Chin. Ann. of Math.
10B (1) 1989

THE UPPER BOUND ESTIMATE OF LARGE DEVIATION FOR INHOMOGENEOUS MARKOV PROCESSES**

LU YUNGANG (卢云刚)*

Abstract

In this paper, the Feynmann-Kac formula is extended to the inhomogeneous case. Using the extended formula, the author studies the upper bound estimate of large deviation for some class of inhomogenous Markov processes. The results in the paper can beconsidered as a generalization of the homogeneous ones.

§1. Introduction

The large deviation theory has been studied expansively by M. D. Donsker S. R. S. Varadhan. In this paper we study the upper bound estimate of large devia for inhomogeneous Markov processes. First we study the estimate for a sequenc independent $r.\ v.$s of which the distributions may be diverse; Second we extend Feynmann-Kac formula to the inhomogeneous case. Then the formula is usec study the upper bound estimate of large deviation for inhomogeneous Mar processes.

Throughout this paper, we assume that E is a Polish space. Let $\mathscr{B}(E)$ be Borel σ-field on E, $B(E)$ be the set of all bounded $\mathscr{B}(E)$-measurable functi $C_b(E)$ be the set of all bounded continuous functions. Finally, denote by $\mathscr{M}_1(E)$ set of all probability measures on $(E, \mathscr{B}(E))$, on which the weak topolog equipped.

§2. The Case of a Sequence of Independent Random Variables

Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of E-valued independent $r.\ v.$s on a probabi space $(\Omega, \mathscr{F}, \mathbb{P})$. We denote by μ_k the distribution of X_k and define a real functio

Manuscript received November 1, 1986.

* Department of Mathematics, Beijing Normal University, Beijing, China.

** The work is supported by the Scientific Fundation from the Chinese Academy of Sciences.

on $B(E)$ as follows

$$T(f)=\overline{\lim_{k\to\infty}}\int f(x)\mu_k(dx),\ f\in B(E). \tag{2.1}$$

Lemma 2.2.

(i) *T is a positive functional on $B(E)$, i.e., for each $0\leqslant f\in B(E)$, we have* $f)\geqslant 0$.

(ii) *T is a non-decreasing functional on $B(E)$, i.e., for each $f,\ g\in B(E)$, $f\leqslant g$, we* ve

$$T(f)\leqslant T(g).$$

(iii) *For each $\alpha,\ \beta\in R^+$ and $f,\ g\in B(E)$, $T(\alpha f+\beta g)\leqslant\alpha T(f)+\beta T(g)$.*

(iv) *Let $\|\cdot\|$ be the uniform norm in $B(E)$. Then T is a bounded continuocs* nctional on $B(E)$ and

$$T(1)=1. \tag{2.2}$$

Proof Every thing is easy to see except the continuity of T. For this, we fix , $f_2\in B(E)$ and set $s=\|f_1-f_2\|$. Without loss of generality, we may and will sume that $0<s<1$. Let $f=(f_1-f_2)/2$, then $f_1=sf+f$, $T(f_1)=T(s(f_1+f_2)$ $(1-s)f_2\leqslant T(f_1+f_2)+(1-s)T(f_2)$ and $S.\ T(f_1)-T(f_2)\leqslant(T(f_1+f_2)-T(f_2))\leqslant$ $\|f_2\|+1)s$. Similarly, $T(f_2)-T(f_1)\leqslant(2\|f_1\|+1)s$, hence

$$|T(f_2)-T(f_1)|\leqslant 2(\|f_1\|+\|f_2\|+1)\|f_1-f_2\|. \tag{2.3}$$

his shows that T is a continuous functional.

Let n be a positive integer. Define L_n:

$$\Omega\to\mathcal{M}_1(E)\ \text{by}$$

$$L_n(\omega,\ A)=\frac{1}{n}\sum_{k=1}^{n}\chi_A(X_k(\omega)),\ A\in\mathscr{B}(E),\ \omega\in\Omega, \tag{2.4}$$

nd set

$$Q_n=\mathbb{P}\circ L_n^{-1}. \tag{2.5}$$

hen, for each positive integer n, Q_n is a probability measure on $\mathcal{M}_1(E)$. Finally, ne function I defined by

$$I(\mu)=\sup_{f\in C_b(E)}\left[\int f\,d\mu-\log T(e^f)\right] \tag{2.6}$$

a convex l. s. c. function on $\mathcal{M}_1(E)$.

Theorem 2.7. *For each compact subset K of $\mathcal{M}_1(E)$, we have*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_n(K)\leqslant-\inf_{\mu\in K}I(\mu). \tag{2.8}$$

Proof The proof is similar to the proof of Lemma V. 2 in [4]. Let $f\in C_b(E)$, $\in R$ and set

$$H_+(f,\ \alpha)=\left\{\mu\in\mathcal{M}_1(E);\int f\,d\mu-\log T(e^f)\geqslant\alpha\right\}. \tag{2.9}$$

Since $\int f\,d\mu-\log T(e^f)$ is a continuous functional in μ, we get

$$\text{int } H_+(f, \alpha) = \left\{ \mu \in \mathcal{M}_1(E); \int f\, d\mu - \log T(e^f) > \alpha \right\}. \tag{2.10}$$

First of all, we assume that

$$\inf_{\mu \in K} I(\mu) \in (0, \infty).$$

Given $s \in (0, \inf_{\mu \in K} I(\mu))$, set

$$\Gamma = \{\mu \in \mathcal{M}_1(E); I(\mu) \leqslant \inf_{\mu \in K} I(\mu) - s\},$$

Then

$$K \subset \Gamma^0 = \bigcup_{f \in C_b(E)} \left\{ \mu \in \mathcal{M}_1(E): \int f\, d\mu - \log T(e^f) > \inf_{\mu \in K} I(\mu) - s \right\}.$$

$$= \bigcup_{f \in C_b(E)} \text{int } H_+(f, \inf_{\mu \in K} I(\mu) - s) \tag{2.}$$

Because K is a compact subset of $\mathcal{M}_1(E)$, there are $f_1, f_2, \cdots f_N \in C_b(E)$ such

$$K \subset \bigcup_{i=1}^{\infty} H_+ (f_i, \inf_{\mu \in K} I(\mu) - s)$$

Hence

$$\begin{aligned} Q_n(K) &= \mathbb{P}\{\omega: L_n(\omega, \cdot) \in K\} \\ &\leqslant \mathbb{P} \left\{ \omega: I_n(\omega, \cdot) \in \bigcup_{i=1}^{N} H+(f_i, \inf_{\mu \in K}(I\mu) - s) \right\} \\ &\leqslant \sum_{i=1}^{N} \mathbb{P}\{\omega: L_n(\omega, \cdot) \in H+(f_i, \inf_{\mu \in K} I(\mu) - s)\} \\ &\leqslant N\, \mathbb{P} \left\{ \omega: \frac{1}{n} \sum_{k=1}^{n} f(X_k(\omega)) \geqslant \log T(e^f) + \inf_{\mu \in K} I(\mu) - s \right\} \\ &\leqslant N \exp[-n(\log T(e^f) + \inf_{\mu \in K} I(\mu) - s)] \prod_{k=1}^{n} \int e^{f(x)} \mu_k(dx), \end{aligned} \tag{2.}$$

where $f = f_{i_0}$ for some $1 \leqslant i_0 \leqslant N$. Therefore we have

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(K) \\ &\leqslant -\log T(e^f) - \inf_{\mu \in K} I(\mu) + s + \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} \log \int e^{f(x)} \mu_k(dx) \end{aligned} \tag{2.}$$

On the other hand, it is easy to cheek that

$$\overline{\text{im}}_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log \int e^{f(x)} \mu_k(dx) \leqslant \log T(e^f),$$

hence

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(K) \leqslant -\inf_{\mu \in K} I(\mu) + s. \tag{2.}$$

This gives our conclusion for the case of

$$\inf_{\mu \in K} I(\mu) \in (0, \infty)$$

Now, we need only to consider the case

$$\inf_{\mu \in K} I(\mu) = \infty.$$

For any $\alpha > 0$, set

$$\Gamma = \{\mu \in \mathcal{M}_1(E): I(\mu) \leqslant \alpha\}.$$

Then

$$K\subset\Gamma^c=\{\mu\in\mathscr{M}_1(E):I(\mu)>\alpha\}$$
$$=\bigcup_{f\in C_b(E)}\left\{\mu\in\mathscr{M}_1(E):\int fd\mu-\log T(e^f)>\alpha\right\}$$
$$=\bigcup_{f\in C_b(E)}\operatorname{int}H_+(f,\alpha).$$

Hence, there exist $f_1, \cdots, f_N\in C_b(E)$ such that

$$K\subset\bigcup_{i=1}^{N}H_+(f_i,\alpha).$$

So we have

$$Q_n(K)\leqslant N\mathbb{P}\left\{\omega: \frac{1}{n}\sum_{k=1}^{n}f(X_k(\omega))\geqslant\log T(e^f)+\alpha\right\}$$
$$\leqslant N\exp[-n(\log T(e^f)+\alpha)]\cdot\prod_{k=1}^{n}\int e^{f(x)}\mu_k(dx), \tag{2.15}$$

where $f=f_{i_0}$ for some $1\leqslant i_0\leqslant N$. Therefore

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_n(K)$$
$$\leqslant-\log T(e^f)-\alpha+\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\log\int e^{f(x)}\mu_k(dx).$$

By the fact

$$\overline{\lim_{n\to\infty}}\int e^{f(x)}\mu_k(dx)\leqslant T(e^f),$$

we have

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\log\int e^{f(x)}\mu_k(dx)\leqslant\log T(e^f),$$

and so

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log\int Q_n(K)\leqslant-\alpha.$$

Our conclusion follows immediately by letting $\alpha\uparrow\infty$.

In order to pass from the case of compact subsets to the case of closed subsets, we make a hypothesison $\{X_k\}$ as follows:

H: There is a compact function φ on E (i.e., for any $d\in R$, $\{x: \varphi(x)\leqslant d\}$ is a compact subset of E), which satisfies

(i) $1\leqslant\varphi\leqslant\infty$,

(ii) $\Phi=\overline{\lim_{k\to\infty}}\int\varphi(x)\mu_k(dx)<\infty.$ (2.16)

Theorem 2.17. *For each $L>0$ and $s>0$,*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_n\{\mu\in\mathscr{M}_1(E),\ \mu(K_L)\leqslant1-s\}$$
$$\leqslant-\log\rho_L-\log\frac{L}{\rho_L},$$

where $\rho_L\in(0,1_A\inf_{x\in K_L}\varphi(x)/\Phi)$ and

$$K_L=\{x: \varphi(x)\leqslant L\Phi\},$$

Proof By the definition of Φ, for any $\delta>0$, one can choose an N such that

$$\frac{\int \varphi(x)\mu_n(dx)}{\Phi}<1+\delta,\ n\geqslant N.$$

Hence

$$E\left[\frac{\varphi(X_1)\cdots\varphi(X_n)}{\Phi^n}\right]\leqslant\left[\prod_{k=1}^{N}\frac{\mathbb{E}\varphi(X_k)}{\Phi}\right](1+\delta)^{n-N}\leqslant M(1+\delta)^n, \tag{2.18}$$

where M is a constant. On the other hand

$$\begin{aligned}&\sum_{k=1}^{n}\log\frac{\varphi(X_k)}{\Phi}\\ &=\sum_{k=1}^{n}[\chi_{K_L^c}(X_k)+(1-\chi_{K_L^c})X_k))]\log\frac{\varphi(X_k)}{\Phi}\\ &\geqslant\Big(\sum_{k=1}^{n}\chi_{K_L^c}(X_k)\Big)\log L+\Big(\sum_{k=1}^{n}(1-\chi_{K_L^c})(X_k)\Big)\log\rho_L\\ &=n\log\rho_L+\Big(\sum_{k=1}^{n}\chi_{K_L^c}(X)\Big)\log\frac{L}{\rho_L}\\ &=n\log\rho_L+n\Big(\log\frac{L}{\rho_L}\Big)L_n(\cdot,K_L^c).\end{aligned} \tag{2.}$$

Using the above two inequalities, we get

$$\mathbb{E}\left[\exp\Big(n\Big(\log\frac{L}{\rho_L}\Big)L_n(\cdot,K_L^c)\Big]\leqslant M(1+\delta)^n\exp\Big(-n\log\frac{L}{\rho_L}\Big), \tag{2}$$

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_n\{\mu\in\mathcal{M}_1(E):\mu(K_L^c)>\varepsilon\}\\ &\leqslant\log(1+\delta)-\log\rho_L-\varepsilon\log\frac{L}{\rho_L}.\end{aligned} \tag{2}$$

This finishes our proof.

The proof of the following two conclusions are similar to the ones of Lemmas and 4.3 in [1], Part III.

Theorem 2.22. *Under the hypothesis* H, *I is a rate function. That is*

(i) *I is a convex l. s. c. function on* $\mathcal{M}_1(E)$;

(ii) *for each* $L\in R$, $A_L=\{\mu: I(\mu)\leqslant L\}$ *is a compact subset of* $\mathcal{M}_1(E)$.

Proof Obviously, (i) follows from the definition of I. Since I is l. s. c., A closed for each $L\in R$. Thus, it suffices for us to prove that A_L is a tight subse $\mathcal{M}_1(E)$. To this end, note that

$$\int f\,d\mu\leqslant L+\log T(e^f) \tag{2}$$

for any $\mu\in A_L$ and $f\in C_b(E)$. On the other hand, since φ is an l. s. c. function o there exists a sequence of continuous functions $\{\varphi_n\}$ such that

$$\varphi_n(x)\uparrow\varphi(x),\quad x\in E,$$
$$\varphi_n(x)\geqslant 1,\quad x\in E.$$

'aking $\bar{\varphi}_n=\varphi_n\wedge n$ instead of φ_n, from Fatou's lemma and (2.23), we get

$$\int \log \varphi d\mu \leqslant \overline{\lim_{n\to\infty}} \int \log \varphi_n d\mu \leqslant L + \overline{\lim_{n\to\infty}} \log T(\bar{\varphi}_n) \leqslant L + \log \Phi, \quad \mu \in A_L. \tag{2.24}$$

Ve now have

$$\int \log \frac{\varphi}{\Phi} d\mu \leqslant L, \quad \mu \in A_L.$$

ext, for each $\sigma \in R$, choose

$$K_\sigma = \{x: \varphi(x) \leqslant \sigma\Phi\}.$$

learly, K_σ is a compact subset of E and

$$L \geqslant \left(\int_{K_\sigma} + \int_{K_\sigma^c}\right) \log \frac{\varphi(x)}{\Phi} \mu(dx) \geqslant (\log \rho_L)\mu(K_\sigma)$$

or each $\mu \in A_L$. Hence, for each $\mu \in A_L$, we get

$$\mu(K_\sigma^c) \leqslant \frac{L-\log \rho_L}{\log \sigma - \log \rho_L}, \tag{2.25}$$

henever $\sigma > \rho_L$. But

$$\lim_{\sigma\to\infty} \frac{L-\log \rho_L}{\log \sigma - \log \rho_L} = 0$$

nd so A_L is tight.

Lemma 2.26. *Under the hypothesis* H, *for each* $A<\infty$, *there exists a tight subset* Γ_A *of* $\mathscr{M}_1(E)$, *such that*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(K_A^c) \leqslant -A. \tag{2.27}$$

Proof Take $\sigma_K \uparrow \infty$, $\varepsilon_K \downarrow 0$ and set

$$K_{\sigma,\varepsilon} = \{\mu \in \mathscr{M}_1(E): \mu(\Gamma_{\sigma_k})(1-\varepsilon_K) \text{ for all } k\}.$$

Since Γ_σ is a compact subset of E, $K_{\sigma,\varepsilon}$ is a tight subset of $\mathscr{M}_1(E)$. Also, by (2.17), we see that

$$\begin{aligned} Q_n(K_{\sigma,\varepsilon}^c) &\leqslant \sum_{k=1}^{\infty} Q_n\{\mu \in \mathscr{M}_1(E): \mu(\Gamma_{\sigma_k})(1-\varepsilon_k)\} \\ &\leqslant M^n \exp(-n\log\rho) \sum_{k=1}^{\infty} \exp\left(-n\varepsilon_K \log \frac{\sigma_K}{\rho}\right) \\ &= \frac{M^n}{\rho^n} \sum_{k=1}^{\infty} \left(\frac{\rho}{\sigma_K}\right)^{n\varepsilon_k}, \end{aligned} \tag{2.28}$$

'here $\rho \in (0, 1_A \inf_{X\in E} \varphi(x)/\Phi$ and

$$\frac{1}{\Phi} \mathbb{E}\varphi(X_K) \leqslant M < \infty$$

or all k.

In particular, if we take

$$\varepsilon_k = \frac{1}{k}, \quad \sigma_k = \left(\frac{A+M}{\rho}\right)^{k^2}$$

and assume that $A>1$, then

$$Q_n(K^C_{\sigma,s}) \frac{M^n}{\rho^n} \sum_{k=1}^{\infty} \frac{1}{(\sigma_k^{1/k})^n}$$
$$= \frac{M^n}{\rho^n} \sum_{k=1}^{\infty} \left[\left(\frac{\rho}{A+M}\right)^n\right]^k = M^n/[(A+M)^n - \rho^n]. \tag{2.29}$$

Hence

$$\lim_{n\to\infty} \frac{1}{n} \log Q_n(K^C_{\sigma,s}) - A.$$

The proof is finished.

Lemma 2.30 *Under the hypothesis* H, *we have.*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(C) \leqslant -\inf_{\mu\in C} I(\mu)$$

for every closed subset C *of* $\mathscr{M}_1(E)$.

Proof By Lemma (2.26), for each $A<\infty$, there exists a subset K_A of $\mathscr{M}$ such that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(K^C_A) \leqslant -A.$$

Since C is a closed, $\overline{C\cap K_A}$ is compact and $\overline{C\cap K_A}\subset C$, we have

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(\overline{C\cap K_A}) \leqslant -\inf_{\mu\in\overline{C\cap K_A}} I(\mu) \leqslant -\inf_{\mu\in C} I(\mu).$$

Hence

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(C)$$
$$= \overline{\lim_{n\to\infty}} \frac{1}{n} \log[Q_n(C\cap K_A) + Q_n(C\cap K^C_A)]$$
$$\leqslant \left[\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n(\overline{C\cap K_A})\right] \vee \overline{\lim_{n\uparrow\infty}} \frac{1}{n} \log Q_n(K^C_A)$$
$$\leqslant (-\inf_{\mu\in C} I(\mu)) \vee (-A).$$

Let $A\to\infty$, the proof is finished.

§3. The Case of The Inhomogeneous Markov Prcesse with Discrete Parameter

Let $\{\pi_k\}_{k=1}$ be a sequence of Feller transition functions on E. For each $x\in$ $(\Omega, \mathscr{F}, \mathbb{P}_x)$ be a probability space and $\{X_k\}_{k=0}^{\infty}$ be a Markov process on $(\Omega, \mathscr{F}$ with transition functions $\{\pi_k\}_{k=1}^{\infty}$ and initial condition $P_x(X_0=x)=1$.

Finally, we define

$$(\pi_i(V)f)(x) = e^{V(x)} \int f(y) \pi_i(x, dy) = e^{V(x)} (\pi_i f)(x) \tag{3.1}$$

for $V\in B(E)$, and define succesively

$$\pi^{n+1}(V_1, \cdots, V_{n+1})f = \pi^n(V_1, \cdots, V_n)[\pi_{n+1}(V_{n+1})f]$$

r $n \geqslant 1$ and $V_1, \cdots, V_{n+1} \in B(E)$. Now, we have the following proposition.

Proposition 3.2. *For each $f \in B(E)$ and $V_k \in B(E)$, $k=1, 2, \cdots, n$,*

$$[\pi^n(V_1 \cdots, V_n)f](x) = \mathbb{E}^{P_x} f(x_n) \exp\Big(\sum_{k=0}^{n-1} V_k(X_k)\Big).$$

Proof For $n=1$, we have

$$\mathbb{E}^{P_x}[f(X_1)\exp(V_1(X_0))]$$

$$= e^{V_1(x)} \cdot \int f(y)\pi_1(x, dy) = (\pi_1(V_1)f)(x) = (\pi^1(V_1)f)(x).$$

ppose that

$$\pi^n(V_1, \cdots, V_n)f(x) = \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big)\Big].$$

ıen, we have

$$\begin{aligned}
&[\pi^{n+1}(V_1, \cdots, V_n, V_{n+1}f](x) \\
&= [(\pi^n(V_1, \cdots, V_n)(\pi_{n+1}(V_{n+1})f)](x) \\
&= \mathbb{E}^{P_x}[\pi_{n+1}(V_{n+1})f](X_n)\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big) \\
&= \mathbb{E}^{P_x}\Big[\exp(V_{n+1}(X_n))(\pi_{n+1}f(X_n)\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big)\Big] \\
&= \mathbb{E}^{P_x}\Big[\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big)\mathbb{E}^{P_x}(f(X_{n+1}) \mid X_n)\Big] \\
&= \mathbb{E}^{P_x}\Big[\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big)\mathbb{E}^{P_x}(f(X_{n+1}) \mid X_0, \cdots, X_n)\Big] \\
&= \mathbb{E}^{P_x}\Big[\exp\Big(\sum_{k=0}^{n-1} V_{k+1}(X_k)\Big)f(X_{n+1})\Big].
\end{aligned}$$

Proposition 3.3. *For each $f \in B^+(E)$, set*

$$V_k^f = \log f - \log \pi_k f, \quad k \geqslant 1,$$

ere $B^+(E) = \{f \in B(E)$: there is a $\delta_f > 0$, such that $f \geqslant \delta_f\}$. Then

$$[\pi^n(V_1^f, \cdots, V_n^f)f](x) = f(x), \quad n \geqslant 1.$$

Proof For every $n \geqslant 1$, we have

$$\begin{aligned}
&\pi^n(V_1^f, \cdots V_n^f)f(x) \\
&= \exp(V_1^f(x)) \int \exp(V_2^f(y_1)) \int \exp) V_3^f(y_2)) \cdots \\
&\quad \exp(V_n^f(y_{n-1})) \int f(y_n)\pi_n(y_{n-1}, dy_n) \cdots \\
&\quad \pi_3(y_2, dy_3)\pi_2(y_1, dy_2)\pi_1(x, dy_1) \\
&= \frac{f}{\pi_1 f}(x) \int \frac{f}{\pi_2 f}(y_1) \int \frac{f}{\pi_3 f}(y_2) \cdots \\
&\quad \frac{f}{\pi_n f}(y_{n-1}) \int f(y_n)\pi_n(y_{n-1}, dy_n) \cdots \\
&\quad \pi_3(y_2, dy_3)\pi_2(y_1, dy_2)\pi_1(x, dy_1) = f(x).
\end{aligned}$$

We now need a hypothesis on $\{\pi_i\}_{i=1}^{\infty}$ as the following:

H_1^*: For each $f \in B^+(E)$,

$$\overline{\lim_{n\geq m\to\infty}} \frac{\pi_m f}{\pi_n f}(x) = 1 \tag{3.4}$$

uniformly in $x \in E$.

Proposition 3.5. *Under* H_1^*, *for each* $x \in E$ *and* $f \in B^+(E)$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{1}{N(n)} \sum_{i=1}^{N(n)} \sum_{k=0}^{n-1} \log \frac{\pi_i f}{f}(X_k)\Big)\Big] \leq 0,$$

where $N(n) \geq n$ *is a positive integer satisfying* $\frac{n}{N(n)} \to 0$.

Proof For each $\varepsilon \in (0, 1)$, by H_1^*, there exists an M such that

$$\frac{\pi_m f}{\pi_n f}(x) \leq 1+\varepsilon, \quad n \geq m \geq M.$$

Therefore

$$\begin{aligned}
&\mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{1}{N(n)} \sum_{i=1}^{N(n)} \sum_{k=0}^{n-1} \log \frac{\pi_i f}{f}(x)\Big)\Big] \\
&= \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{1}{N(n)}\Big(\sum_{i=1}^{M}\sum_{k=0}^{M-1} + \sum_{i=1}^{M}\sum_{k=M}^{n-1} + \sum_{i=M+1}^{N(n)}\sum_{k=0}^{M=1} \\
&\quad + \sum_{i=M+1}^{N(n)}\sum_{k=M}^{n-1}\Big)\log \frac{\pi_i f}{f}(X_k)\Big)\Big] \\
&\leq \Big(\frac{\|f\|}{\delta_f}\Big)^{3M} \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{1}{N(n)} \sum_{i=M+1}^{N(n)} \sum_{k=M}^{n-1} \log \frac{\pi_i f}{f}(X_k)\Big)\Big] \\
&= \Big(\frac{\|f\|}{\delta_f}\Big)^{3M} \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{N(n)-M}{N(n)} \sum_{k=M}^{n-1} \log \frac{\pi_{k+1} f}{f}(x_k)\Big) \\
&\quad \cdot \exp\Big(-\frac{1}{N(n)} \sum_{i=M+1}^{N(n)} \sum_{k=M}^{n-1} \log \frac{\pi_i f}{\pi_{k+1} f}(X_k)\Big)\Big] \\
&\leq \Big(\frac{\|f\|}{\delta_f}\Big)^{3M} (1+\varepsilon)^{(N(n)-n)(n-M)/N(n)} \Big(\frac{\|f\|}{\delta_f}\Big)^{(n-M)^2/N(n)} \\
&\quad \cdot \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\frac{N(n)-M}{N(n)} \sum_{k=0}^{n-1} \log \frac{\pi_{k+1} f}{f}(X_k)\Big) \\
&\quad \cdot \exp\Big(\frac{N(n)-M}{N(n)} \sum_{k=1}^{M-1} \log \frac{\pi_{k+1} f}{f}(X_k)\Big)\Big] \\
&\leq \Big(\frac{\|f\|}{\delta_f}\Big)^{4M+(n-M)^2/N(n)} (1+\varepsilon)\frac{(N)n(-n)\,(n-M)}{N(n)} \cdot \\
&\quad \cdot \mathbb{E}^{P_x}\Big[f(X_n)\exp\Big(-\sum_{k=0}^{n-1} \log \frac{\pi_{k+1} f}{f}(X_k)\Big) \\
&\quad \cdot \exp\Big(\frac{M}{N(n)} \sum_{k=0}^{n-1} \log \frac{\pi_{k+1} f}{f}(X_k)\Big)\Big] \\
&\leq f(x)\Big(\frac{\|f\|}{\delta_f}\Big)^{5M+(n-M)^2/N(n)} (1+\varepsilon)^{(N(n)-n)(n-M))/N(n)}.
\end{aligned}$$

Then, we have

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log \mathbb{E}^{\mathbb{P}_x}\Big[f(X_n)\exp\Big(-\frac{1}{N(n)}\sum_{i=1}^{N(n)}\sum_{k=0}^{n-1}\log\frac{\pi_i f}{f}(X_k)\Big)\Big]\leqslant\log(1+\varepsilon).$$

et $\varepsilon\downarrow 0$, the proof is finished.

Set

$$L_n(\omega, A)=\frac{1}{n}\sum_{k=0}^{n-1}\chi_A(X_k(\omega)),\quad A\in\mathscr{B}(E),\ \omega\in\Omega,\ n\in Z^+, \tag{3.6}$$

$$Q_{n,x}=\mathbb{P}_x\circ L_n^{-1},\qquad x\in E,\ n\in Z^+, \tag{3.7}$$

nd

$$I(\mu)=-\inf_{f\in C_b^+(E)}\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{i=1}^{n}\int\log\frac{\pi_i f}{f}(x)\mu(dx),\ \mu\in\mathscr{M}_1(E), \tag{3.8}$$

here $C_b^+(E)=C_b(E)\cap B^+(E)$.

Theorem 3.9. *Under H_1^*, for each compact subset K of $\mathscr{M}_1(E)$ we have*

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_{n,x}(K)\leqslant-\inf_{\mu\in K}I(\mu).$$

Proof

$$\begin{aligned}Q_{n,x}(K)=&\int_K\exp\Big(n\int\log\big[\prod_{k=1}^{N(n)}\pi_k f/f^{N(n)}\big]^{1/N(n)}(x)\mu(dx)\Big)\\&\cdot\exp\Big(n\int\log\Big[(f^{N(n)})\Big/\prod_{k=1}^{N(n)}\pi_k f\Big]^{1/N(n)}(x)\mu(dx)\Big)Q_{n,x}(d\mu)\\\leqslant&\exp\Big(n\sup_{\mu\in K}\log\Big[\Big(\prod_{k=1}^{N(n)}\pi_k f\Big)/f^{N(n)}\Big]^{1/N(n)}(x)\mu(dx)\Big]\\&\cdot\int\exp\Big(n\int\log\Big[f^{N(n)}\Big/\prod_{k=1}^{N(n)}\pi_k f\Big]^{1/N(n)}(x)\mu(dx)\Big)Q_{n,x}(d\mu)\end{aligned}$$

d by Proposition 3.5, we have

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\frac{1}{n}\log\int\exp\Big(n\int\log\Big[f^{N(n)}\Big/\prod_{k=1}^{N(n)}\pi_k f\Big]^{1/N(n)}(x)\mu(dx)\Big)Q_{n,x}(d\mu)\\&=\overline{\lim_{n\to\infty}}\frac{1}{n}\log\int\exp\Big(-\frac{1}{N(n)}\sum_{i=1}^{N(n)}\sum_{k=0}^{n-1}\log\frac{\pi_i f}{f}(X_k)\Big)d\mathbb{P}_x\leqslant 0.\end{aligned}$$

Since

$$\int\log\Big(\frac{\pi_i f}{f}\Big)(x)\mu(dx)$$

continuous in μ, K is compact, there is a $\mu_{n,f}\in K$ such that

$$\begin{aligned}&\sup_{\mu\in K}\frac{1}{N(n)}\sum_{i=1}^{N(n)}\int\log\frac{\pi_i f}{f}(x)\mu(dx)\\&=\frac{1}{N(n)}\sum_{i=1}^{N(n)}\int\log(\pi_i f/f)(x)\mu_{n,f}(dx).\end{aligned}$$

en

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_{n,x}(K)\\&\leqslant\overline{\lim_{n\to\infty}}\frac{1}{N(n)}\sum_{i=1}^{N(n)}\int\log(\pi_i f/f)(x)\mu_{n,f}(dx).\end{aligned}\tag{3.10}$$

Furthermore

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_{n,x}(K)$$

$$\leqslant \inf_{f\in C_b^+(E)} \overline{\lim_{n\to\infty}} \frac{1}{N(n)} \sum_{i=1}^{N(n)} \int \log \frac{\pi_i f}{f}(x)\, \mu_{n,f}(dx)$$

$$\leqslant \sup_{\mu\in K} \inf_{f\in C_b^+(E)} \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \int \log \frac{\pi_i f}{f}(x)\, \mu(dx) = -\inf_{\mu\in K} I(\mu). \quad (3.11)$$

For passing from the case of compact subset to the case of closed subset, we would give a hypothesis on $\{X_k\}_{k=1}^{\infty}$ as follows:

H_2^*: There is a function U and a sequence $\{u_n\}\subset C_b(E)$ such that

(i) $u_n(x)\geqslant 1$; $n\geqslant 1$, $x\in E$;

(ii) $\sup_n u_n(x)<\infty$, $x\in E$;

(iii) $\overline{\lim_{k\to\infty}}\, \overline{\lim_{n\uparrow\infty}} \frac{1}{n} \sum_{i=1}^{n} \log \frac{\pi_i u_k}{u_k}(x) = U(x)$, $x\in E$;

(iv) There exists a constants $A\in R^+$ such that

$$\sup_{k,x} \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \log \frac{\pi_i u_k}{u_k}(x) \leqslant A; \quad (3.1$$

(v) $-U$ is a compact function on E.

Lemma 3.13. *Let $\{\pi_i\}_{i=1}^{\infty}$ be a sequence of transition functions satisfying* H_1^* *a* H_2^*.

(i) *For each $L\in R$, $\{\mu: I(\mu)\leqslant L\}$ is a tight subset of $\mathscr{M}_1(E)$.*

(ii) *For each $N<\infty$, there exists a tight subset K_N of $\mathscr{M}_1(E)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \log Q_{n,x}(K_N^C) \leqslant -N. \quad (3.1$$

Proof Take $\mu\in\mathscr{M}_1(E)$ and $I(\mu)\leqslant L$. Then

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \int \log(\pi_i f/f)(x)\mu(dx) \geqslant -L$$

for each $f\in C_b^+(E)$. In particular

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \int \log \frac{\pi_i u_k}{u_k}(x)\, \mu(dx) \geqslant -L,\ k=1, 2\cdots.$$

By Fatou's lemma and (iv) in H_2^*,

$$\int U(x)\mu(dx) \geqslant -L.$$

Then, for every $h<0$,

$$-L\leqslant \int U(x)\mu(dx) \leqslant h\mu\{x: U(x)<h\}+A.$$

Therefore

$$\mu\{x: U(x)<h\} \leqslant \frac{1}{|h|}(L+A).$$

Set

$$\Gamma_h = \{x: U(x)\geqslant h\}$$

r (v) of H_2^*, Γ_h is a compact subset of E and

$$\{\mu\colon I(\mu)\leqslant L\}\subset\{\mu\colon \mu(\Gamma_h^c)\leqslant(L+A)/|h|\}.$$

his shows that $\{\mu\colon I(\mu)\leqslant L\}$ is a tight subset of $\mathscr{M}_1(E)$.

For $n\geqslant 1$, $\sigma\in R$ and $\varepsilon>0$, set

$$J_n^\sigma=\{x\colon U(x)\geqslant -n\sigma\},$$

$$M_{n,\sigma}^\varepsilon=\{\mu\colon\mu((J_n^\sigma)^c)>\varepsilon\}.$$

hen, J_n^σ is a compact subset of E and for $\mu\in M_{n,\sigma}^\varepsilon$

$$\begin{aligned}\int U(x)\mu(dx)&=\Big(\int_{J_n^\sigma}+\int_{(J_n^\sigma)^c}\Big)U(x)\mu(dx)\\ &\leqslant A\mu(J_n^\sigma)-n\sigma\mu((J_n^\sigma)^c)\\ &=A[1-\mu((J_n^\sigma)^c)]-n\sigma\mu((J_n^\sigma)^c)\leqslant A-\varepsilon n\sigma. \end{aligned}\tag{3.14}$$

On the other hand, by Proposition 3.5 and (i), (ii) and (iv) of H_2^*, for each >0, when n is large enough, we have

$$\begin{aligned}&\int \exp\Big(-n\int U(x)\mu(dx)\Big)Q_{n,x}(d\mu)\\ &\leqslant\int \exp\Big(-n\int \overline{\lim_{k\to\infty}}\,\overline{\lim_{r\to\infty}}\frac{1}{rN(n)}\sum_{i=1}^{rN(n)}\log\frac{\pi_i u_k}{u_k}(x)\mu(dx)\Big)Q_{n,x}(d\mu)\\ &\leqslant\int \exp\Big(-n\,\overline{\lim_{k\to\infty}}\,\overline{\lim_{r\to\infty}}\frac{1}{rN(n)}\sum_{i=1}^{rN(n)}\log\frac{\pi_i u_k}{u_k}(x)\mu(dx)\Big)Q_{n,x}(d\mu)\\ &\leqslant\int \underline{\lim_{k\to\infty}}\,\underline{\lim_{r\to\infty}}\exp\Big(-\frac{1}{rN(n)}\sum_{i=1}^{rN(n)}\sum_{j=0}^{n-1}\log\frac{\pi_i u_k}{u_k}(X_j)\Big)d\mathbb{P}_x\\ &\leqslant\underline{\lim_{k\to\infty}}\,\underline{\lim_{r\to\infty}}\int \exp\Big(-\frac{1}{rN(n)}\sum_{i=1}^{rN(n)}\sum_{j=0}^{n-1}\log\frac{\pi_i u_k}{u_k}(X_j)\Big)d\mathbb{P}_x\leqslant Ce^{\eta n},\end{aligned}\tag{3.15}$$

ıere C is a constant. By (3.14) and (3.15), we have

$$\begin{aligned}Q_{m,x}(M_{n,\sigma}^\varepsilon)&=\int_{M_{n,\sigma}^\varepsilon}\exp\Big(mU(x)\mu(dx)\Big)\exp\Big(-m\int U(x)\mu(dx)\Big)Q_{m,x}(d\mu)\\ &\leqslant Ce^{m(A+\eta)}e^{-mn\sigma\varepsilon},\end{aligned}$$

: m large enough. In particular

$$Q_{m,x}M_{n^2,\sigma}^{1/n})\leqslant Ce^{mA}e^{-mn\sigma}e^{m\eta}.$$

is implies

$$Q_{m,x}\Big(\bigcup_{n=1}^\infty M_{n^2,\sigma}^{1/n}\Big)\leqslant Ce^{m(A+\eta)}\frac{e^{-m\sigma}}{1-e^{-m\sigma}}.\tag{3.16}$$

;

$$K_\sigma=\Big(\bigcup_{n=1}^\infty M_{n^2,\sigma}^{1/n}\Big)^c.$$

en

$$K_\sigma=\Big\{\mu\colon \mu((J_{n^2}^\sigma)^c)\leqslant\frac{1}{n}\text{ or all }n\Big\}$$

is a tight subset of $\mathscr{M}_1(E)$ and

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\log Q_{n,x}(K_\sigma^c)\leqslant A-\sigma+\eta.\tag{3.17}$$

With the lemma in mind, it is easy to obtain the following theorem.

Theorem 3.18. *Let* $\{\pi_k\}_{k=1}^{\infty}$ *be a sequence of transition functions satisfying* H_1^* *and* H_2^*. *Then*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_{n,x}(C) \leqslant -\inf_{\mu\in C} I(\mu), \quad x\in E,$$

for every closed subset C *of* $\mathscr{M}_1(E)$.

§ 4. The Case of the Inhomogeneous Markov Processes with Continuous Parameter

Let $\{p(s, x, t, dy): 0\leqslant s\leqslant t<\infty\}$ be a family of the standard Feller transit functions on E. For $x\in E$, let $(\Omega, \mathscr{F}, \mathbb{P}_x)$ be a probability space and $\{X(t)\}_{t\geqslant 0}$ b right continuous process on $(\Omega, \mathscr{F}, \mathbb{P}_x)$, $\{p(s, x, t, dy): 0\leqslant s\leqslant t<\infty\}$ be the transit functions of $\{X(t)\}_{t\geqslant 0}$ and $\mathbb{P}_x(X(0)=x)=1$. Set

$$L_t(\omega, A)=\frac{1}{t}\int_0^{t_0} \chi_A(X(s, \omega))ds, t>0, \ \omega\in\Omega, \ A\in\mathscr{B}(E), \tag{4}$$

$$Q_{t,x}=\mathbb{P}_x\circ L_t^{-1}, \ t>0,$$

$$(T(s, t)f)(x)=\int f(y)p(s, x, t, dy), \ s\leqslant t, \ f\in C_b(E).$$

It is obvious that for any $0\leqslant s\leqslant t<\infty$, $T(s, t)$ is linear operator form $C_b(E)$ to its $\|T(s, t)\|\leqslant 1$, $T(t, t)$ is an identity and

$$T(s, t)=T(s, t)\cdot T(u, t), \ 0\leqslant s\leqslant u\leqslant t<\infty. \tag{4}$$

$$D(A_s)=\left\{f\in C_b(E): \text{there exists the limit } A_s f\triangleq\lim_{t\downarrow s}\frac{T(s, t)f-f}{t-s}\in C_b(E)\right\}, \tag{4}$$

$$D^+(A_s)=\{f\in D(A_s): \text{there exists } \delta_f>0, \text{ such that } f\geqslant\delta_f\},$$

$$D^+(A)=\bigcap_{s\geqslant 0} D^+(A_s).$$

Clearly, $D^+(A)$ is not empty.

We make a hypothesis on $\{A_s\}_{s\geqslant 0}$ as the following:

H_1^{**}: For each $f\in D^+(A)$ and $T>0$

(i) $\overline{\lim_{t\geqslant s\to\infty}} [A_t f(x)-A_s f(x)]=0$ uniformly in $x\in E$;

(ii) $\lim_{t\downarrow s}\frac{T(s, t)f-f}{t-s}=A_s f$ uniformly in $s\in[0, T]$;

(iii) $\sup_{t\in[0, T]}\|A_t f\|<\infty$;

(iv) $A.f(X(\cdot))$ is R-integrable on $[0, T]$. (4.

By (iii) of H_1^{**}, we know that for each $f\in D^+(A)$ there exists an $N(t)$ such that

$$\lim_{t\to\infty}\overline{N}(t)=0, \tag{4.5}$$

here
$$\overline{N}(t)\triangleq\frac{1}{N(t)}(t+\sup_{s\in[0,t]}\|A_sf\|).$$
imply take
$$N(t)=t^2(1+\sup_{s\in[0,t]}\|A_sf\|)^2.$$

Proposition 4.6. *Under* H_1^{**}, *for each* $f\in D(A)$, *we have*
$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\frac{1}{N(t)}\int_0^{N(t)}\Big(\int_0^t\frac{A_sf}{f}(X(\tau))d\tau\Big)ds\Big)\leqslant 0,$$
here $N(t)$ *satisfies* (4.5).

Proof By H_1^{**} and (4.5), for each $\varepsilon\in(0,1)$, there exists an $M>0$ such that
$$A_sf(x)-A_tf(x)<\varepsilon$$
uniformly in $x\in E$ and
$$\overline{N}(t)=(t+\sup_{s\in[0,t]}\|A_sf\|)/N(t)<\varepsilon,\ t\geqslant s\geqslant M.$$
Then, we obtain
$$\begin{aligned}
&\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\frac{1}{N(t)}\int_0^{N(t)}\Big(\int_0^t\frac{A_sf}{f}(X(\tau))d\tau\Big)ds\Big)\Big]\\
&=\mathbb{E}_{P^x}\Big[f(X(t))\exp\Big(-\frac{1}{N(t)}\Big(\int_0^M\int_0^M+\int_0^M\int_M^t\\
&\quad+\int_M^{N(t)}\int_0^M+\int_M^{N(t)}\int_M^t\frac{A_sf}{f}(X(\tau))\,ds\,d\tau\Big)\Big]\\
&\leqslant C_1\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\frac{1}{N(t)}\int_M^{N(t)}\int_M^t\frac{A_sf}{f}(X(\tau))d\tau ds\Big]\\
&=C_1\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\frac{N(t)-M}{N(t)}\int_M^t\frac{A_\tau f}{f}(X(\tau))d\tau\Big)\\
&\quad\cdot\exp\Big(-\frac{1}{N(t)}\int_M^{N(t)}\int_M^t\frac{A_sf-A_\tau f}{f}(X(\tau))dsd\tau\Big)\\
&\leqslant C_2\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\int_0^t\frac{A_\tau f}{f}(X(\tau))d\tau\Big)\\
&\quad\cdot\exp\Big((M/N(t))\int_0^t\frac{A_\tau f}{f}(X(\tau))\,d\tau\Big)\\
&\quad\cdot\exp\Big(\frac{1}{N(t)}\Big(\int_M^t\int_M^t+\int_t^{N(t)}\int_M^t\Big)\frac{A_\tau f-A_sf}{f}(X(\tau))d\tau ds\Big)\\
&\leqslant\exp\Big(\frac{(N(t)-t)\,(t-M)}{\delta_fN(t)}\,\varepsilon\Big)C_3^{t\overline{N}(t)}\\
&\quad\cdot\mathbb{E}^{P_x}\Big[f(X(t))\exp\Big(-\int_0^t\frac{A_\tau f}{f}(X(\tau))d\tau\Big)\Big].
\end{aligned}$$
For each k, $n\in Z^+$ and $f\in D^+(A)$,
$$\frac{T((k-1)t/n,kt/n)f}{f}=1+\frac{t}{n}\frac{A_{(k-1)t/n}f}{f}+o_k\Big(\frac{t}{n}\Big)$$
$$\log\Big(T\Big(\frac{k-1}{n}t,\frac{k}{n}t\Big)f/f\Big)=\frac{t}{n}(A_{(k-1)t/n}f/f)+o_k\Big(\frac{t}{n}\Big).$$

On the other dand,

$$\left\{X\left(\frac{k}{n}t\right)\right\}_{k=0}^{\infty}$$

is a Markov process on $(\Omega, \mathscr{F}, \mathbb{P}_x)$ and its transition function is $\{\pi_k^{(n)}\}_{k=1}^{\infty}$, where

$$\pi_k^{(n)}(x, dy) = p\left(\frac{k-1}{n}t, x, \frac{k}{n}t, dy\right)$$

for $k, n \in Z^+$. Therefore, by Propositions (3.2), (3.3) and (ii), (iv) of H_1^{**}, we have

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}_x}\left[f(X(t))\exp\left(-\int_0^t \frac{A_\tau f}{f}(X(\tau))d\tau\right)\right] \\
&= \mathbb{E}^{\mathbb{P}_x}\left[f(X(t))\lim_{n\to\infty}\exp\left(-\sum_{k=0}^{n-1}\frac{t}{n}(A_{kt/n}f/f)\left(X\left(\frac{k}{n}t\right)\right)\right)\right] \\
&\leqslant \lim_{n\to\infty}\mathbb{E}^{\mathbb{P}_x}\left[f(X(t))\exp\left(-\sum_{k=0}^{n-1}\log\frac{\pi_{k+1}^{(n)}f}{f}\left(X\left(\frac{k}{n}t\right)\right)\right)\exp(0(1))\right] \\
&\leqslant (1+\eta)f(x)
\end{aligned}$$

for each $\eta > 0$. Letting $\eta \to 0$, we have

$$\mathbb{E}^{\mathbb{P}_x}\left[f(X(t))\exp\left(-\int_0^t \frac{A_\tau f}{f}(X(\tau))d\tau\right)\right] \leqslant f(x).$$

Hence

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log\mathbb{E}^{\mathbb{P}_x}\left[f(X(t))\exp\left(-\frac{1}{N(t)}\int_0^{N(t)}\left(\int_0^t \frac{A_s f}{f}(X(\tau))d\tau\right)ds\right)\right] \leqslant s.$$

Let $s \downarrow 0$, the proof is finished.

Setting

$$I(\mu) = -\inf_{f\in D^+(A)}\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t\left(\int\frac{A_s f}{f}(x)\mu(dx)\right)ds, \quad \mu\in\mathscr{M}_1(E),$$

we have the following theorem.

Theorem 4.7. *Under* H_1^{**},

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log Q_{t,x}(K) \leqslant -\inf_{\mu\in K} I(\mu)$$

for each compact subset K *of* $\mathscr{M}_1(E)$.

Proof

$$\begin{aligned}
Q_{t,x}(K) &= \int_K \exp\left(\frac{t}{N(t)}\int_0^{N(t)}\left(\int\frac{A_s f}{f}(x)\mu(dx)\right)ds\right) \\
&\quad \cdot\exp\left(-\frac{t}{N(t)}\int_0^{N(t)}\left(\int\frac{A_s f}{f}(x)\mu(dx)\right)ds\right)Q_{t,x}(d\mu) \\
&\leqslant \exp\left(\sup_{\mu\in K}\frac{t}{N(t)}\left(\int_0^{N(t)}\frac{A_s f}{f}(x)\mu(dx)\right)ds\right) \\
&\quad \cdot\mathbb{E}^{\mathbb{P}_x}\left[\exp\left(-\frac{1}{N(t)}\int_0^{N(t)}\left(\int\frac{A_s f}{f}(X(\tau))d\tau\right)ds\right)\right] \\
&\leqslant \exp\left(\frac{t}{N(t)}\int_0^{N(t)}\left(\int\frac{A_s f}{f}(x)\mu_{t,f}(dx)\right)ds\right) \\
&\quad \cdot\mathbb{E}^{\mathbb{P}_x}\left[\exp\left(-\frac{1}{N(t)}\int_0^{N(t)}\left(\int_0^t\frac{A_s f}{f}(X(\tau))d\tau\right)ds\right)\right],
\end{aligned}$$

ɀere $\mu_{t,f}\in K$. By (4.5)

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log Q_{t,x}(K)\leqslant\lim_{t\to\infty}\frac{1}{N(t)}\int_0^{N(t)}\frac{A_sf}{f}(x)\mu_{t,f}(dx)\Big)ds.$$

orever

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log Q_{t,x}(K)$$

$$\leqslant\inf_{f\in D^+(A)}\overline{\lim_{t\to\infty}}\frac{1}{N(t)}\int_0^{N(t)}\Big(\frac{A_sf}{f}(x)\mu_{t,f}(dx)\Big)ds$$

$$\leqslant\sup_{\mu\in K}\inf_{f\in D^+(A)}\overline{\lim_{t\to\infty}}\frac{1}{N(t)}\int_0^{N(t)}\Big(\int\frac{A_sf}{f}(x)\mu(dx)\Big)ds$$

$$\leqslant\sup_{\mu\in K}\inf_{f\in D^+(A)}\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t\Big(\int\frac{A_sf}{f}(x)\mu(dx))ds=-\inf_{\mu\in K}I(\mu).\tag{4.8}$$

In oader to pass from the case of compact subset to the case of closed subset: we ›ed a hypothesis as the following: H_2^{**}: There is a function U and a sequence $\{u_n\}\in$ $^+(A)$ such that

(i) $u_n(x)\geqslant 1$, $n\geqslant 1$, $x\in E$,

(ii) $\sup\limits_n u_n(x)<\infty$, $x\in E$,

(iii) $\overline{\lim\limits_{n\to\infty}}\,\overline{\lim\limits_{t\to\infty}}\frac{1}{t}\int_0^t\frac{A_su_n}{u_n}(x)ds=U(x)$, $x\in E$,

(iv) there is $B<\infty$ such that

$$\sup_{n,x}\overline{\lim_{t\to\infty}}\frac{1}{t}\int_0^t\frac{A_su_n}{u_n}(x)ds\leqslant B,$$

(v) $-U$ is a compact function on E. (4.9)

Lemma 4.10. *Under H_1^{**} and H_2^{**}, for erch $L\in R$, $\{\mu: I(\mu)\leqslant L\}$ is a tight subset $\mathscr{M}_1(E)$ and for each $N<\infty$, there exists a tight subset K_N of $\mathscr{M}_1(E)$ such that*

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\log Q_{t,x}(K_N^C\leqslant -N.\tag{4.11}$$

Proof The proof is similar to that of Theorem (3.9). It is enough to show, for ch 0 and t largh enough, that

$$\int\exp\Big(-t\int U(x)\mu(dx)\Big)Q_{t,x}(d\mu)$$

$$=\int\exp\Big(-t\int\overline{\lim_{n\to\infty}}\,\overline{\lim_{T\to\infty}}\frac{1}{T}\int_0^T\frac{A_su_n}{u_n}(x)\mu(dx)\Big)Q_{t,x}(d\mu)$$

$$\leqslant\int\exp\Big(-t\int\overline{\lim_{n\to\infty}}\,\overline{\lim_{r\to\infty}}\frac{1}{rN(t)}\int_0^{rN(t)}\frac{A_su_n}{u_n}(x)\mu(dx)\Big)Q_{t,x}(d\mu)$$

$$\leqslant\int\lim_{n\to\infty,}\lim_{r\to\infty}\int\exp\Big(-t\int\frac{1}{rN(t)}\int_0^{rN(t)}\frac{A_su_n}{u_n}(x)\mu(dx)\Big)Q_{t,x}(d\mu)$$

$$=\lim_{n\to\infty}\lim_{r\to\infty}\int\exp\Big(-\frac{1}{rN(t)}\int_0^t\int_0^{rN(t)}\frac{A_su_n}{u_n}(X(\tau)d\tau)\Big)d\mathbb{P}_x\leqslant Me^{st}.$$

With the lemma in mind it is easy to obtain the following theorem.

Theorem 4.12. *Under H_1^{**} and H_2^{**}, we have*

$$\overline{\lim_{t\to\infty}} \frac{1}{t} Q_{t,\alpha}(C) \leqslant -\inf_{\mu\in C} I(\mu)$$

for each closed subset C of $\mathscr{M}_1(E)$.

Acknowledgment. The auther wishes to express thanks to professors S. J Yan and M. F. Chan for their guidance.

References

[1] Donsker M. D. and Varadhan, S. R. S., Asymptotic Evaluation of Certain Markov Process Expectai for Large Time, *I. Comm. Pure. App. Math.*, **28** (1975), 1—47; *II. Comm. Pure. App. Math.*, **28** (1 279—301; *III. Comm. Pure. App. Math.*, **29** (1976), 389—461; *IV. Comm. Pure. App. Math.*, **36** (19 183—212.

[2] Stroock, D. W. An Introduction to the Theory of Large Dexiations, Springer-Verlg, New York 19

[3] Stroock, D. W. and Varadhan, S. R. S. Multidimensimal Diffusion Processes, Springer-Verlag, York, 1979.

[4] EillsR. S., Large Deviations for General Class of Random Vectors, *Ann. Prob.*, **12:1** (1984), 1—12.

[5] Accardi, L. and Olla. S. Donsker and Varadhan's Theory for Stationary Processes (preprint).

[6] Donsker.M. D. and Varadhan, S. R. S. Asymptotics for The Wiener Sauasge *Comm. Pure. A* **28** (1975), 525—565.

[7] Donsker, M. D. and Varadhan, S. R. S. Large Deviation Asymptoties and The Polaron, In Stoch Processes in Quantum Theory and Statistial Mechanies, Eb. s Albeverio, Ph. Combe, M, Sirugue-C Springer Lecture Notes in Physics, No. 173.

[8] Donsker, M. D. and Varadhan, S. R. S. Asymptotics for the Polaron, *Comm. Pure Appl.*, **36** (1 505—528.

[9] Donsker, M. D. and Varadhan, S. R. S. Large Deviation for Stationary Gaussian Processes, *Co Math. Phys.*, **97** (1985), 187—211.

[10] Kac, M., Integration in Function Space and Some of its Applications Lezioni Fermiane, Scuola Norı Superlore Pisa, (1980).