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TORSION THEORIES OVER *N*-RINGS

LUO JIANGANG (罗建刚)*

Abstract

In this paper the author studies some properties concerned with torsion theories over an N-ring. Among other things, it is provedthat N-rings are stable.

§0. Introduction

All rings in this paper are assumed to be commutative with identity, while all modules unitary.

In [1], [2] and [3], N-rings have been studied extensively by Gilmer. Heinzer and Lantz. An N-rnig, by definition, is aring R with the following property: for every ideal I of R, there exists a unitary extension T of R (i. e. $R \subseteq T$ and T has the same identity as R) such that T is noetherian and $IT \cap R = I$. Such rings can be viewed as ageneralization of noetherian rings. Many examples of nonnoetherian N-rings are offered in the papers mentioned above.

The purpose of the present paper is to study torsion theories over N-rings and consider some related questions. In § 1, we first extend to N-rings the well-known characterization of idempotent filters of commutative noetherian rings (see [4]) and then extend the result in Gabriel's paper [5] (also see [6]) that commutative noetherian rings are stable. In § 2, we discuss some miscellaneous questions. In § 3, we show that the fixed subring of an N-ring R under a finite automorphism group of R is itself an N-ing. In a more general setting, we probe into the localization rings of fixed subrings.

Throughout this paper, all terminologies and notations concerning torsion theories (resp. commutative algebra) are kept in accordance with [7] (resp. [8]). We use$E_R(M)$, or $E(M)$ when R is clear from the context, to denote the injective hull of an R-module M. As usual, Spec R stands for the prime spectrum of R.

§1.

We start by recalling that a nonempty set A of ideals ofa (commutative) ring

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* Department of Mathematics, Beijing Normal University, Beijing, China.

R is an idempotent filter if

A1. If $I \in A$ and $a \in R$, then $(I:a) \in A$,

A2. If I is an ideal of R for which there is an $H \in A$ such that $(I:h) \in A$ for all $h \in H$, then $I \in A$.

Proposition 1.1. *Let $S=\{I_i\}_{i\in\Lambda}$ be a nonempty set of ideals of an N-ring R. Then the following set is an idempotent filter of R*

$$\mathscr{L}=\{I\lhd R \mid I\supseteq I_1I_2\cdots I_n,\ I_i\in S\},$$

where for $i\neq j$, I_i and I_j are not necessarily different.

Proof Suppose that $I\in\mathscr{L}$, $I\supseteq I_1I_2\cdots I_n$, $I_i\in S$. Given any $x\in R$, we l $(I:x)\supseteq I\supseteq I_1I_2\cdots I_n$. Therefore, $(I:x)\in\mathscr{L}$. So A1 holds for $\mathscr{L}$.

To check A2 for $\mathscr{L}$, let $H\in\mathscr{L}$, or $H\supseteq I_1I_2\cdots I_n$ for some $I_i\in S$ and I be an i of R such that for all $h\in H$, $(I:h)\in\mathscr{L}$. We shall show that $I\in\mathscr{L}$. Take a noethe unitary extension T of R such that $IT\cap R=I$. Since T is noetherian, we have

$$I_1I_2\cdots I_nT=x_1T+\cdots+x_tT$$

with x_i in $I_1I_2\cdots I_n$. Then we know from the hypothesis that for each x_i there e $J_1^{(i)},\cdots,J_{n_i}^{(i)}\in S$ such that $J_1^{(i)}J_2^{(i)}\cdots J_{n_i}^{(i)}\subseteq(I:x_i)$. Let J be the product of all t $J_k^{(i)}$s. Then it follows that $J\supseteq(I:x_1)\cap(I:x_2)\cap\cdots\cap(I:x_t)$. Thus, $JI_1I_2\cdots I_n\subseteq JI_1$ $I_nT\cap R=\left(\sum_{i=1}^{t}Jx_iT\right)\cap R\subseteq IT\cap R=I$, i. e. $I\in\mathscr{L}$.

We refer to the torsion theory determined by S as the S-adic torsion theory denote it by τ_s.

Definition. *A primary ideal I of a ring R is said to be strongly primar some power of the radical is containd in I. A ring R is said to be strongly Laske if every proper ideal of R can be represented as the intersection of a finite numbe strongly primary ideals.*

Proposition [GH] (Proposition 2.14 in [1]). *N-rings are strongly Lasker*

This result, which plays a fundamental role in this paper, indicates that prin decomposition are practicable in N-rings. The following results reveal to us tha bring all the idempotent filters of R under control we have only to know all the me ideals of R and their behavior under product.

Proposition 1.2. *Let R be an N-ring and τ a nontrivial torsion theory ove Then we have $\tau=\tau_{\mathscr{P}}$ where $\mathscr{P}=\mathscr{L}_\tau\cap\operatorname{Spec}R$.*

Proof We are going to show that $\mathscr{L}_\tau=\mathscr{L}_{\tau_{\mathscr{P}}}$. It is obvious that $\mathscr{P}$ is nonem According to Proposition 1.1, $\mathscr{P}$ determines a torsion theory $\tau_{\mathscr{P}}$. Since $\mathscr{P}\subseteq\mathscr{L}_\tau$ have $\mathscr{L}_{\tau_{\mathscr{P}}}\subseteq\mathscr{L}_\tau$. For the other direction, let $I\in\mathscr{L}_\tau$ and $I\neq R$. Then

$$I=\bigcap_{i=1}^{n}I_i$$

by Proposiotion [GH], where I_i is P_i-primary for some $P_i\in\operatorname{Spec}R$ and $P_i^{m_i}\subseteq I_i$ for ome m_i. From $P_i \supseteq I_i \supseteq I$, we deduce that $P_i \in \mathscr{L}_\tau \cap \text{Spec } R$. Noting that $P_1^{m_1} P_2^{m_2} \cdots$ $P_n^{m_n} \subseteq I_1 \cap I_2 \cap \cdots \cap I_n = I$, we have $I \in \mathscr{L}_{\tau_{\mathscr{P}}}$, i. e., $\mathscr{L}_\tau \subseteq \mathscr{L}_{\tau_{\mathscr{P}}}$.

Now we start to prove that N-rings are seminoethrian. We begin with the ollowing useful lemma.

Lemma 1.3. *Let R be a ring and $\tau \in R$-tors. Given $P \in \text{Spec } R$, then R/P is ither τ-torsion or τ-torsion-free.*

Proof If R/P is not τ-torsion-free, then there mustbe an $r \in R/P$ and an $I \in \mathscr{L}_\tau$ such that $Ir \subseteq P$. Since P is prime, we have $I \subseteq P$, so $P \in \mathscr{L}_\tau$, namely, R/P is τ-tortion.

Definition. *Let R be a ring and $\tau \in R$-tors. A nonzero R-module M is said to be τ-cocritical if M is τ-torsion-free and every proper quotient module of M is τ-torsion. A ring R is said to be seminoetherian if everyproper torsion theory τ over R has a τ-cocritical R-module.*

Theorem 1.4. *N-rings are seminoetherian.*

Proof Let R be an N-ring. We must show that every proper torsion theory τ over R has a τ-cocritical module.

By Proposition [GH], $0 = I_1 \cap I_2 \cap \cdots \cap I_n$ where I_i is P_i-primary and $P_i^{m_i} = I_i$ for some natural number m_i. Then $P_1^{m_1} P_2^{m_2} \cdots P_n^{m_n} \subseteq I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n = 0$ i. e., $P_1^{m_1} P_2^{m_2} \cdots P_n^{m_n} = 0$. By the assumption that τ is proper, we must have $P_i \notin \mathscr{L}_\tau$ for ome i. Therefore $R \backslash P_i$ has to be τ-torsion-free in view of Lemma 1.3. Thus the set

$$\mathscr{P} = \{P \in \text{Spec} R \mid R/P \text{ is } \tau\text{-torsion-free}\}$$

s not empty. Using Proposition [GH] and the fact that strongly Laskerian rings havd a. c. c. for prime ideals (see Theorom 4 in [9]), we can find a maximal element P in $\mathscr{P}$. We claim that R/P is τ-cocritical. From our choice we know that R/P is τ-torsion-free. To complete the proof, itsuffices to show that for an ideal $I \neq R$ $I \not\subseteq P$, R/I is τ-torsion. Again using Proposition [GH], we have $I \supseteq P_1 P_2 \cdots P_k$ where $P_i \in$ Spec R and $P_i \supseteq I \supseteq P$. Be the maximality of P in $\mathscr{P}$, we know that P_i is not τ-torsion-free, so it is τ-torsion by Lemma 1.3. Hence, $P_1 P_2 \cdots P_k \in \mathscr{L}_\tau$ and therefore $I \in \mathscr{L}_\tau$.

Definition. *Let R be a ring. $\tau \in R$-tors. τ is said to be stable if $\mathscr{T}_\tau$ is closed under taking injective hulls. R is said to be stable if every torsion theory over R is stable.*

Proposition 1.5. *Let R be a seminoetherian ring. Then R is stable if and only if every prime torsion theory over R is stable.*

Proof An intersection of stable torsion theories is stable. According to Raynaud's Theorem (see Proposition 20.12 in [7] where "seminoethrian" is misprinted as "semiartinian"), each torsion theory over a seminoetherian ring is an intersection of prime torsion theories.

From this, the desired result is clear. Next, we give a lemma which assures us of certain "boundedness".

Lemma 1. 6. *Let R be an N-ring, $\tau\in R$-tors. Given a pair of ideals of $RI\subseteq J$ such that $J\backslash I\in\mathscr{T}_\tau$, then we can find an $H\in\mathscr{L}_\tau$ such that $HJ\subseteq I$.*

Proof Take a noetherian unitary extension T of R such that $IT\cap R=I$. Suppose that $x_1T+x_2T+\cdots+x_nT=JT$ with x_i in J. Since $J/I\in\mathscr{T}_\tau$, by assumption, there exist $H_1, H_2, \cdots, H_n\in\mathscr{L}_\tau$ such that $H_ix_i\subseteq I$. Set $H=H_1H_2\cdots H_n$. Then $H\in\mathscr{L}_\tau$ and we have $HJ\subseteq H(\sum x_iT)\cap R\subseteq(\sum(Hx_i)T\cap R\subseteq IT\cap R=I$.

Theorem 1. 7. *N-rings are stable.*

Proof Let R be an N-ring. By Theorem 1.4 and Proposition 1.5. it suffice show that prime torsion theories over R are stable.

In the commutative case, prime torsion theories coincide with the us localization at a prime ideal (Example 3 of Chapter VII in [7]). So we have in i to show the stableness for $\tau=\chi(R/P)$ where P is prime. Suppose, on the contra that τ is not stable, or equivalently, that there exists an R-module M such t $M\in\mathscr{T}_\tau$ but $E(M)\notin\mathscr{T}_\tau$. Since $T_\tau(E(M))\nsubseteq M$, we may assume that $T_\tau(M)=$ without loss of generality. Take any $m\in E(M)\backslash M$, we have $R_m=R/I\notin\mathscr{T}_\tau$, but

$$T_\tau(Rm)=Rm\cap M=J/I\in\mathscr{T}_\tau$$

and that J/I is essential in R/I as R-modules. By Lemma 1.6 we have an $H\in$ such that $HJ\subseteq I$, or equivalently, there is an $s\in R\backslash P$ such that $sJ\subseteq I$. Suppose t $sp\in sP\cap J$ for some $p\in P$. Since $R/J\in\mathscr{F}_\tau$, it follows that $J\cap R\backslash P=\varnothing$ and he $J\subseteq P$. So $P/J\subseteq R/J$ and $P/J\in\mathscr{F}_\tau$. Hence, $sp\in J$ implies that $p\in J$. Thus, obtain $sP\cap J\subseteq sJ\subseteq I$, in other words, $(sP+I)/I\cap J/I=0$. Noting that J/I essential in R/I, we have $sP\subseteq I\subseteq J$. Again using the fact that $R/J\in\mathscr{F}_\tau$, we h $P\subseteq J$. In summary, we have proved that $P=J$. Then, $sR\cap P=sP=sJ\subseteq I$. Simila we have $sR\subseteq I\subseteq J=P$. In particular, $s\in P$, a contradiction. Therefore, we h finishe d the proof that R is stable.

Corollary 1.8. *Commutative noetherian rings are stable.*

§2.

First, we consider N-rings of Krull dimension zero. Here, the Krull dimens of a ring R is the largest length of prime ideals of R.

Theorem 2. 1. *If R is an N-ring of Krull dimension zero, then R is isomorp to a direct product of a finite number of local N-rings of Krull dimension zero and t decomposition is unique up to isomorphism.*

Proof By using Proposition [GH], the proof is similar to that of the Structu Theorem for Artinian Rings (see, for instance, [8]). The only thing that needs more explanation is that each direct summand in the decomposition is an N-ring. This is true because the class of N-rings is closed under taking homomorphic images (see Corollary 2.3 in [1])

We now givean application of Theorem 2.1.

Proposition 2.2. *Suppose that R is an N-ring of Krull dimension zero. Then there is a ring $S=F_1[x]\oplus\cdots\oplus F_n[x]$ where F_is are fields and x an indeterminate satisfying the condition that $R[x]$-tors and S-tors are isomorphic as lattices.*

Proof By Proposition 2.1, $R=R_1\oplus R_2\oplus\cdots\oplus R_n$, where R_is are local N-rings of Krull dimension zero. Let A_i be the unique maximal ideal of R_i and $F_i=R_i/A_i$. We have $R[x]=R_1[x]\oplus\cdots\oplus R_n[x]$. The zero ideal of R_i can be represented as a product of prime ideals of R_i by Proposition [GH]. Since A_i is the only prime ideal of R_i, $A_i^{k_i}=0$ for some k_i. Consequently, we know that $A_1[x]+\cdots+A_n[x]$ is nilpotent. It is known that if T is a ring and I an ideal of T, then we have a one-to-one correspondence between T/I-tors and the subset of T-tors $\{\tau\in T\text{-tors}\,|\,\tau<\xi(T/I)\}$. Moreover, this correspondence is order-preserving (see [10]). Since I is now nilpotent, $I^k=0$ for some k, thus $\xi(R/I)=\chi$. It followsthat $\{|\tau\in T\text{-tors}\,|\,\tau<\xi(R/I)\}=$ T-tors where T is $R_1[x]\oplus\cdots\oplus R_n[x]$ and $I=A_1[x]\oplus\cdots\oplus A_n[x]$. Note that $T/I\cong F_1[x]\oplus\cdots\oplus F_n[x]=S$, therefore $R[x]$-tors is isomorphic to S-tors as lattices

Proposition 2.4 indicates that the lattice $R[x]$-tors for an N-ring R of Krull dimension zero has been determined to a great extent because the torsion theories over a direct product of a finite number of rings can be constructed from that over each of its direct summand (see the remark at the end of Chapter II in [7]). And $F_i[x]$, as a noetherian ring, is much more concret as far as the lattice of torsion theories is concerned.

In general, $R[x]$-tors for an N-ring R need not be an N-ring. In fact, it is an N-ring if and only if R is noetherian.[1]

We know that semiartinian noetherian rings are artinian (see Proposition 12.8 in [7]). Nevertheless, semiartinian N-rings need not be artinian. We may assure ourselves of this fact by examining the example presented in [1]: Let $R=F(t_1, t_2, \cdots]/(\{t_it_j\})$, where F is a field and $t_1, t_2, \cdots$ a set of an infinite number of indeterminates over F.

It is well-know that a commutative noetherian ring R is artinian if and only if the Krull dimension of R is zero. Similarly, we have the following theorem.

Theorem 2.3. *An N-ring R is semiartinian if and only if the Krull dimension of R is zero.*

Proof We select a complete set of representatives of the isomorphism classes of simple R-modules and denote it by R-simp. We use $\mathfrak{m}$ to denote the set of maximal ideals of R.

Suppose that R is semiartinian. Then $\chi=\chi(0)=\xi(R\text{-simp})$ by definition.

Remember that $\tau_{\mathfrak{m}}$ is the $\mathfrak{m}$-adic torsion theory determined by $\mathfrak{m}$, we have

$$\xi(R\text{-simp})=\tau_{\mathfrak{m}}.$$

Hence $\tau_{\mathfrak{m}}=\chi$ and consequently $0=M_1M_2\cdots M_k$ for some $M_i\in\mathfrak{m}$. For any $P\in\text{Spec}R$, we have $M_1M_2\cdots M_k=0\subseteq P$ and therefore $M_j\subseteq P$ for some j, thus, $M_j=P$. This is equivalent to saying that the Krull demension of R is zero.

Conversely, suppose that R is zero-demensional. Again, by Proposition [GH], $0=P_1P_2\cdots P_k$ where $P_i\in\text{S peo } R=\mathfrak{m}$. Therefore, $0\in\mathscr{L}_{\tau_m}=\mathscr{L}_{\xi(R\text{-simp})}$, i. e.,

$$\xi(R\text{-simp})=\chi,$$

namely, R is semiartinian.

The characterization of idempotent filters among ideal classes is always interesting problem. Here we contribute some items to the list of idempotent filt over N-rings.

Definitions. *We define the Gabriel filtration of a ring R as a chain $\{\tau_i\}_{i=-1}^{\infty}$ torsion theories over R satisfying*

(i) $\tau_{-1}=\xi=\xi(0)$,

(ii) *if i is not a limit ordinal*

$$\tau_i=\tau_{i-1}\vee\xi(M\in R\text{-mod } M \text{ is } \tau_{i-1}\text{-cocriticle}),$$

(iii) *if i is a limit ordinal*

$$\tau_i=\vee\{\tau_j\,|\,j<i\}.$$

Theorem 2.4. *Let $\tau_{-1}<\tau_0<\tau_1<\cdots$ be the Gabriel filtration of an N-ring Then for any integer $h\geqslant 0$, we have $\tau_n=\tau_{\mathscr{P}_n}$ where $\mathscr{P}_n$ is the set of all prime ideals o whose depths do not exceed n.*

Proof We use induction on n.

If $n=0$, it is clear.

Suppose that $\tau_n=\tau_{\mathscr{P}_n}$ hold for $n\leqslant k$. Consider the case when $n=k+1$. Let $P\in$ S R, depth $P=k+1$. We claim that R/P is τ_k-cocritical. First, we have that $R/$ τ_k-torsion-free, for, otherwise, by Lemma 1.3, R/P would be τ_k-torsion, then $P_1P_2\cdots P_m$ for some prime ideals P_i of depth not exceeding k. So $P\supseteq P_j$ for som This contradicts the fact that depth $P=k+1$. Second, suppose that $P\not\subseteq I\subseteq$ According to Proposition [GH], $I\supseteq Q_1Q_2\cdots Q_t$ for some $Q_i\in\text{Spec } R$ and $Q_i\not\supseteq P$. implies that depth $Q_i\leqslant k$. So we deduce that $I\in\mathscr{L}_{\mathscr{P}_k}$, by the induction hypoth $\mathscr{L}_{\tau\mathscr{P}_k}=\mathscr{L}_{\tau_k}$, so $R/I\in\mathscr{T}_{\tau_k}$. Thus $\tau_{\mathscr{P}_{k+1}}\subseteq\tau_{k+1}$. For the inverse inclusion, let M τ_k-cocritical R-module. It suffices to show that every cyclic submodule of M is τ torsion. Suppose that $0\neq R_m=R/P$ is a cyclic submodule. Then P must be prime Proposition 18.2 and Corollary 18.8 in [7]). We claim that depth $P=k+1$. Firs cannot be smaller than $k+1$, or else we would have $R/P\in\mathscr{T}_{\tau\mathscr{P}_k}=\mathscr{T}_{\tau_k}$, a contradic to the fact that R/P is τ_k-torsion-free. Second, the depth of P cannot be larger than $k+1$ either. Suppose, on the contrary, that there is a chain of prime ideals

$P\nsupseteq P_0\nsupseteq P_1\nsupseteq\cdots\nsupseteq P_{k+2}$. Note that R/P is τ_k–cocritical, so $P_1\in\mathscr{L}_{\tau_k}$. This tells us that $P_1\supseteq Q_1Q_2\cdots Q_s$ with $Q_i\in\operatorname{Spec} R$ and depth $Q_i\leqslant k$, so $P\supseteq Q_j$ for some j. Tnis contradicts the fact that depth $P_1\geqslant k+1$. Thus the depth of P must be $k+1$. The proof is completed.

Corollary 2.5. *If R is an N–ring.The nthe Gabriel dimension of R G–dim$R$ is finite if and only if the Krull dimension of R K–dim$R$ is finite. In addition, if they are indeed finite, they must be equal.*

Proof For any integer $n\geqslant 0$, G–dim$R\leqslant n$ implies that $\tau_n=\chi$, i. e., $\mathscr{L}_{\tau_n}$ contains all the prime ideals of R. By Theorem 2.4, $\tau_{\mathscr{P}_n}=\tau_n=\chi$. This is the same as saying that depth $P\leqslant n$ for all $P\in\operatorname{Spec}R$. Therefore, K–dim$R\leqslant n$. Note that the deduction above is reversible, so the first statement has been proved. When the two dimensions are both finite, the above proot also indicates that for any integer $n\geqslant 0$, G–dim$R\leqslant n$ if and only if K–dim$R\leqslant n$. This forces the two dimensions to be equal.

§3.

In this section, we always assume that R is a ring, G a finite automorshism group of R such that the order of G, $n=|G|$, is an invertible element in R and $R^G=\{r\in R\mid \alpha(r)=r \text{ for all } \alpha\in G\}$, the fixed subring of R under G.

Proposition 3.1. *For any ideal I of R^G, $IR\cap R^G=I$.*

Proof It is clear that $I\subseteq IR\cap R^G$.

Suppose that $r\in IR\cap R^G$. Then

$$r=\sum_{i=1}^{n} r_i a_i$$

for some $a_i\in I$, $r_i\in R$. Write G as $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Then for each $j=1, 2, \cdots, n$,

$$r=\alpha_j(r)=\sum\alpha_j(r_i)\alpha_j(a_i=\sum\alpha_j(r_i)a_i.$$

Taking summation about j, we obtain

$$nr=\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\alpha_j(r_i)\ \Big)a_i.$$

It is easy to see that $n^{-1}(\sum_j\alpha_j(r_i))\in R^G$ for all i. Thus, $IR\cap R^G=I$.

Corollary 3.2. *If R is an N–ring, then R^G is also an N–ring.*

Proof Given any ideal I of R^G, by assumption, we can find a noetherian unitary extension T of R such that $IRT\cap R=IR$. According to Proposition 3.1, $IR\cap R^G=I$, so

$$IT\cap R^G=(IRT\cap R)\cap R^G=IR\cap R^G=I.$$

Definition. *A torsion theory τ over R is said to be R^G–lifted if $I\in\mathscr{L}_\tau$ and $J\supseteq I\cap R^G$ for $J\lhd R$ imply $\in\mathscr{L}_\tau$.*

Proposition 3.3. *There is a one–to–one correspondence between R^G–tors and R^G–*

lifted torsion theories over R.

Proof Given $\sigma\in R^G$–tors, send it to $\sigma^e\in R$–tors determined by $\mathscr{L}_{\sigma_e}=\{I\lhd R\,|\,I\supseteq HR$ for some $H\in\mathscr{L}_\sigma\}$.

It is known that $\mathscr{L}_{\sigma_e}$ is an idempotent filter (see section 9 in [7]). σ^e is obviously R^G–lifted. Next, given any R^G–lifted $\tau\in R$–tors, we define $\tau^c\in R^G$–tors by $\mathscr{L}_{\tau_c}=\{I\lhd R^G\,|\,I=H\cap R^G$ for some $H\in\mathscr{L}_\tau\}$. Using Proposition 3.1, we can prove that $\mathscr{L}_{\tau_c}$ is indeed an idempotent filter and that $\sigma^{ec}=\sigma$ and $\tau^{ce}=\tau$. We leave out the proof since it is routine.

For the rest of this paper, we assume, once and for all, that $\sigma\in R^G$–tors and $\sigma^e\in R$–tors.

Lemma 3.4. *If M is an R–module, then $T_\sigma(M)=T_\tau(M)$ as R^G–modules.*

Proof It is clear since $\tau=\sigma^e$.

Proposition 3.5. *Suppose, in addition, that σ is perfect. Then we can m $Q_\sigma(R)$ into an R–module so that the R–module structure on $Q_\sigma(R)$ preserves the canon R–module structure on $\bar{R}=R/T_\sigma(R)$. Moreover, $Q_\sigma(R)$ and $Q_\tau(R)$ are isomorphi R–modules and this isomorphism extends the identity map on $\bar{R}$.*

Proof Since σ is perfect, we have $Q_\sigma(R)=(R^G)_\sigma\otimes_{R^G}R$. This makes $Q_\sigma(R)$ an R–module.

From

$$\hat{\sigma}_R\colon R\to Q_\sigma(R)\quad r\mapsto i\otimes r$$

and $\operatorname{Ker}\hat{\sigma}_R=T_\sigma(R)$, we see directly that the R–module structure on $Q_\sigma(R)$ prese the R–module structure on $\bar{R}$.

Note that $Q_\sigma(R)$ is an essential extension of $\bar{R}$ as R–module. By Lemma 3.4 is τ–dense in $Q_\sigma(R)$. Using the τ–injectiveness of $Q_\tau(R)$, we can extend embedding of $\bar{R}$ into $Q_\tau(R)$ to an R–homomorphism α: $Q_\sigma(R)\to Q_\tau(R)$. Since restriction of α to $\bar{R}$ is a monomorphism, so is α. We assert that it is also epic suffices to prove that $Q_\sigma(R)$ is a τ–injective R–module. Suppose that $I\in\mathscr{L}_\tau$ an $I\to Q_\sigma(R)$ an R–homomorphism. By Lemma 3.4, I is a σ–dense submodule of R^G–module R. By use of the σ–injectiveness of $Q_\sigma(R)$, β can be extended to R^G–homomorphism $\bar{\beta}$: $R\to Q(R)$. The proof will be finished if we can prove th is also an R–homomorphism. To do this, let $r\in R$. Set

$$f_r\colon R\to Q_\sigma(R)\qquad s\mapsto\bar{\beta}(rs)-r\bar{\beta}(s).$$

Then f_r is an R^G–homomorphism. For any $a\in I$, $f_r(a)=\bar{\beta}(ra)-r\bar{\beta}(a)=\beta(ra)-r$ $=0$. So f_r induces an R^G–homomorphism $\bar{f}_r$ from R/I to $Q_\sigma(R)$ such that $\bar{f}_r(s+$ $f_r(s)$. $Q_\sigma(R)$ is σ–torsion–free while, by Lemma 3.4, R/I is σ–torsion, so $\bar{f}_r$, h f_r, must be the zero map, i. e., $\bar{\beta}(rs)=r\bar{\beta}(s)$ for all $s\in R$. Since r is arbitr chosen, is $\bar{\beta}$ an R–homomorphism.

Theorem 3.6. *Suppose, as before, that σ is perfect. Then an automorphism of R*

which belongs to G induces an automorphism of R. Therefore, G induces a finite automorphism group H of R satisfying

$$(R^G)_\sigma \cong (R_\tau)^H = \{x \in R_\tau \mid \alpha(x) = x \text{ for all } \alpha \in H\}.$$

Proof First of all, we show that $(R^G)_\sigma$ can be embedded into R_τ as rings. Since is perfect, we have, as R^G-modules, $Q_\tau(R) \cong Q_\sigma(R) \cong (R^G)_\sigma \otimes_{R^G} R$. We can make $(R^G)_\sigma \otimes_{R^G} R$ into a ring by first stipulating that $(a\otimes b)(c\otimes d) = ac\otimes bd$ and then tending it linearly to $(R^G)_\sigma \otimes_{R^G} R$ (see, for example, [8]).

Now look at the map $\hat{\sigma}_R$: $R \to (R^G)_\sigma \otimes_{R^G} R$ $r \mapsto 1\otimes r$. We can identify R with its age under the embedding inducesed by $\hat{\sigma}_R$. We see that the multiplication defined ove is consistent with the R-module structure on $(R^G)_\sigma \otimes_{R^G} R$. Because the iqueness of such multiplications, we know that the multiplication we defined is st the one of $(R^G)_\sigma \otimes_{R^G} R$ as the localizati on of R at τ. So $(R^G)_\sigma \otimes_{R^G} R$ is isomorphic R_τ as rings. Next suppose that

$$g: (R^G)_\sigma \to (R^G)_\sigma \otimes^{R^G} R,\ x \mapsto x\otimes 1.$$

It is easy to see that g is a ring homomorphism and the restriction of g to $\bar{R}^G = {}^G/T(R^G)$ is the identity map. We claim that there is a $0 \neq x \in (R^G)_\sigma$ such that $x\otimes = 0$. Then there is an $r \in R^G$ such that $0 \neq rx \in R^G$. Write rx as $\bar{s}$ for $s \in R^G$. Then

$$0 = r(x\otimes 1) = rx\otimes 1 = \bar{s}\otimes 1 = s(1\otimes 1) = 1\otimes s.$$

nce $\hat{\sigma}_R(s) = 1\otimes s$ and $\mathrm{Ker}\hat{\sigma}_R = T_\sigma(R)$, $s \in T_\sigma(R^G)$ and hence $\bar{s} = 0$, a contradiction. herefore, g is an embedding.

Now given an $\alpha \in G$, then α becomes automatically an R^G-endomorphism of R. induces an abelian group homomorphism

$$1\otimes\alpha:\ (R^G)_\sigma \otimes_{R^G} R \to (R^G)_\sigma \otimes_{R^G} R$$
$$a_i\otimes b_i \mapsto a_i\otimes\alpha(b_i).$$

bviously, $1\otimes\alpha$ is invertible and $(1\otimes\alpha)^{-1} = 1\otimes\alpha^{-1}$.

It is readily seen that $1\otimes\alpha$ is also a ring homomorphism and hence an itomorphism group of R_τ. Set $H = \{1\otimes\alpha_1, \cdots, 1\otimes\alpha_n\}$. It is easy to see that $(R^G)_\sigma \subseteq (R^H)$. Conversely, for a given element $\sum a_i\otimes b_i$ in $(R^G)_\sigma \otimes_{R^G} R$ which is fixed by H, e.,

$$\sum a_i\otimes\alpha_j(b_i) = \sum a_i\otimes b_i,$$

nce $n^{-1}(\sum_j \alpha_j(b_i)) \in R^G$, $\sum a_i\otimes b_i = \sum a_i(n^{-1}\sum_j \alpha_j(b_i))\otimes 1 \in (R^G)_\sigma$. This completes the roof of the theorem.

Corollary 3.7. *If σ is perfect, then so is τ. Moreover, for any R-module M,*

$$Q_\tau(M) = (R^G)_\sigma \otimes_{R^G} M.$$

Proof Given any $I \in \mathscr{L}_\tau$, then $I \cap R^G \in \mathscr{L}_\sigma$.

Since σ is perfect, $(R^G)_\sigma(I \cap R^G) = (R^G)_\sigma$. So $1 = \sum r_i a_i$ for some $r_i \in (R^G)_\sigma$ and $a_i \in I \cap R^G$. Since $(R^G)_\sigma$ is a subring of R_τ, we have $1 \in R_\tau I$, i. e., $IR_\tau = R_\tau$. This proves that τ is perfect (see Proposition 17.1 in [7]). And what is more,

$$Q_\tau(M)=R_\tau\otimes_R M=((R^G)_\sigma\otimes_{R^\sigma}R)\otimes_R M=(R^G)_\sigma\otimes_{R^\sigma}(R\otimes_R M)=(R^G)_\sigma\otimes_{R^\sigma}M.$$

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