.

Theorem 8. *Let $f \in C^1(S^1, M)$. If there is a compact set $K \subset M$, such tha*

$$E(f) \leqslant \frac{1}{\pi} i(M \backslash K)^2,$$

then the homotopy class $[f]$ contains a minimal closed geodesic.

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