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COHOMOLOGY OF GRADED LIE ALGEBRAS OF CARTAN TYPE $S(n, m)$**

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Abstract

Let F be an algebraically closed field of characteristic $p>3$. This paper studies the cohomology of graded Lie algebras $S(n, \boldsymbol{m})$ of Cartan type over F. The author determines the structures of $H^1(S(3, \boldsymbol{m}), \tilde{V}_0)$, where $\tilde{V}_0$ is the graded $S(3, \boldsymbol{m})$-module which is induced by an irreducible sl (3)-module V_0, the structures of $H^1(S(3, \boldsymbol{m}), V)$, where V is an irreducible $S(3,\boldsymbol{m})$-module, and the structures of the restricted cohomology $H^1_*(S(3, (1, 1, 1)), V)$, where V is an irreducible restricted $S(3, (1, 1, 1))$-module.

§0. Introduction

In [3], Dzhumadil′ daev gave the structure of the cohomology groups $H^1(W$ $\boldsymbol{n}), U_t)$ of Zassenhaus algebra $W(1, \boldsymbol{n})$. In [2], we gave the structure of cohomology groups $H^1(L, \tilde{V}_0)$, where $L=W(1, \boldsymbol{m})$, $W(3, \boldsymbol{m})$or $H(2, \boldsymbol{m})$ and $\tilde{V}_($ a graded L-module, and the structure of the restricted cohomology groups $H^1_*(L,$] where $L=W(2, (1, 1))$, $W(3, (1, 1, 1))$ or $H(2, (1, 1))$ and V is an irreduci restricted L-module.

In this paper we study the cohomology of graded Lie algebras $S(n, \boldsymbol{m})$ of Car type. In particular, we determine the structure of the first cohomology of the ran graded Lie algebra $S(3, \boldsymbol{m})$ of Cartan type with coefficients in a graded $S(3, \boldsymbol{n}$ module $\tilde{V}_0$ which is constructed from an irreducible sl(3)-module V_0(cf.[9]), structure of $H^1(S(3, \boldsymbol{m}), V)$, where V is an arbitrary irreducible $S(3, \boldsymbol{m})$-modu and the structure of $H^1_*(S(3, (1, 1, 1)), V)$, where V is an arbitrary irreduci restricted $S(3, (1, 1, 1))$-module. Since $W(1, \boldsymbol{n})$ is the only rank one graded] algebras of Cartan type, $W(2,\boldsymbol{m})$ and $H(2, \boldsymbol{m})$ are the only rank two graded] algebras of Cartan type, and $W(3,\boldsymbol{m})$ and $S(3, \boldsymbol{m})$ are the only rank three grac Lie alebras of Cartan type of depth 1, the first cohomology groups of all rank n graded Lie algebra of Cartan type of depth 1 with the above coefficients are obtain

In § 1, we shall review the notions and the results in [2]. In § 2, we disc some cohomology properties of $S(n, \boldsymbol{m})$ and reduce the computation o $H^1(S(n, \boldsymbol{n}$

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$\bar{V}_0$) to the computation of $H^1(\mathrm{sl}(n),\ V_0)$. In § 3, we determine the structure of $H^1(S(3,\ \boldsymbol{m}),\ \bar{V}_0)$ (see Theorem 3.1). In § 4, we determine the structure of $H^1(S(3,\ \boldsymbol{n}),\ V)$ (see Theorem 4.1) and $H^1_*(S(3,\ (1,\ 1,\ 1)),\ V)$ (see Corollary 4.1).

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§ 1. Notational Preliminaries

Let F be an algebraically closed field, char $F=p>0$. All Lie algebras and modules treated in the present article are assumed to be finite-dimensional. A Lie algebra L is a graded Lie algenbra if

$$L=\bigoplus_{i\in Z}L_{[i]},$$

where $L_{[i]}$ are subspaces of L and $[L_{[i]},L_{[j]}]\subset L_{[i+j]}$. An L-module V is graded if

$$V=\bigoplus_{i\geq 0}V_i$$

and $L_{[j]}V_i\subset V_{i+j}$ (We assume $V_0\neq 0$ if $V\neq 0$ and V_0 is called the base space of V).

We now give a brief description of Lie algebras of Cartan type $W(n,\ \boldsymbol{m})$ and $S(n,\ \boldsymbol{m})$. Let $A(n)$ be the set of n-tuples of non-negative integers. For

$$\alpha=(\alpha_1,\ \cdots,\ \alpha_n)\in A(n),$$

let

$$|\alpha|=\sum_{i=1}^{n}\alpha_i.$$

Set $\varepsilon_i=(\delta_{1i},\ \cdots,\ \delta_{ni})\in A(n)$. Let $\mathfrak{U}(n)$ be the divided power algebra with basis $\{x^{(\alpha)}\mid\alpha\in A(n)\}$ and multiplication

$$x^{(\alpha)}x^{(\beta)}=\binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)},\quad \alpha,\ \beta\in A(n),$$

where

$$\binom{\alpha}{\beta}=\prod_{i=1}^{n}\binom{\alpha_i}{\beta_i}$$

for $\alpha=(\alpha_1,\ \cdots,\ \alpha_n)$, $\beta=(\beta_1,\ \cdots,\ \beta_n)\in A(n)$.

If $\boldsymbol{m}=(m_1,\ \cdots,\ m_n)$ is an n-tuple of positive integers and

$$A(n,\ \boldsymbol{m})=\{\alpha\in A(n)\mid\alpha_i<p^{m_i}\},$$

then $\mathfrak{U}=\mathfrak{U}(n,\ \boldsymbol{m})=\langle x^{(\alpha)}\mid\alpha\in A(n,\ \boldsymbol{m}\rangle$ is a subalgebra of $\mathfrak{U}(n)$. Write $\pi=(p^{m_1}-1,\ \cdots,\ p^{m_n}-1)\in A(n,\ \boldsymbol{m})$. Define derivations D_i, $i=1,\ \cdots,\ n$, of $\mathfrak{U}(n,\ \boldsymbol{m})$ by

$$D_ix^{(\alpha)}=x^{(\alpha-\varepsilon_i)}$$

(We set $x^{(\alpha)}=0$ if $\alpha\notin A(n)$). Then

$$W=W(n,\ \boldsymbol{m}):=\left\{\sum_{j=1}^{n}a_j D_j\ \middle|\ a_j\in\mathfrak{U}(n,\ \boldsymbol{m})\right\}$$

is a derivation algebra of $\mathfrak{U}(n,\ \boldsymbol{m})$. The bracket operation of $W(n,\ \boldsymbol{m})$ is

$$[\sum_i a_iD_i,\ \sum_i b_iD_i]=\sum_i\sum_j(a_jD_j(b_i)-b_jD_j(a_i))D_i. \tag{1.1}$$

Set $\mathfrak{U}_{[i]}=\langle x^{(\alpha)}\mid|\alpha|=i\rangle$ and $W_{[i]}=\langle x^{(\alpha)}D_j\mid x^{(\alpha)}\in\mathfrak{U}_{[i+1]},\ j=1,\cdots,n\rangle$. Then

$$W=\bigoplus_{i\geqslant -1}W_{[i]}$$

is a graded Lie algebra of depth 1.

The subspace $S(n, \boldsymbol{m})$ spanned by

$$D_{i,j}(f):=D_j(f)D_i-D_i(f)D_j,\ f\in\mathfrak{U}(n,\boldsymbol{m}),\ i,j=1,\cdots,n,$$

is a Lie subalgebra of $W(n, \boldsymbol{m})$ (see [7]). We have the bracket products

$$[D_{i,j}(f),\ D_{i,j}(g)]=D_{i,j}(D_j(f)D_i(g)-D_i(f)D_j(g)), \qquad (1\ ^{\text{?}})$$

and

$$[D_{i,j}(f),\ D_{j,k}(g)]$$
$$=D_{j,j}(D_i(f)D_k(g))+D_{k,j}(D_j(f)D_i(g))+D_{i,k}(D_j(f)D_j(g))$$
$$(k\neq i), \qquad (1$$

for $f, g\in\mathfrak{U}(n, \boldsymbol{m})$ and $i, j, k=1, \cdots, n$. Let $S_{[i]}=S(n, \boldsymbol{m})\cap W_{[i]}$. Then

$$S=S(n,\boldsymbol{m})=\bigoplus_{i\geqslant -1}S_{[i]}$$

is a graded Lie algebra of depth 1 and under the linear map $x^{(\varepsilon_i)}D_j\mapsto E_{ij}$, $S_{[0]}$ isomorphic to $\mathrm{sl}(n)$, where E_{ij} is the matrix whose (k, l)-component is $\delta_{ik}\delta_{jl}$. Let be a representation of $S_{[0]}$ in the module V_0 and $\widetilde{V}_0=\mathfrak{U}\otimes V_0$. If $D=\Sigma\,\alpha_iD_i\in S$, th $\widetilde{D}:=\Sigma\,D_i(\alpha_j)\otimes E_{ij}\in\mathfrak{U}\otimes S_{[0]}$ (see [9]). Let $\widetilde{D}=\Sigma\,g_i\otimes l_i$, where $g_i\in\mathfrak{U}$, $l_i\in S_{[0]}$. Def a linear transformation $\tilde{\rho}_0(D)$ of $\widetilde{V}_0$ by

$$\tilde{\rho}_0(D)\,(f\otimes v)=D(f)\otimes v+\Sigma\,g_if\otimes\rho_0(l_i)v,f\in\mathfrak{U},\ v\in V_0.$$

Then $\tilde{\rho}_0$ is a representation of $S(n, \boldsymbol{m})$ in $\widetilde{V}_0$ and

$$\widetilde{V}_0=\bigoplus_{i\geqslant 0}V_i$$

is a graded $S(n, \boldsymbol{m})$-module where $V_i=\langle x^{(\alpha)}\otimes V_0\mid|\alpha|=i\rangle$ and the base space of is $1\otimes V_0\cong V_0$ (cf. [9, 10, 11]). By [2, Theorems 1. 1 and 1. 2], we have the fo wing theorem.

Theorem 1.1. *Let $S=S(n, \boldsymbol{m})$ and V be an irreducible S-module. Then*

$$H^*(S,\ V)=0,$$

where the S-module V is not isomorphic to a graded module or $V=\widetilde{V}_0$ where V_0 is irreducible $S_{[0]}$-module but is not an integral-highest weight module.

§2. Properties of Cohomology of $S(n, \boldsymbol{m})$

Let char $F=p>3$, $S=S(n, \boldsymbol{m})$, $\mathfrak{U}=\mathfrak{U}(n, \boldsymbol{m})$ and

$$S_i=\bigoplus_{j\geqslant i}S_{[j]}.$$

We identify V_0 with the subspace $1\otimes V_0$ of $\widetilde{V}_0$. Suppose that S_1 acts trivially on $S_{[0]}$-module V_0. Then we may regard V_0 as S_0-module. Now we generalize Lem 2. 1 in [2] and obtain the following lemma.

Lemma 2. 1. *The relative cohomology $H^*(S, S_{[-1]}, \widetilde{V}_0)$ is a direct summand of*

$H^*(S, \tilde{V}_0)$ *and*

$$H^*(S, S_{[-1]}, \tilde{V}_0) \cong H^*(S_0, V_0).$$

Proof We denote the projection from $\tilde{V}_0$ onto V_0 by Pr_{V_0}. Let A: $C^*(S, \tilde{V}_0) \to C^*(S_0, V_0)$ be a linear map such that $A\tilde{v} = Pr_{V_0}(\tilde{v})$ for $\tilde{v} \in C^0(S, \tilde{V}) = \tilde{V}_0$, and

$$A\psi(l_1, \cdots, l_k) = Pr_{V_0}(\psi(l_1, \cdots, l_k)),$$

for $l_1, \cdots, l_k \in S_0$, where $k>0$ and $\psi \in C^k(S, \tilde{V}_0)$. We shall show that t e following diagram is commutative.

$$\begin{array}{ccc} C^k(S, \tilde{V}^0) & \xrightarrow{d} & C^{k+1}(S, \tilde{V}_0) \\ A\uparrow & & A\downarrow \qquad k\geqslant 0. \\ C^k(S_0, V_0) & \xrightarrow{d} & C^{k+1}(S_0, V_0) \end{array}$$

It is clear that $Pr_{V_0}(l(\tilde{v})) = lPr_{V_0}(\tilde{v}))$, for $l \in S_0$, $\tilde{v} \in \tilde{V}_0$, and $l(A\psi(\cdots, \hat{l}, \cdots)) = A(l(\psi(\cdots, \hat{l}, \cdots))$, for $\psi \in C^k(S, \tilde{V}_0)$ and $l \in S_0$. Thus we have

$$d\, A\psi = A\, d\psi.$$

We now prove that $A|C^0(S, S_{[-1]}, \tilde{V}_0)$ is injective. By the definition of relative cohomology and Ann $S_{[-1]} = 1 \otimes \tilde{V}_0$(s:e [10]), we have $C^0(S, S_{[-1]}, \tilde{V}_0) = \tilde{V}_0^{S_{[-1]}} = V_0$. Thus $A|C^0(S, S_{[-1]}, \tilde{V}_0)$ is the identity map. For $k>0$, let $\psi \in C^k(S, S_{[-1]}, \tilde{V}_0)$ and $A\psi = 0$. Assume $Pr_{V_{j_1}}(\psi(\cdots)) \neq 0$, where j_1 is the smallest positive integer such that the inequality is valid. Since $\psi \in C^k(S, S_{[-1]}, \tilde{V}_0)$, we have

$$D_j(\psi(l_1, \cdots, l_k)) = \sum_{i=1}^{k} (-l)^i \psi([D_j, l_i], l_1, \cdots, \hat{l}_i, \cdots, l_k),$$

or $l_1, \cdots l_k \in S_0$ and $j = 1, 2, \cdots, n$. Applying $Pr_{V_{j_1-1}}$ to the both sides of the equality, the right side becomes zero, but the left side does not, and we get a contradition. Thus $\psi = 0$.

Next we define a linear map A': $C^*(S_0, V_0) \to C^*(S_0, \tilde{V}_0)$. Thus we set

$$\begin{aligned} &(A'\varphi)(D_{(1)}(x^{(\alpha_{(1)})}), \cdots, D_{(k)}(x^{(\alpha_{(k)})})) \\ &= \sum_{|\beta_{(1)}|, \cdots, |\beta_{(k)}| \geqslant 2} D^{\beta_{(1)}}(x^{(\alpha_{(1)})}) \cdots D^{\beta_{(k)}}(x^{(\alpha_{(k)})}) \\ &\quad \otimes \varphi(D_{(1)}(x^{(\beta_{(1)})}), \cdots, D_{(k)}(x^{(\beta_{k})})) \end{aligned}$$

for $k>0$ and $\varphi \in C^k(S_0, V_0)$, where $\alpha_{(i)}, \beta_{(i)} \in A(n, m)$, $D_{(i)} \in \{D_{j,k} | j, k = 1, 2, \cdots, n\}$ and let $\beta_{(i)} = (\beta_{i1}, \cdots, \beta_{in})$, then $D^{\beta_{(i)}} = D_1^{\beta_{i1}} \cdots D_n^{\beta_{in}}$, and

$$A'v = v, \text{for } v \in C^0(S_0, V_0) = V_0.$$

It is clear that $A'C^*(S_0, V_0) \sqsubseteq C^*(S, S_{[-1]}, \tilde{V}_0)$. We need only to show that AA': $C^*(S_0, V_0) \to C^*(S_0, V_0)$ is the identity map. Then $C^*(S, S_{[-1]}, \tilde{V}_0) \cong C^*(S_0, V_0)$ and the lemma is proved.

For $l \in S_0$, if $l \in S_{[i]}$, we write $|l| = i$. For $r \geqslant 0$ and $k>0$, let

$$C_r^k(S_0, V_0) = \{\varphi \in C^k(S_0, V_0) \mid \text{if} |l_1| + \cdots + |l_k| \neq r, \text{ then } \varphi(l_1, \cdots, l_k) = 0\}.$$

Then $C_r^k(S_0, V_0) = \bigoplus_{r\geqslant 0} C_r^k(S_0, V_0)$. If $\varphi \in C_r^k(S_0, V_0)$, then

$$(A'\varphi)(D_{(1)}(x^{(\alpha_{(1)})}),\cdots,D_{(k)}(x^{(\alpha_{(k)})}))$$
$$=\sum_{|\beta_{(1)}|+\cdots+|\beta_{(k)}|-2k=r} D^{\beta_{(1)}}(x^{(\alpha_{(1)})})\cdots D^{\beta_{(k)}}(x^{(\alpha_{(k)})})\otimes\varphi(D_{(1)}(x^{(\beta_{(1)})}),\cdots,D_{(k)}(x^{(\beta_{(k)})})).$$

It is obvious that $A'\varphi(l_1,\cdots,l_k)=0$ if $|l_1|+\cdots+\{l_k|<r$,

$$AA'\varphi(l_1,\cdots,l_k)=0 \text{ if } |l_1|+\cdots+|l_k|>r, \text{ and}$$
$$AA'\varphi(l_1,\cdots,l_k)=\varphi(l_1,\cdots,l_k) \text{ if } |l_1|+\cdots+|l_k|=r.$$

Now we shall reduce the computation of $H^1(S,\tilde{V}_0)$ to the computation of $H^1(\mathrm{sl}(n),V_0)$.

Lemma 2.2. (1) *If V_0 is nontrivial irreducible $sl(n)$-module with a high weight, then*

$$H^1(S,\tilde{V}_0)\cong H^1(S,S_{[-1]},\tilde{V}_0).$$

(2) *If $V_0=F$, then*

$$H^1(S,\mathfrak{U})\cong\langle[\beta_1]\rangle\oplus\cdots\oplus\langle[\beta_n]\rangle\oplus H^1(S,S_{[-1]},\mathfrak{U}),$$

where $\mathfrak{U}=\tilde{V}(0)$,

$$\beta_i(D_j)=\begin{cases}x^{(0,\cdots,0,p^{m_i}-1,0,\cdots,0)}, & if\ j=i,\\ 0, & if\ j\neq i,\end{cases}$$

for $i, j=1,\cdots,n$, and $[\beta_i]$ is the cohomogy class of β_i.

Proof It is obvious that there is an exact sequence

$$0\to H^1(S,S_{[-1]},\tilde{V}_0)\to H^1(S,\tilde{V}_0)\to H^1(S_{[-1]},\tilde{V}_0). \qquad (2.1)$$

We factor the restriction map from S as follows.

$$H^1(S,\tilde{V}_0)\to H^1(S_{[0]}\oplus S_{[1]},\tilde{V}_0)\to H^1(S_{[-1]},\tilde{V}_0).$$

By the cohomology five-term sequence, we have

$$0\to H^1(S_{[0]},\tilde{V}_0^{S_{[-1]}})\to H^1(S_{[0]}\oplus S_{[-1]},\tilde{V}_0)$$
$$\to H^1(S_{[-1]},\tilde{V}_0)^{S_{[0]}}\to H^2(S_{[0]},\tilde{V}_0^{S_{[-1]}})\to\cdots.$$

Hence the restriction map $H^1(S,\tilde{V}_0)\to H^1(S_{[-1]},\tilde{V}_0)$ takes values in $H^1(S_{[-1]},\tilde{V}_0)^{sl(n)}$.

If $V_0=F$, then any 1-cocycle $\psi\in Z^1(S_{[-1]},\mathfrak{U})$ can be extended to a 1-cocycle of S via

$$\psi(D_{i,j}(f))=D_j(f)\psi(D_i)-D_i(f)\psi(D_j),\ \text{for } f\in\mathfrak{U},\ i,j=1,\cdots,n.$$

It implies that the restriction map $H^1(S,\mathfrak{U})\to H^1(S_{[-1]},\mathfrak{U})$ in (2.1) is surjective. Furthermore we show that $H^1(S_{[-1]},\mathfrak{U})^{sl(n)}=H^1(S_{[-1]},\mathfrak{U})$ and the natural action of $S_{[0]}=sl(n)$ on $H^1(S_{[-1]},\mathfrak{U})$ is trivial. Hence we have an exact sequence

$$0\to H^1(S,S_{[-1]},\mathfrak{A})\to H^1(S,\mathfrak{A})\to H^1(S_{[-1]},\mathfrak{A})\to 0.$$

Therefore we have

$$H^1(S,\mathfrak{A})\cong H^1(S_{[-1]},\mathfrak{A})\oplus H^1(S,S_{[-1]},\mathfrak{A}).$$

Since $S_{[-1]}=\langle D_1,\cdots,D_n\rangle$ is commutative, we can easily show that

$$H^1(S_{[-1]},\mathfrak{A})\cong\langle[\beta_1]\rangle\oplus\cdots\oplus\langle[\beta_n]\rangle$$

(cf. [2, Lemma 3.1]). Then the assertion in the case (2) is valid.

If $V_0\neq F$, then

$$H^1(S_{[-1]}, \tilde{V}_0)^{sl(n)} = (H^1(S_{[-1]}, \mathfrak{A}(\otimes V_0)^{sl(n)}$$
$$= H^1(S_{[-1]}, \mathfrak{A}) \otimes V_0^{sl(n)} = 0.$$

Hence the restriction map $H^1(S, \tilde{V}_0) \to H^1(S_{[-1]}, \tilde{V}_0)$ is 0. By (2.1), we have

$$H^1(S, \tilde{V}_0) \cong H^1(S, S_{[-1]}, \tilde{V}_0).$$

This completes the proof of the case (1).

Now we shall compute $H^1(S, S_{[-1]}, \tilde{V}_0)$. If $V_0 = F$, then by Lemma 2.1, we have

$$H^1(S, S_{[-1]}, \mathfrak{A}) \cong H^1(S_0, F) \cong S_0/[S_0, S_0]. \tag{2.2}$$

For $n=3$, using (1.2) and (1.3), by direct computation, we have

$$S_0/[S_0, S_0] \cong 0. \tag{2.3}$$

Using the cohomology five-term sequence, we have

$$0 \to H^1(sl(n), V_0^{S_1}) \to H^1(S_0, V_0) \to H^1(S_1, V_0)^{sl(n)} \to H^2(sl(n), V_0^{S_1}) \to H^2(S_0, V_0). \tag{2.4}$$

Now we shall discuss $H^1(S_1, V_0)^{sl(n)}$ for $V_0 \neq F$. This is equal to

$$\mathrm{Hom}_{sl(n)}(S_1/[S_1, S_1], V_0) = \bigoplus_{i\geq 1} \mathrm{Hom}_{sl(n)}(Y_i, V_0), \tag{2.5}$$

where Y_i is the contribution to $S_1/[S_1, S_1]$ coming from $S_{[i]}$. Let $\mathfrak{h} = \langle E_{11}, \cdots, E_{nn}\rangle$ and Λ_i, $i=1, \cdots, n$, be linear functions on $\mathfrak{h}$ such that $\Lambda_i(E_{jj}) = \delta_{ij}$, $\lambda_0 = 0$ and

$$\lambda_i = \sum_{j=1}^{i} \Lambda_j, \ i=1, \cdots, n.$$

Then $\mathfrak{h} \cap sl(n)$ is the Cartan standard subalgebra of $sl(n)$ and $\lambda_i|_{\mathfrak{h}\cap sl(n)}$, $i=1, \cdots, n-1$, is the fundamental weight of $sl(n)$. For convenience, we still denote it by λ_i, $i=0, 1, \cdots, n-1$. If $S_{[1]}$ is regarded as an $sl(n)$-module, then we can now prove the following lemma.

Lemma 2. 3. *Let $V(\lambda)$ be an irreducible (restricted) $sl(n)$-module with a highest weight λ. Then*

$$S_{[1]} \cong V(2\lambda_1 + \lambda_{n+1}).$$

Proof We can easily verify that the linear map

$$D_{i,j}(f) \mapsto D_j(f) \otimes D_i - D_i(f) \otimes D_j, \ f \in \mathfrak{A}, \ i, j = 1, \cdots, n,$$

is an $sl(n)$-module isomorphism of $S_{[1]}$ into $\mathfrak{A}_2 \otimes S_{[-1]}$. As $sl(n)$-modules we have $\mathfrak{A}_{[2]} \cong V(2\lambda_1)$ and $S_{[-1]} \cong V(\lambda_{n-1})$, If $p \geq \langle 2\lambda_1 + \lambda_{n-1} + \delta, \alpha_0^\vee\rangle = n+2$, where δ equals half the sum of positive roots and $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}$ is the highest root, then the Weyl modules $V(2\lambda_1 + \lambda_{n-1})$ and $V(\lambda_1)$ are simple. Hence we have

$$\mathfrak{A}_{[2]} \otimes S_{[-1]} \cong V(2\lambda_1 + \lambda_{n-1}) \oplus V(\lambda_1).$$

Since $\dim V(\lambda_1) = n$ (see [6, Ex. 21. 11]), we have

$$\dim V(2\lambda_1 + \lambda_{n-1}) = (n + n(n-1)/2)n - n = n^3/2 + n^2/2 - n = \dim S_{[1]}.$$

It forces $S_{[1]} \cong V(2\lambda_1 + \lambda_{n-1})$.

By Lemma 2.3, we have

$$\mathrm{Hom}_{sl(n)}(S_{[1]}, V_0) \cong \begin{cases} F, & \text{if } V_0 \cong V(2\lambda_1 + \lambda_{n-1}), \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Then by (2.4), (2.5) and (2.6), we may reduce the computation of $H^1(S_0, V_0)$ to the computation of $H^1(sl(n), V_0)$ and $S_1/[S_1, S_1]$. By (1.2), (1.3) and Lucas theorem, that is

$$\binom{\sum_{\mu\geq 0} b_\mu p^\mu}{\sum_{\nu=0}^{t} a_\nu p^\nu} \equiv \prod_{\mu=0}^{t} \binom{b_\mu}{a_\mu} \pmod p,$$

where $\sum_{\mu\geq 0} b_\mu p^\mu$ and $\sum_{\nu=0}^{t} a_\nu p^\nu$ are the p-adic expressions, we can directly verify th... $n=3$, then

$$S_1/[S_1, S_1] = \overline{S}_{[1]} \cong S_{[1]},$$

where the congruent classes modue $[S_1, S_1]$ are denoted by the bars.

§3. $H^1((3, m), \widetilde{V}_0)$

Let S be S (3, m) over an algebraically closed field F, char $F=p>3$. In section we determine the structure of $H^1(S(3, m), \widetilde{V}_0)$, where V_0 is an irreduc $S_{[0]}$-module. Thanks to Theorem 1.1, we need to consider only the cases where V an irreducible highest weight madule of $S_{[0]}$.

Now we shall compute $H^1(sl(3), V_0)$, where V_0 is irreducible and restricted. [5, p.575], we have

$$H^1(sl(3), V_0) \cong H^1_*(sl(3), V_0). \quad (3$$

Let G be the algebraic group SL (3) over F and G_1 the first Frobenius kernel o Fix a Borel subgroup B in G and a maximal torus T in B. Let U be the unipo radical of B. Let $\Delta=\{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots, $X(T)$ the lattice of weights of T, $X(T)^+$ the set of dominant weights in $X(T)$, and

$$X_1 = \{\lambda \in X(T)^+ \mid 0 \leqslant \langle\lambda, \alpha\rangle < p \text{ for all } \alpha\in\Delta\}.$$

More precisely we need to replace X_1 by $X(T)/pX(T)$. Then X_1 is the set of weights for $sl(3)$. Let Φ^+ and W be the set of the positive roots and the Weyl gro respectively. By [6, Ex. 23.4] and [3, Theorem 3], we have

$$H^*(sl(3), V(\lambda)) = 0, \text{ for } \lambda\in X_1 \text{ and } \lambda \notin W.0, \quad (3$$

where $V(\lambda)$ is an irreducible (restaicted) sl (3)-module with a highest weight λ $w\in W$ has length $l(w)$ and $\lambda\in X(T)^+$ with $w.\ 0+p\lambda \in X(T)^+$, then by [1, Coroll 5.5] we have

$$H^1(G_1, H^0(w.\ 0+p\lambda)) \cong \begin{cases} H^0(S^{(1-l(w))/2}(U^*)\otimes\lambda) \text{ if } l(w)\equiv 1 \bmod 2, \\ 0, \text{ otherwise,} \end{cases} \quad (3$$

where $S(u^*)$ is the symmetric algebraon $u^* = (\text{Lie } U)^*$.

Let $X_t = \{a_1\lambda_1 + a_2\lambda_2 \mid a_1 + a_2 > p-2 \text{ and } 0 < a_1, a_2 < p-1\}$,

and $X_b = \{a_1\lambda_1 + a_2\lambda_2 \mid a_1 + a_2 < p-2 \text{ and } 0 \leqslant a_1, a_2 < p-1\}$.

ecall that if $\lambda\in X_b$ then $H^0(\lambda)=V(\lambda)$ and if $\lambda=a_1\lambda_1+a_2\lambda_2\in X_t$ then we have a ort exact sequence

$$0\to V(\lambda)\to H^0(\lambda)\to V(\lambda')\to 0, \tag{3.5}$$

iere $\lambda'=(p-2-a_2)\lambda_1+(p-2-a_1)\lambda_2\in X_b$. Let $V(\lambda)$ be an irreducible (restricted) 3)-module with a highest weight $\lambda\in X_1$. Then by (3.2)—(3.5) we have

$$H^1(sl(3),\ V(\lambda))\cong\begin{cases}H^0(\lambda_1)^{(1)}, \text{ if } \lambda=(p-2)\lambda_1+\lambda_2,\\ H^0(\lambda_2)^{(1)}, \text{ if } \lambda=\lambda_1+(p-2)\lambda_2,\\ F, \text{ if } \lambda=(p-2)(\lambda_1+\lambda_2),\\ 0, \text{ otherwise.}\end{cases} \tag{3.6}$$

y (2.4), (2.6), (2.7) and (3.6), we have

$$H^1\ (S_0,\ V(\lambda))\cong\begin{cases}H_0(\lambda_1)^{(1)}, \text{ if } \lambda=(p-2)\lambda_1+\lambda_2,\\ H^0(\lambda_2)^{(1)}, \text{ if } \lambda=\lambda_1+(p-2)\lambda_2,\\ F, \text{ if } \lambda=2\lambda_1+\lambda_2 \text{ or } (p-2)(\lambda_1+\lambda_2),\\ 0, \text{ otherwise.}\end{cases}$$

y Lemmas 2.1 and 2.2, (2.2)—(2.6) and the remark at the beginning of this secon, we have the following theorem.

Theorem 3. 1. *Suppose char $F=p>3$. Let V_0 be an irreducible sl (3)-module ith a highest weight. Then*

$$H^1(S(3,\ \boldsymbol{m}),\ \tilde{V}_0)\cong\begin{cases}\langle[\beta_1]\rangle\oplus\langle[\beta_2]\rangle\oplus\langle[\beta_i]\rangle, \text{ if } V_0=F,\\ H^0(\lambda_1)^{(1)}, \text{ if } V_0=V((p-2)\lambda_1+\lambda_2),\\ H_0(\lambda_2)^{(1)}, \text{ if } V_0=V(\lambda_1+(p-2)\lambda_2),\\ F, \text{ if } V_0=V(2\lambda_1+\lambda_2) \text{ or } V((p-2)(\lambda_1+\lambda_2)),\\ 0, \textit{ therwiseo.}\end{cases}$$

§4. $H^1(S(3, \boldsymbol{m}), V)$ and $H_*^1(S(1, 1, 1)), V)$

In this section we determine the structure of $H^1(S(3,\ \boldsymbol{m}),\ V)$ where V is an rreducible $S(3,\ \boldsymbol{m})$-module and the structure of $H^1_*(S(3,\ (1,\ 1,\ 1),\ V)$ where V is n irreducible restricted $S(3,\ (1,\ 1,\ 1))$-module. All irreducible graded modules of $(n,\ \boldsymbol{m})(=S)$ are determined in [11]. We have (cf. [9, 10, 11]) the following roposition.

Proposition 4. 1. (1) *If V is an irreducible graded module, then $V\mapsto V_0$ is, p to isomorphism, a bijective map of the class of irreducible graded S-modules onto the lass of irreducible $S_{[0]}$-modules.* (2) *If V_0 is $S_{[0]}$-irreducible, then $\tilde{V}_0$ is S-rreducible unless V_0 is trivial or a highest weight module with a fundamental weight as its highest weight, that is, unless $V_0=V(\lambda_i)$, $i=0,\ 1,\ \cdots,\ n-1$.* (3) *If V_0 is an irreducible $S_{[0]}$-module, then the irreducible graded S-module with base space V_0 isomorphic to the (unique) minimum submodule $(\tilde{V}_0)_{\min}$ of $\tilde{V}_0$.* (4) *If $\boldsymbol{m}=(1,\ 1,\ \cdots,$*

1), *i.e.*, *S is restricted, then every irreducible restricted S-module V is graded and* $V \mapsto V_0$ *is, up to isomorphism, a bijective map of the class of irreducible restricted* $S_{[0]}$*-modules onto the class of irreducible restricted* $S_{[0]}$*-modules such that* $(\tilde{V}_0)_{\min}$ *is the unique irreducible restricted S-module whose base space is isomorphic to* V_0.

For convenience, denote $\tilde{V}(\lambda_i)$ and $(\tilde{V}(\lambda_i))_{\min}$ by $\tilde{V}_i$ and M_i respectively, $i=0, 1, \cdots, n-1$. By [11, Theorem 2.2, Proposition 2.1 and Lemma 2.1], we have exact sequences

$$\begin{cases} 0 \to F \to \tilde{V}_0^* \to M_1 \to 0, \\ 0 \to F^{n+1} \to \tilde{V}_1/M_1 \to M_2 \to 0, \\ 0 \to F^{\binom{n}{i}} \to \tilde{V}_i/M_i \to M_{i+1} \to 0, \ i>1, \end{cases} \quad (4$$

where $\tilde{V}_0^* = \bigoplus_{j<||\pi|} \tilde{V}(\lambda_0)_j$ and $M_n = 0$.

Let char $F=p>3$ and $S=S(3, \boldsymbol{m})$. Now we shall compute $H(S, M_i)$, $i=1, 2$. the exact sequence $0 \to M_i \to \tilde{V}_i \to \tilde{V}/M_i \to 0$, we have the long exact sequences

$$H^0(S, \tilde{V}_i) \to H^0(S, \tilde{V}_i/M_i) \to H^1(S, M_i) \to H^1(S, \tilde{V}_i), \ i=1, 2. \quad (4$$

By Theorem 3.1, we have $H^1(S, \tilde{V}_i)=0$, $i=1, 2$. Then by (4.1) and (4.2), we h

$$H^1(S, M_i) \cong H^0(S, \tilde{V}_i/M_i) \cong \begin{cases} F^4, \text{ if } i=1, \\ F^3, \text{ if } i=2. \end{cases} \quad (4$$

By Proposition 4.1, Theorem 3.1 and (4.3), we have the following theorem

Theorem 4.1. *Suppose char* $F=p>3$. *Let V be an irrducible* $S(3, \boldsymbol{m})$*-modu Then*

$$H^1(S(3, \boldsymbol{m}), V) \cong \begin{cases} H^0(\lambda_1)^{(1)}, \ if\ V=\tilde{V}((p-2)\lambda_1+\lambda_2), \\ H^0(\lambda_2)^{(1)}, \ if\ V=\tilde{V}(\lambda_1+(p-2)\lambda_2), \\ F, \ if\ V=\tilde{V}(2\lambda_1+\lambda_2) \text{ or } \tilde{V}((p-2))\lambda_1+\lambda_2)), \\ F^4, \ if\ V=(\tilde{V}(\lambda_1))_{\min} \\ F^3, \ if\ V=(\tilde{V}(\lambda_2))_{\min}, \\ 0, \ otherwise. \end{cases}$$

By [5, p. 575], we have the following corollary.

Corolary 4.1. *Suppose char* $F=p>3$. *Let V be an irreducibe restricted* $S(3, 1, 1))$*-module. Then*

$$H^1_*(S, (3, 1,1, 1), V) \cong \begin{cases} H^0(\lambda_1)^{(1)}, \ if\ V=\tilde{V}((p-2)\lambda_1+\lambda_2), \\ H^0(\lambda_2)^{(1)}, \ if\ V=\tilde{V}(\lambda_1+(p-2)\lambda_2), \\ F, \ if\ V=\tilde{V}(2\lambda_1+\lambda_2) \text{ or } \tilde{V}((p-2)(\lambda_1+\lambda_2)), \\ F^4, \ if\ V=(\tilde{V}(\lambda_1))_{\min}, \\ F^3, \ i\ \ V=(\tilde{V}(\lambda_2))_{\min}, \\ 0, \ otherwise. \end{cases}$$

References

[1] Andersen, H. H. Jantzen, J. C., Cohomology of induced representations for algebraic groups, *Math.*

Ann. **269** (1984), 487—525.

] Chiu Sen and Shen Guangyu, Cohomology of graded Lie algebras of Cartan type of characteristic p, *Abh. Math. Sem. Univ.* Hamburg, **57** (1987), 139—156.

] Dzhumadil'daev, A. S., On cohomology of modular Lie algebras (Russian), *Matem. sb.* **119(161)**, (1981), 132—149.

] Hochschild, G. and Serre, J. P., Cohomology of Lie algebras, *Ann. of Math.*, **57** (1953), 591—603.

] Hochschild, G., Cohomology of restricted Lie algebras, *Amer J. Math.*, **76** (1954), 555—580.

] Humghreys, J., Intrduction to Lie algebras and representation theory, Springer, Nem York, 1972.

] Kostrikin, A. I., and Šafarevic, I. R. Graded Lie algebras of finite characteristic, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **33** (1969), 251—322.

] Rotman, J., An introduction to homological algebra, New York, Academic Press, 1979.

] Shen Guaugyu, Graded modules of graded Lie algebras of Cartan type (I) —mixed product of modules, *Scientia Sinica* (*Ser. A*), **29**: 6 (1986), 570—581.

] Shen Guangzu, Graded modules of graded Lie algebras of Cartan type (II)—positive and negative graded modules *Scientia Sinica* (*Ser. A*), **29**: 10 (1986), 1009—1019.

] Shen Guangyu, Graded modules of graded Lie algebras of Cartan type (III)—irreducible modules, (to appear).

] Wan Zhexian, Lie Algebra, Science Publishing House, Beijing, 1964 (in Chinese).

] Wilson, R., A structural characterization of the simple Lie algebras of generalized Cartan type over fields of prime characteristic, *J. Algebra*, **40** (1976), 418—465.