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ASYMPTOTICALLY OPTIMAL EMPIRICAL BAYES ESTIMATION FOR PARAMETERS OF TWO-SIDED TRUNCATION DISTRIBUTION FAMILIES**

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Abstract

Consider the two-sided truncation distrbution families written in the form

$$f(x,\ \theta)dx=w(\theta_1,\ \theta_2)h(x)I_{[\theta_1,\ \theta_2]}(x)dx,\ \text{where}\ \theta=(\theta_1,\ \theta_2).$$

$$T(x)=(t_1(x),\ t_2(x))=(\min(x_1,\ \cdots,x_m),\ \max(x_1,\ \cdots,\ x_m))$$

is a sufficient statistic and its marginal density is denoted by $f(t)d\mu^T$. The prior distribution of θ belongs to the family

$$\mathscr{F}=\left\{G:\ \iint\limits_{\Theta}\|\theta\|^2\,dG(\theta)<\infty\right\}.$$

In this paper, the author constructs the empirical Bayes estimator (EBE) of θ, $\phi_n(t)$, by using the kernel estimation of $f(t)$. Under a quite general assumption imposed upon $f(t)$ and $h(x)$, it is shown that $\phi_n(t)$ is an asymptotically optimal EBE of θ.

§ 1. Introduction and Summary

Asymptotically optimal (a. o.) empirical Bayes estimation (EBE) of parameter out uniform distribution families $U(0,\ \theta)$ was considered by R. J. Fox in [1]. author studied the convergence rates of this EBE in [2]. Furthermore the or also discussed the EBE problem about general one-sided truncation ribution families in [3]. But the EBE problem for multi-parameter has little dealt with. In this paper the author exhibits the a. o. EBE for two-parameter, nd θ_2, under two-sided truncation distribution families.

Consider the two-sided truncation distribution families written in the following

$$f(x;\ \theta_1,\ \theta_2)\,dx=w(\theta_1,\ \theta_2)\,h(x)\,I_{[\theta_1,\ \theta_2]}(x)\,dx,$$

$$\theta=(\theta_1,\ \theta_2)\in\Theta,\ \Theta=\{(\theta_1,\ \theta_2):-\infty\leqslant a<\theta_1<\theta_2<b\leqslant+\infty\},\tag{1}$$

re a and b are fixed constants, $h(x)>0$, $a.\ e.$ (for Lebesgue measure) on $(a,\ b)$,

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and it is a Lebesgue integrable function on $[\theta_1, \theta_2]$ for any $\theta \in \Theta$. But

$$w(\theta_1, \theta_2) = \left[\int_{\theta_1}^{\theta_2} h(x)\,dx\right]^{-1}.$$

Let $X=(X_1, \cdots, X_m)$ $(m \geqslant 2)$ be the iid sample drawing from above distribution families. It is obvious that,

$$(t_1(X)\ t_2(X)) = (\min(X_1, \cdots, X_m),\ \ \max(X_1, \cdots, X_m))$$

is a ufficient statistic for this families. Let

$$T(X) = t(X) = (t_1(X), t_2(X))$$

and denote its valued spaceby $\mathscr{T}$. In this paper we denote the actual observed val random vector $T(X)$ by $t(x)$ or t. The conditional distribution density of $T(X)$ given θ is written in the folliwong form

$$g(t|\theta)\,d\mu^T = m(m-1)\,w^m(\theta_1, \theta_2)\left[\int_{t_1}^{t_2} h(y)\,dy\right]^{m-2} \cdot h(t_1)h(t_2)\,I_{[\theta_1<t_1<t_2<\theta_2]}(t)\,d\mu^T.$$

Let the loss function be

$$L(\theta, d) = \|\theta - d\|^2 = (\theta_1 - d_1)^2 + (\theta_2 - d_2)^2,$$

where $d=(d_1, d_2) \in \mathscr{D}$, $(\mathscr{D}, \mathscr{B}_{\mathscr{D}})$ is called an action space.

Let the prior distribution families as follows

$$\mathscr{F} = \left\{G:\ \iint_{\Theta} \|\theta\|^2\,dG(\theta) < \infty\right\}.$$

Suppose that $g(\theta_1, \theta_2)\,d\theta_1 d\theta_2$ is the density function of $G(\theta)$. Then the marg density of $T(X)$ for the pair $(T(X), \theta)$ is

$$f(t) = \iint_{\Theta} g(t|\theta)\,dG(\theta) = \left\{m(m-1)\left[\int_{t_1}^{t_2} h(y)\,dy\right]^{m-2} h(t_1)h(t_2)\right\}$$

$$\cdot\left[\int_{t_2}^{b} d\theta_2 \int_a^{t_1} w^m(\theta_1, \theta_2)\,g(\theta_1, \theta_2)\,d\theta_1\right] = u(t)\,v(t),$$

where

$$u(t) = m(m-1)\left[\int_{t_1}^{t_2} h(y)\,dy\right]^{m-2} h(t_1)h(t_2),$$

$$v(t) = \int_{t_2}^{b} d\theta_2 \int_a^{t_1} w^m(\theta_1, \theta_2)\,g(\theta_1, \theta_2)\,d\theta_1.$$

If we denote $\dfrac{\partial^2\varphi(y_1, y_2)}{\partial y_1 \partial y_2}$ and $\dfrac{\partial\varphi(y_1, y_2)}{\partial y_i}$ $(i=1, 2)$ by $\varphi''_{12}(y_1, y_2)$ and $\varphi'_i(y_1, y_2)$ 1, 2) respectively, then under the prior distribution G the Bayes estimator of θ_1

$$\phi_{1G}(t) = \iint_{\Theta} \theta_1 g(t|\theta)\,dG(\theta)/f(t) = \frac{u(t)}{f(t)}\int_{t_2}^{b} d\theta_2 \int_a^{t_1} \theta_1 w^m(\theta_1, \theta_2)\,g(\theta_1, \theta_2)\,d\theta_1$$

$$= \frac{u(t)}{f(t)}\int_{t_2}^{b} d\theta_2 \int_a^{t_1} \theta_1[-v''_{12}(\theta_1, \theta_2)]\,d\theta_1 = \frac{u(t)}{f(t)}\left[t_1 v(t) - \int_a^{t_1} v(\theta_1, t_2)\,d\theta_1\right]$$

$$\triangleq t_1 - \frac{u(t)}{f(t)}\,r_1(t) \triangleq t_1 - \psi_1(t), \tag{8}$$

where

$$r_1(t)=\int_a^{t_1}\frac{f(y,t_2)}{u(y,t_2)}\,dy,\text{ and} \tag{9}$$

$$\psi_1(t)=\frac{u(t)}{f(t)}\,r_1(t). \tag{10}$$

Similarly we get

$$\phi_{2G}(t)=t_2+\frac{u(t)}{f(t)}\,r_2(t)=t_2+\psi_2(t), \tag{11}$$

where

$$r_2(t)=\int_{t_2}^{b}\frac{f(t_1,y)}{u(t_1,y)}\,dy,\text{ and} \tag{12}$$

$$\psi_2(t)=\frac{u(t)}{f(t)}\,r_2(t). \tag{13}$$

The Bayes estimate of $\theta=(\theta_1, \theta_2)$ is defined by

$$\phi_G(t)=(\phi_{1G}(t),\ \phi_{2G}(t)). \tag{14}$$

Let R_G be theBayes risk versus G, i. e.,

$$R_G=R(\phi_G, G)=E_{(T,\theta)}\,\|\phi_G(T)-\theta\|^2=E_{(T,\theta)}(\phi_{1G}(T)-\theta_1)^2+E_{(T,\theta)}(\phi_{2G}(T)-\theta_2)^2, \tag{15}$$

where $E_{(\cdot)}$ denotes the expectation with respect to the joint distribution of random vector $(\cdot)$, and we shall always use this symbol in this paper.

In the EBE framework, we make the following assumptions: Let $(X^{(1)}, \theta^{(1)}), \cdots, (X^{(n)}, \theta^{(n)}), \cdots$ be a sequence of independent random vectors and let $(X,\theta)=(X^{(n+1)}, \theta^{(n+1)})$, then the $\theta^{(1)}, \cdots, \theta^{(n)}$ and θ have a common prior distribution $G(\theta)$, where $X^{(i)}=(X_{i1}, \cdots, X_{im})$, $\theta^{(i)}=(\theta_{i1}, \theta_{i2})$, $i=1, 2, \cdots, n$, and $X=(X_1, \cdots, X_m)$, $\theta=(\theta_1, \theta_2)$. Usually $X^{(1)}, \cdots, X^{(n)}$ denote the historical samples and X is the present sample. Let

$$T_i=T_i(X^{(i)})=(t_1(X^{(i)})\ t_2(X^{(i)})),\ i=1, 2, \cdots, n.$$

$$T(X)=(t_1(X), t_2(X))=t(X)\ \text{as above.}$$

Therefore $T_1, \cdots, T_n$ are also called the historical samples of random vector $T(X)$, and t is the present sample. Obviously $T_1, \cdots, T_n$ and T are mutually independent and each T_i possesses the same distribution as T, given by (5).

In order to establish the EBE ofθ, we use the class of kernel function defined as follows:

Let $k_0(x)$ $(x\in R_1)$ be a borel measurable bounded function vanishing off $(-1, 1)$ and satisfying the condition

$$\frac{(-1)^j}{j!}\int_{-1}^{1}y^j k_0(y)\,dy=\begin{cases}1, & \text{if } j=0,\\ 0, & \text{if } j=1, 2, \cdots.\end{cases}$$

It is not difficult to find a $k_0(x)$ that satisfies above condition[4].

Let $k(u)=k(u_1, u_2)=k_0(u_1)k_0(u_2)$, $(u_1, u_2)\in R_2$.

It is easy to find

(1) $\frac{(-1)^l}{l_1!l_2!}\int_{R_2} k(u)u_1^{l_1}u_2^{l_2}du=\begin{cases}1, \text{if } l_1=l_2=0,\\ 0, \text{otherwise,}\end{cases}$ $l=l_1+l_2$;

(2) $|k(u)|\leqslant M,\ u\in R_2$;

(3) $\int_{R_2}|k(u)|du\leqslant M$.

In order to establish the kernel estimation of $f(t)$, we define

$$f_n(t)=\frac{1}{nh_n^2}\sum_{i=1}^{n}k\left(\frac{t-T_i}{h_n}\right), \tag{1}$$

where $h_n>0$ and $\lim_{n\to\infty}h_n=0$.

Let

$$\hat{f}_n(t)=\begin{cases}f_n(t) \text{ if } |f_n(t)|>\delta_n,\\ \delta_n \text{ if} |f_n(t)|\leqslant\delta_n\end{cases} \tag{1}$$

be a kernel estifmate of $f(t)$, where $\{\delta_n\}$ is a sequence of positive numbeas surh tl $\lim_{n\to\infty}\delta_n=0$.

Let

$$r_{1n}(t)=\int_a^{t_1}\frac{f_n(y,t_2)}{u(y,t_2)}dy,\quad r_{2n}(t)=\int_{t_2}^{b}\frac{f_n(t_1,y)}{u(t_1,y)}dy. \tag{1}$$

The EBE of θ_1 and θ_2 are defined by

$$\phi_{1n}(t)=t_1-\frac{u(t)}{\hat{f}_n(t)}r_{1n}(t) \tag{1}$$

and

$$\phi_{2n}(t)=t_2+\frac{u(t)}{\hat{f}_n(t)}r_{2n}(t) \tag{2}$$

respectively. Therefore we define the EBE of θ by

$$\phi_n(t)=(\phi_{1n}(t),\ \phi_{2n}(t)). \tag{2}$$

LetE_* and E be the expectation withfrespect to the joint distribution of $(T_1,\cdots T_n;\ (T,\ \theta))$ and $(T_1,\ \cdots,\ T_n)$ respectively in this paper. Then the "over-all" Bay risk of $\phi_n(t)$ is given by

$$R_n=R_n(\phi_n,\ G)=E_*\|\phi_n(T)-\theta\|^2=E_*(\phi_{1n}(T)-\theta_1)^2+E_*(\phi_{2n}(T)-\theta_2)^2\ . \tag{2}$$

By definition, $\phi_n(t)$ is said to be an a. o. EBE of θ with respect tothe prior fam $\mathscr{F}^*$ if

$$\lim_{n\to\infty}R_n=R_G,\ \text{for any } G\in\mathscr{F}^*. \tag{2}$$

The main result in this paper can be formulated in the following theorem.

Let $D_{C\alpha}$ be a probability density on R_2. Every density function in $D_{0\alpha}$ satisfi one-order Lipschitz condition and its absolute value is bounded by α.

Theorem 1. *Let a and b be fixed finite constants, $a<b$. If*

(i) *$h(x)\leqslant M$ and $[h(x)]^{-1}$ is a Lebesgue integrable function on $[a,\ b]$,*

(ii) *$f(t)\in D_{C\alpha}$.*

then the $\phi_n(t)$, defined by (21) with $h_n=o(\delta_n)$ and $h_n=n^{-\nu}$, $0<\nu\leqslant1/4$, is an a. o. EBE *of θ under the loss function* (3) *and the prior family* (4).

§ 2. Several Lemmas

In this paper, c and M denote positive constants that do not depend on n. They can be taken different values in their each appearance even within the same expression.

Lemma 1. *Let $\delta\geqslant 1$. If $\iint_{\Theta}\|\theta\|^{\delta}dG(\theta)<\infty$, then*

$$E_*|\phi_{iG}(T)|^{\delta}<\infty,\ i=1,2, \tag{24}$$

and

$$E_*|\psi_i(T)|^{\delta}<\infty,\ i=1,2. \tag{25}$$

Proof By Jensen inequality of convex function we get

$$E_*|\phi_{1G}(T)|^{\delta}=\int_{\mathscr{T}}|\phi_{1G}(t)|^{\delta}f(t)\,d\mu^T\leqslant\int_{\mathscr{T}}E_{(\theta|t)}(|\theta_1|^{\delta})f(t)\,d\mu^T$$
$$=\int_{\mathscr{T}}\iint_{\Theta}|\theta_1|^{\delta}f(t|\theta)\,dG(\theta)\,d\mu^T$$
$$=\iint_{\Theta}|\theta_1|^{\delta}\left(\int_{\mathscr{T}}f(t|\theta)\,d\mu^T\right)dG(\theta)$$
$$=\iint_{\Theta}|\theta_1|^{\delta}dG(\theta)\leqslant\iint_{\Theta}\|\theta\|^{\delta}dG(\theta)<\infty.$$

Similarly we obtain

$$E_*|\phi_{2G}(T)|^{\delta}<\infty.$$

Since a and b are finite constants,

$$E_*|t_i|^{\delta}<|b|^{\delta}<\infty.$$

From (8) and (11) we have

$$E_*|\psi_i(T)|^{\delta}<\infty.$$

Lemma 2. *If $f(t)\in D_{0a}$, then for the $f_n(t)$, defined by* (16) *with*

$$h_n=n^{-\nu},\ 0<\nu\leqslant 1/4,$$
$$E|f_n(t)-f(t)|^2\leqslant ch_n^2.$$

Proof Since

$$E|f_n(t)-f(t)|^2\leqslant 2[\mathrm{Var}(f_n(t))+(Ef_n(t)-f(t))^2]\triangleq 2(P_1+P_2),$$

from (16) we have

$$P_1=\mathrm{Var}(f_n(t))=\frac{1}{nh_n^4}\mathrm{Var}\left[k\left(\frac{t-T_1}{h_n}\right)\right]\leqslant\frac{1}{nh_n^4}E\left[k^2\left(\frac{t-T_1}{h_n}\right)\right]$$
$$=\frac{1}{nh_n^4}\int_{R_2}k^2\left(\frac{t-z}{h_n}\right)f(z)\,dz.$$

Let $u=\dfrac{t-z}{h_n}$. Since $f(t)$ is bounded, we get

$$P_1 \leq \frac{1}{nh_n^2}\int_{R_2} k^2(u) f(t-h_n u)\,du \leq c(nh_n^2)^{-1} \leq ch_n^2 \tag{26}$$

if $h_n = n^{-\nu}$, $0<\nu\leq 1/4$. Also

$$E[f_n(t)] = \frac{1}{h_n^2} E\left[k\left(\frac{t-T_1}{h_n}\right)\right] = h_n^{-2}\cdot\int_{R_2} k\left(\frac{t-z}{h_n}\right) f(z)\,dz.$$

Let $u = \frac{t-z}{h_n}$. Similarly

$$E[f_n(t)] = \int_{R_2} k(u) f(t-h_n u)\,du.$$

Since $f(t)$ satisfies one-order Lipschitz condition, by the properties of $k(u)$ have

$$|Ef_n(t) - f(t)| = \left|\int_{R_2} k(u)\,(f(t-h_n u) - f(t))\,du\right|$$

$$\leq \int_{R_2} |k(u)|\cdot M\cdot \|h_n u\|\,du \leq ch_n. \tag{27}$$

Therefore

$$P_2 = |Ef_n(t) - f(t)|^2 \leq ch_n^2.$$

From (26) and (27) we obtain

$$E(f_n(t) - f(t))^2 \leq ch_n^2.$$

Lemma 3. *Let $(h(x))^{-1}$ be Lebesgue integrable and $h(x)\leq M$ on $[a, b]$. If $f \in D_{0a}$, then for the $r_{in}(t)$ ($i=1, 2.$), defined by* (18) *with $h_n = n^{-\nu}$, $0<\nu\leq 1/4$,*

$$u^2(t) E[r_{in}(t) - r_i(t)]^2 \leq c_1\cdot h_n^2, \quad i=1, 2. \tag{28}$$

Proof We prove Lemma 3 only for $i=1$. It is similar for $i=2$.

$$E[r_{1n}(t) - r_1(t)]^2 = E\left[\int_a^{t_1} \frac{f_n(y, t_2)}{u(y, t_2)}\,dy - \int_a^{t_1} \frac{f(y, t_2)}{u(y, t_2)}\,dy\right]^2$$

$$= E\left[\int_a^{t_1} \frac{1}{\sqrt{u(y, t_2)}}\left(\frac{f_n(y, t_2) - f(y, t_2)}{\sqrt{u(y, t_2)}}\right) dy\right]^2$$

$$\leq \int_a^{t_1} \frac{1}{u(y, t_2)}\,dy\cdot\int_a^{t_1} \frac{1}{u(y, t_2)} E[f_n(y, t_2) - f(y, t_2)]^2 dy \triangleq Q_1\cdot Q_2$$

From (6) we have

$$Q_1 = \int_a^{t_1} \frac{1}{m(m-1)\left[\int_y^{t_2} h(z)\,dz\right]^{m-2} h(y) h(t_2)}\,dy$$

$$\leq \frac{M}{m(m-1) h(t_2)\left[\int_{t_1}^{t_2} h(z)\,dz\right]^{m-2}}$$

$$= \frac{c_1}{h(t_2)\left[\int_{t_1}^{t_2} h(z)\,dz\right]^{m-2}}.$$

By Lemma 2 we get

$$E[f_n(y, t_2) - f(y, t_2)]^2 \leq ch_n^2. \tag{30}$$

Therefore

$$Q_2=\int_a^{t_1}\frac{1}{u(y,\ t_2)}E[f_n(y,\ t_2)-f(y,\ t_2)]^2dy$$

$$\leqslant c_1h_n^2\int_a^{t_1}\frac{dy}{u(y,\ t_2)}\leqslant\frac{c_1h_n^2}{\left[\int_{t_1}^{t_2}h(z)dz\right]^{m-2}h(t_2)}. \tag{31}$$

ence

$$u^2(t)E[r_{1n}(t)-r_1(t)]^2\leqslant c_1h^2(t_1)\cdot h_n^2\leqslant ch_n^2.$$

milarly we have

$$u^2(t)E[r_{2n}(t)-r_2(t)]^2\leqslant ch_n^2.$$

§3. The Proof of Theorem 1

From trigometric inequality we have

$$0\leqslant\sqrt{E_*\|\phi_n(T)-\theta\|^2}-\sqrt{E_*\|\phi_G(T)-\theta\|^2}\leqslant\sqrt{E_*\|\phi_n(T)-\phi_G(T)\|^2},$$

e., $0\leqslant\sqrt{R_n}-\sqrt{R_G}\leqslant\sqrt{E_*\|\phi_n(T)-\phi_G(T)\|^2}$. If we can prove

$$\lim_{n\to\infty}E_*\|\phi_n(T)-\phi_G(T)\|^2=0,$$

en

$$\lim_{n\to\infty}\sqrt{R_n}-\sqrt{R_G}=0.$$

e know $R_G<\infty$ by (4), therefore

$$\lim_{n\to\infty}R_n=R_G.$$

ıe proof of $\lim E_*\|\phi_n(T)-\phi_G(T)\|^2=0$ is as folows

$$E_*\|\phi_n(T)-\phi_G(T)\|^2=E_*(\phi_{1n}(T)-\phi_{1G}(T))^2+E_*(\phi_{2n}(T)-\phi_{2G}(T))^2\triangleq J_1+J_2. \tag{32}$$

definition, $\phi_n(t)$ is said to be an a.o. EBE of θ with respect to the prior family if

$$\lim_{n=\infty}J_i=0,\quad i=1,\ 2. \tag{33}$$

e prove (33) first for $i=1$.

By (32) and the dominant convergence theorem, one sees that in order to ablish the a.o. property of $\phi_n(t)$ one must verify that

(a) for any $G\in\mathscr{F}$ there exists a function $M(T,\ \theta)$ not depending on n such at

$$E[\phi_{1n}(T_1,\ \cdots,\ T_n,\ T)-\phi_{1G}(T))^2]\leqslant M(T,\ \theta),\text{for } n=1,\ 2,\ \cdots,$$

$$E_{(T,\ \theta)}(M(T,\ \theta))=E_*(M(T,\ \theta))<\infty;$$

(b) for fixed $T\in\mathscr{T}$ and $\theta\in\Theta$,

$$\lim_{n\to\infty}E[\phi_{1n}(T_1,\ \cdots,\ T_n,\ T)-\phi_{1G}(T))^2]=0$$

First we prove (a). Since

$$E(\phi_{1n}(t)-\phi_{1G}(t))^2=E\left\{u^2(t)\left[\frac{r_{1n}(t)}{\hat{f}(t)}-\frac{r_1(t)}{f(t)}\right]^2\right.$$

$$\leqslant 2E\left[u^2(t)\left(\frac{r_{1n}(t)-r_1(t)}{\hat{f}_n(t)}\right)^2\right]$$

$$+2E\left[\psi_1^2(t)\cdot\left(\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right)^2\right]$$

$$\leqslant 2\delta_n^{-2}u^2(t)\cdot E[r_{1n}(t)-r_1(t)]^2$$

$$+2\psi_1^2(t)\cdot E\left[\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right]^2\triangleq 2(J_{11}+J_{12}),$$

by Lemma 3 we have

$$J_{11}\leqslant c\delta_n^{-2}h_n^2. \quad (\text{[illegible]}$$

We consider J_{12}. Since

$$E\left[\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right]^2$$

$$=E\left\{\left(\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right)^2 I_{[f(t)>\delta_n]}\right\}+E\left\{\left(\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right)^2 I_{[f(t)<\delta_n]}\right\}$$

$$\triangleq J_{12}^{(1)}+J_{12}^{(2)},$$

where

$$f(t)-\hat{f}_n(t)=\begin{cases} f(t)-f_n(t) & \text{if } |f_n(t)|\geqslant\delta_n,\\ f(t)-\delta_n\leqslant f(t)-f_n(t) & \text{if } |f_n(t)|<\delta_n,\end{cases}$$

by Lemma 2 we get

$$J_{12}^{(1)}\leqslant\delta_n^{-2}E(f_n(t)-f(t))^2\leqslant c\cdot\delta_n^{-2}h_n^2.$$

Since $\left|\frac{f(t)}{\hat{f}_n(t)}\right|\leqslant 1$ if $f(t)<\delta_n$, we have

$$J_{12}^{(2)}=E\left\{\left(1-\frac{f(t)}{\hat{f}_n(t)}\right)^2 I_{(f(t)<\delta_n)}\right\}\leqslant 4P(f(t)<\delta_n).$$

It is obvious that $J_{12}^{(2)}\leqslant 4$ for all n.

Since $\lim\limits_{n\to\infty}\delta_n^{-2}h_n^2=0$, there exists an N, such that if $n>N$,

$$J_{12}^{(1)}\leqslant 1 \text{ and } J_{11}\leqslant 1.$$

Hence

$$E\left[\frac{\hat{f}_n(t)-f(t)}{\hat{f}_n(t)}\right]^2=J_{12}^{(1)}+J_{12}^{(2)}\leqslant 5, \text{ if } n>N.$$

Therfeore we get

$$E(\phi_{1n}(t)-\phi_{1G}(t))^2\leqslant 2+10\psi_1^2(t)\triangleq M(t,\theta) \text{ if } n>N. \quad (\text{[illegible]}$$

By Lemma 1 we have

$$E_{(T,\theta)}(\psi_1^2(T))=E_*(\psi_1^2(T))<\infty.$$

Thus

$$E_{(T,\theta)}(M(T,\theta))\leqslant 2+10E_*(\psi_1^2(T))<\infty. \quad (\text{[illegible]})$$

This proves (a).

It is obvious that

$$\lim_{n\to\infty} E(\phi_{1n}(t)-\phi_{1G}(t))^2 \leqslant \lim_{n\to\infty}[2J_{11}+2\psi^2(t)(J_{12}^{(1)}+J_{12}^{(2)})]$$
$$\leqslant \lim_{n\to\infty}[c\delta_n^{-2}h_n^2+2\psi_1^2(t)(c\delta_n^{-2}h_n^2+4P(f(t)<\delta_n))]=0$$
$$\text{for any fixed } t\in\mathscr{T} \text{ and } \theta\in\Theta. \tag{37}$$

'herefore (b) follows, and we get $\lim_{n\to\infty} J_1=0$.

Similarly we can get $\lim_{n\to\infty} J_2=0$. The proof of Theorem 1 is concluded.

§ 4. An Especial Example

Let $h(x)=1$ on (a, b) in (1). Then we get the special case, uniform distrbution amilies $U(\theta_1, \theta_2)$. It is an especial example of two-sided truncation distribution milies. In this case the expressions in § 1 are changed correspondingly. For example, ormulas (1), (2), (6) and (7) are changed as follows:

$$f(x,\theta)=\begin{cases}\dfrac{1}{\theta_2-\theta_1}, & \text{if } a<\theta_1<x<\theta_2<b,\\ 0, & \text{otherwise,}\end{cases} \tag{1'}$$

/here $-\infty\leqslant a<b\leqslant+\infty$.

$$g(t|\theta)=\begin{cases}\dfrac{m(m-1)(t_2-t_1)^{m-2}}{(\theta_2-\theta_1)^m}, & \text{if } \theta_1<t_1<t_2<\theta_2.\\ 0 & \text{, otherwise.}\end{cases} \tag{2'}$$

$$u(t)=m(m-1)(t_2-t_1)^{m-2}. \tag{6'}$$

$$v(t)=\int_{t_2}^{b} d\theta_2 \int_a^{t_1} \frac{1}{(\theta_2-\theta_1)^m}\, g(\theta_1, \theta_2)\, d\theta_1. \tag{7'}$$

In this case the conclusion of Theorem 1 follows obviously. If we change the fixed inite constants a and b of Theorem 1 into arbitrary constants, i. e., $-\infty\leqslant a<b\leqslant$ $+\infty$, then we obtain the following theorem.

Theorem 2. *Let a and b in* (1) *be arbitrary constants, i. e.,* $-\infty\leqslant a<b\leqslant+\infty$. *f $f(t)\in D_{\infty}$. then the $\phi_n(t)$, defined by* (21) *with $h_n=o(\delta_n)$ and $h_n=n^{-\nu}$,* $0<\nu\leqslant1/4$, *s an a. o. EBE of θ under loss function* (3) *and prior family* (4).

In order to prove Theorem 2, Lemmas 1 and 3 need to be changed lightly as olloms.

Lemma 1'. *Let a and b in* (1) *be arbitrary constants, i. e.,* $-\infty\leqslant a<b\leqslant+\infty$, *nd let $\delta\geqslant1$. If*

$$\iint_{\Theta} \|\theta\|^{\delta}\, dG(\theta)<\infty,$$

then
$$E_*|\phi_{iG}(T)|^{\delta}<\infty,\ i=1,2, \tag{24'}$$
$$E_*|t_i|^{\delta}<\infty \text{ and } E_*|\psi_i(T)|^{\delta}<\infty,\ i=1,2. \tag{25'}$$

Proof The proof of (24') is the same as (24). Since $\theta_1<t_1<t_2<\theta_2$, $|t_i|\leqslant$

$\max(|\theta_1|, |\theta_2|) \leqslant \|\theta\|$ and $E_*|t_i|^\delta \leqslant E_*\|\theta\|^\delta < \infty$. From (8) and (11) we have

$$E_*|\psi_i(T)|^\delta \leqslant c_\delta(E_*|\phi_{iG}(T)|^\delta + E_*|t_i|^\delta) < \infty.$$

Thus (25′) is proved.

Lemma 3′. *If* $f(t) \in D_{0\alpha}$, *then for the* $r_{in}(t)$, *defined by* (18) *with* $h_n = n^{-\nu}$, $0 < \nu \leqslant 1/4$,

$$u^2(t) E[r_{in}(t) - r_i(t)]^2 \leqslant c(t_2 - t_1)^2 h_n^2, \quad i = 1, 2. \tag{28′}$$

Proof In order to prove Lemma 3′, we change the Q_1 and Q_2 in Lemma 3 into

$$Q_1 = \int_a^{t_1} \frac{1}{u(y, t_2)} dy = \int_a^{t_1} \frac{dy}{m(m-1)(t_2 - y)^{m-2}}$$
$$= \frac{1}{m(m-1)(m-3)} \left[\frac{1}{(t_2 - t_1)^{m-3}} - \frac{1}{(t_2 - a)^{m-3}} \right]$$
$$\leqslant \frac{1}{m(m-1)(m-3)(t_2 - t_1)^{m-3}}, \tag{2}$$

$$Q_2 = \int_a^{t_1} \frac{1}{u(y_1 t_2)} E[f_n(y, t_2) - f(y, t_2)]^2 dy \leqslant \frac{ch_n^2}{m(m-1)(m-3)(t_2 - t_1)^{m-3}}. \tag{3}$$

Hence

$$u^2(t) E[r_{1n}(t) - r_1(t))^2 \leqslant u^2(t) Q_1 \cdot Q_2 \leqslant c(t_2 - t_1)^2 h_n^2.$$

Similarly we obtain

$$u^2(t) E[r_{2n}(t) - r_2(t)]^2 \leqslant c(t_2 - t_1)^2 h_n^2.$$

The proof of Theorem 2 is similar to that of Theorem 1. We only need to mak few changes as follows.

By Lemma 3′ we have

$$J_{11} \leqslant c(t_2 - t_1)^2 \delta_n^{-2} h_n^2. \tag{3}$$

Change (35) into

$$E(\phi_{1n}(t) - \phi_{1G}(t))^2 \leqslant c(t_2 - t_1)^2 + 10\psi_1^2(t) \triangleq M(t, \theta), \quad \text{if } n > N. \tag{3}$$

From Lemma 1′, we get

$$E_{(T, \theta)}(\psi_1^2(T)) = E_*(\psi_1^2(T)) < \infty,$$
$$E_{(T, \theta)}(t_2 - t_1)^2 = E_*(t_2 - t_1)^2 \leqslant 2(E_*|t_1|^2 + E_*|t_2|^2) < \infty.$$

Hence by changing (36) into

$$E_{(T, \theta)}(M(T, \theta)) = E_*(c(t_2 - t_1)^2) + 10 E_*(\psi_1^2(T)) < \infty, \tag{3}$$

the (a) is proved.

In this case (37) is changed as follows

$$\lim_{n\to\infty} E(\phi_{1n}(t) - \phi_{1G}(t))^2$$
$$\leqslant \lim_{n\to\infty} [2J_{11} + 2\psi_1^2(t)(J_{12}^{(1)} + J_{12}^{(2)})]$$
$$\leqslant \lim_{n\to\infty} [c(t_2 - t_1)^2 \delta_n^{-2} h_n^2 + 2\psi_1^2(t)(c\delta_n^{-2} h_n^2 + 4P(f(t) \leqslant \delta_n))] = 0. \tag{3}$$

The proof of Theorem 2 is finished.

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