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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF n ORDER NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract

This paper considers oscillatory and asymptotic behaviour of following n order neutra functional differential equation:

$$\frac{d^n}{dt^n}[x(t)-cx(t-\tau)]+(-1)^{n-1}\int_{-\tau^*}^{0}x(t+\theta)d\eta(\theta)=0, \tag{1}$$

where $\tau>0$, $\tau^*>0$, $1>c\geqslant 0$, $\eta(\theta)$ is nondecreasing function with bounded variation oı $[-\tau^*, 0]$.

In this paper the author obtains some results for any integer n and $c\in[0, 1)$. Wheı $c=0$ or $n=1$, these results coincide with the results in G. Ladas's paper [4] and the author's papers [1, 2].

§1. Introduction

In this paper we consider oscillatory and asymptotic behaviour of followin order neutral functional differential equation:

$$\frac{d^n}{dt^n}[x(t)-cx(t-\tau)]+(-1)^{n-1}\int_{-\tau^*}^{0}x(t+\theta)d\eta(\theta)=0,$$

where $\tau>0$, $\tau^*>0$, $1>c\geqslant 0$, $\eta(\theta)$ is nondecreasing function with bounded vari on on $[-\tau^*, 0]$.

The author[1,2] gave some results about oscillatory and asymptotic behavi of first order NDDE. Xu-Yuan Tong[3] gave some sufficient conditions, under wh n order linear RFDE is oscillatory or nonoscillatory. G. Ladas and I. Stavralaki O. Arino, I. Gyori, A. Jawhari[5] gave some results about oscillatory behaviour o order RDDE.

When $c=0$, the results in this paper coincide with the results of RFI Existence of solution on $(-\infty, +\infty)$ for equation $(1)_n$ has been given by J. Hale[6]. The results of nonautonomous n order NFDE will be given by the autl in another paper.

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§2. Main Results and Proofs

Suppose $\{t_k, k=1, 2, \cdots, m\}$, $0>t_1>t_2>\cdots>t_m\geqslant-\tau^*$, is a sequence in $[-\tau^*, 0]$ and $\eta(\theta)$ has positive damp on $\{t_k\}$.

Theorem 1. *In the case $n=1$, each of the following conditions is a sufficient condition for all the solution of $(1)_1$ to be oscillating:*

(A)
$$\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)](-t_k)>E_1^*(c), \tag{2}$$

$$E_1^*(c)=\max_{z>0}[ze^{-z}(1-ce^{\mu z})]=z_1^*(c)\exp(z_1^*(c))[1-c\exp(\mu z_1^*(c))],$$

where $z_1^*(c)$ satesfies the equation

$$(1-z_1^*(c)[1-c\exp(\mu z_1^*(c))]-c\mu z_1^*(c)\exp(\mu z_1^*(c))=0,$$

$$\mu=\tau\Big[\sum_{k=1}^{m}(\eta(t_k^+)-\eta(t_k^-)\Big]\Big/\Big[\sum_{k=1}^{m}(\eta(t_k^+)-\eta(t_k^-)(-t_k);$$

$$m\tau\geqslant\sum_{k=1}^{n}(-t_k),$$

(B)
$$\Big[\sum_{k=1}^{m}(\eta(t_k^+)-\eta(t_k^-))\Big]^{1/m}\Big[\sum_{k=1}^{m}(-t_k)\Big]>E_1(c), \tag{3}$$

where $E_1(c)=\max_{z>0}[ze^{-z}(1-ce^{z})]=z_1(c)e^{-z_1(c)}[1-ce^{z_1(c)}]$,

$z_1(c)$ sctisfies the equation

$$1-z_1(c)-ce^{z_1(c)}=0;$$

(C) *For some k_0, $k_0\in\{1, 2, \cdots, m\}$, we have*

$$[\eta(t_{j_0}^+)-\eta(t_{k_0}^-)](-t_{k_0})>E_1^*(c); \tag{4}$$

(D) *For some p, $p>1$, we have*

$$(1/m)^{p-1}\Big\{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]^{1/p}(-t_k)^{1/p}\Big\}^p>E_1^*(c); \tag{5}$$

(E) *Set* $\zeta=\max_{k=1,2,\cdots,m}\{t_k\}=t_1$,

$$\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]\cdot(-\zeta)>E_1^*(c). \tag{6}$$

Proof (I) Taking $x(t)=e^{-\lambda t}$ in $(1)_1$, we obtain a characteristic equation

$$f_1(\lambda)=-\lambda(1-ce^{\lambda\tau})+\int_{-\tau^*}^{0}e^{-\lambda\theta}\,d\eta(\theta)=0. \tag{7}$$

Now we prove $f_1(\lambda)>0$ for all real λ. If this is proved, then $(1)_1$ is oscillatory, according to J. K. Hale's book. Now we are in a position to prove this result.

Assume the characteristic equation $f_1(\lambda)=0$ has no real root. Because the neutral operator for $(1)_1$ is the stable operator $(0<c<1)$, $f_1(\lambda)=0$ has finite number of roots with nonpositive real part, $\lambda_j=\alpha_j+i\beta_j$, $j=1, 2, \cdots, N$, $\alpha_j\leqslant0$, $e^{-\alpha_j t}$ are positive exponential functions. Set $|\alpha_1|=\max_{j=1,2,\cdots,N}|\alpha_j|=a$, i. e, $a+\alpha_j>0(j=2, \cdots, N)$. Take $\mu\in(0, \max_{j=2,\cdots,N}(a+\alpha_j))$. It is obvious that if t is a sufficient large number, then any solution $x(t)$ of $(1)_1$ can be written as (Ref. 7.4, 12.10 in [6]).

$$\sum_{j=1}^{N} A_j t^{k_j} e^{-\alpha_j t}\cos(\beta_j t+\gamma_j)+O(e^{-\gamma t}) \quad (\gamma>0)$$

$$=e^{at}\left\{A_1 t^{k_1}\cos(\beta_1 t+\gamma_1)+\sum_{j=1}^{N} A_j t^{k_j} e^{-(a+\alpha_j)t}\cos(\beta_j t+\gamma_j)\right\}+O(e^{-\gamma t})$$

$$=e^{at}\{A_1 t^{k_1}\cos(\beta_1 t+\gamma_1)+O(e^{-\mu t})\}+O(e^{-\gamma t}),$$

where k_j concerns the multiplicity of root λ_j. Since the function $e^{at}A_1 t^{k_1}\cos(\beta_1 t+\gamma_1)$ is oscillatory, any solution $x(t)$ of $(1)_1$ is also oscillatory.

(II) Otherwise, assume there exists $\lambda_0\in\mathbb{R}$, such that $f_1(\lambda_0)\leqslant 0$, i. e.,

$$-\lambda_0(1-ce^{\lambda_0\tau})+\int_{-\tau^*}^{0} e^{-\lambda_0\theta}\,d\eta(\theta)\leqslant 0,$$

$$\int_{-\tau^*}^{0} e^{-\lambda_0\theta}d\eta(\theta)\leqslant\lambda_0(1-ce^{\lambda_0\tau}).$$

In fact $\lambda_0\notin(-\infty, 0]$. Then, we suppose $\lambda_0>0$, $1-ce^{\lambda_0\tau}>0$. We easily obtain

$$\sum_{k=1}^{m} e^{-\lambda_0 t_k}[\eta(t_k^+)-\eta(t_k^-)]\leqslant\lambda_0(1-ce^{\lambda_0\tau}).$$

(III) (A) We first prove sufficient condition (2). Becanuse

$$\sum_{i=1}^{m} k_i e^{z_i}\geqslant\exp\left[\sum_{i=1}^{m} k_i z_i\right] \quad \left(\sum_{i=1}^{m} k_i=1\right),$$

we heve

$$\frac{\lambda_0(1-ce^{\lambda_0\tau})}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}\geqslant\sum_{i=1}^{m}\frac{[\eta(t_i^+)-\eta(t_i^-)]e^{-\lambda_0 t_i}}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}$$

$$\geqslant\exp\left[\sum_{i=1}^{m}\frac{[\eta(t_i^+)-\eta(t_i^-)](-\lambda_0 t_i)}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}\right],$$

$$\frac{\lambda_0(1-ce^{\lambda_0\tau})}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}\left[\sum_{i=1}^{m}(\eta(t_i^+)-\eta(t_i^-))(-t_i)\exp\left[\frac{-\lambda_0\sum_{i=1}^{m}(-t_i)[\eta(t_i^+)-\eta(t_i^-)]}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}\right.\right.$$

$$\geqslant\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-)](-t_i).$$

Set

$$z=\lambda_0\sum_{i=1}^{m}(-t_i)[\eta(t_i^+)-\eta(t_i^-)]\Big/\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-)],$$

$$\mu=\tau\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-)]\Big/\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-](-t_i),$$

i. e., $\lambda_0\tau=\mu z$. We have $0<\lambda_0<-\lg c/\tau$, $0<z<-\lg c/\mu$.

$$\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-)](-t_i)\leqslant ze^{-z}(1-ce^{\mu z}).$$

Set $G^*(z)=ze^{-z}(1-ce^{\mu z})$, we obtain $G^*(z)=e^{-z}p(z)=0$, where

$$p(z)=(1-z)(1-ce^{\mu z})-c\mu ze^{\mu z}.$$

Because $p(0)=1-c>0$, $p(-\lg c/\mu)=\lg c<0$, there exists $z_0^*(c)\in(0, \ -\lg c/\mu)$ such that $p(z_0^*(c))=0$, i. e. $1-c\exp(\mu z_0^*(c))>0$. Also we have

$$1-z_0^*(c)=c\mu z_0^*(c)\exp(\mu z_0^*(c))/(1-c\exp(\mu z_0^*(c))>0,$$

e., $z_0^*(c)<1$. Notice that

$$\begin{aligned}G^{*\prime\prime}(z_0^*(c))&=\exp(-z_0^*(c))[p'(z_0^*(c))-p(z_0^*(c))]\\&=\exp(-z_0^*(c))p'(z_0^*(c))\\&=-\exp(-z_0^*(c))\left[\frac{1-c\exp(\mu z_0^*(c))}{z_0^*(c)}\right]\\&\quad[z_0^*(c)+[(1-\mu)z_0^*(c)-2][z_0^*(c)-1]]<0.\end{aligned}$$

(For $\mu\geqslant 1$, $(1-\mu)z_0^*(c)-2\leqslant -2<0$; For $\mu<1$, $(1-\mu)z_0^*(c)-2<1-\mu-2<0$).

Ve have

$$\begin{aligned}G^*(z_0^*(c))&=z_0^*(c)\exp(-\mu z_0^*(c))[1-c\exp(-\mu z_*^0(c))]=E_1^*(c)\\&=\max_{z\in(0,-\lg c/\mu)}G^*(z),\end{aligned}$$

.e.,

$$\left[\sum_{i=1}^{m}(\eta(t_i^+)-\eta(t_i^-)\right](-t_i)\leqslant E_1^*(c).$$

'his contradicts the hypothesis (2). So $(1)_1$ is oscillatory.

(B) The p roof of sufficient condition (3). Notice

$$\begin{aligned}&\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]e^{-\lambda_0 t_k}\\&\geqslant m\left[\prod_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]\right]^{1/m}\exp\left(\lambda_0\sum_{k=1}^{m}(-t_k)/m)\right),\end{aligned}$$

.e.,

$$\left[\prod_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]\right]^{1/m}\leqslant\frac{\lambda_0(1-ce^{\lambda_0\tau})}{m\cdot\exp\left(\lambda_0\sum_{k=1}^{m}(-t_k)/m\right)},$$

$$\begin{aligned}&\left[\sum_{k=1}^{m}(-t_k)\right]\left[\prod_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]\right]^{1/m}\\&\leqslant\frac{\lambda_0\left(\sum_{k=1}^{m}(-t_k)\right)}{m}\cdot\frac{\left[1-c\exp\left(\lambda_0\sum_{k=1}^{m}(-t_k)/m\right)\right]}{\exp\left[\lambda_0\sum_{k=1}^{m}(-t_k)/m\right]},\end{aligned}$$

Jet

$$z_0=\lambda_0\left[\sum_{k=1}^{m}(-t_k)\right]\Big/m,\quad G(z_0)=z_0e^{-z_0}(1-ce^{z_0}).$$

We have

$$\left[\prod_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]\right]\cdot\left[\sum_{k=1}^{m}(-t_k)\right]\leqslant\max_{z_0}G(z_0)=E_1(c).$$

This contradicts the hypothesis (3). So if (3) is true, then $(1)_1$ is oscillatory.

(C) From sufficient condition (2), we can prove sufficient conditions (4), (6). Notice Cauchy inequality

$$\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]^{1/p}(-t_k)^{1/p}\leqslant\left[\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)](-t_k)\right]^{1/m}m^{1-1/p}.$$

We can prove sufficient condition (5). Theorem 1 is proved.

It is obvious that some results in the author's papers[1, 2] coinside with Theorem 1 of this paper.

Theorem 2. *If n is odd, then each of the following conditions is a sufficient* *condition for all the solution of* $(1)_n$ *to be oscillating. If* n *is even, then each of the following conditions is a sufficient condition for all bounded solution of* $(1)_n$ *to be oscillating.*

$$\text{(A)} \qquad \sum_{k=1}^{m} [\eta(t_k^+) - \eta(t_k^-)](-t_k/n)^n > E_n(c), \tag{9}$$

where

$$E_n(c) = E_n^*(c)(\mu/\tau)^{n-1},$$
$$E_n^*(c) = \max_z \{[ze^{-z}]^n[1-ce^{\mu z}]\}$$
$$= [z_n^*(c)\exp(-z_n^*(c))]^n(1-c\exp(\mu z_n^*(c))],$$

$z_n^*(c)$ *satisfies the equation*

$$g^*(z) = n(1-z)[1-ce^{\mu}] - c\mu z e^{\mu z} = 0,$$

$$\mu = \tau\Big[\sum_{k=1}^{m} [\eta(t_k^+ - \eta(t_k^-)]\Big] \Big/ \Big[\sum_{i=1}^{m}(-t_i/n)^n[\eta(t_i^+) - \eta(t_i^-)]\Big].$$

(B) *For some* k_0, $k_0 \in \{1, 2, \cdots, m\}$, *we have*

$$[\eta(t_{k_0}^+) - \eta(t_{k_0}^-)](-t_{k_0}/n)^n > E_n(c);$$

(C) *For* $\zeta = \max\limits_{k=1,2,\cdots,m} \{t_k\} = t_1$, *we have*

$$(-\zeta/n)^n \sum_{k=1}^{m} [\eta(t_k^+) - \eta(t_k^-)] > E_n(c):$$

$$\text{(D)} \qquad [\eta(0) - \eta(-\tau^*)]c\tau^n > \bar{E}_n(c),$$

where

$$\bar{E}_n(c) = \max_z z^n(e^{-z} - c) = [z_0^*(c)]^n[\exp(-z_0^*(c) - c],$$

$z_0^*(c)$ *satfisies the equation*

$$g(z) = n(e^{-z} - c) - ze^{-z} = 0.$$

Proof Taking $e^{-\lambda t}$ in $(1)_n$, we obtain a characteristic equation

$$f_n(\lambda) = (-\lambda)^n(1 - ce^{\lambda\tau}) + (-1)^{n-1}\int_{-\tau^*}^{0} e^{-\lambda\theta}\, d\eta(\theta) = 0.$$

(I) When n is odd, (13) is changed into

$$f_n(\lambda) = -\lambda^n(1 - ce^{\lambda\tau}) + \int_{-\tau^*}^{0} e^{-\lambda\theta}\, d\eta(\theta) = 0.$$

If we can prove the equation $f_n(\lambda) = 0$ has no real root, then $(1)_n$ is oscillatory. can prove this result by a similar argument as in the proof of Theorem 1. Nov prove $f_n(\lambda) > 0$ for all real λ. Otherwise, there exists $\lambda_0 \in \mathbb{R}$ such that $f_n(\lambda_0) \leqslant 0$,

$$\int_{-\tau^*}^{0} e^{-\lambda_0\theta} d\eta(\theta) \leqslant \lambda_0^*(1 - ce^{\lambda_0\tau}).$$

(A) We first prove sufficient condition (9). From (15) we know $\lambda_0 > 0$ and

$$\sum_{i=1}^{m} [\eta(t_i^+) - \eta(t_i^-)]e^{-\lambda_0 t_i} \leqslant \int_{-\tau^*}^{0} e^{-\lambda_0\theta}\, d\eta(\theta)$$

Similarly, as in the proof of Theorem 1, we can obtain

$$\frac{\lambda_0(1-ce^{\lambda_0\tau})}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-]}\geqslant\exp\left\{\left[\sum_{i=1}^{m}\frac{[\eta(t_i^+)-\eta(t_i^-)](-t_i)}{\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)]}\cdot\frac{\lambda_0}{n}\right]n\right\}$$

et

$$z=\lambda_0\left[\sum_{i=1}^{m}(-t_i/n)^n[\eta(t_i^+)-\eta(t_j^-)]\right]\Big/\left[\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-])\right],$$

e., $\lambda_0\tau=\mu z$,

$$\sum_{i=1}^{m}[\eta(t_i^+)-\eta(t_i^-)](-t_i/n)^n\leqslant z^ne^{-nz}(1-ce^{\mu z})(\mu/\tau)^{n-1}$$

$$\leqslant\max_z[z^ne^{-nz}(1-ce^{\mu z})](\mu/\tau)^{n-1}$$

$$=E_n^*(c)(\mu/\tau)^{n-1}=E_n(c),$$

$$E_n^*(c)=[z_n^*(c)\exp(-z_n^*(c))]^n[1-c\exp(\mu z_n^*(c))],$$

here $z_n^*(c)$ = satisfies

$$g^*(z)=n(1-z)(1-ce^{\mu z})-c\mu ze^{-\mu z}$$

otice $g^*(0)=n(1-c)>0$, $g^*(-\lg c/\mu)=\lg c<0$. So we can prove that there exists $(c)\in(0^0-\lg c/\mu)$ such that $g^*(z_n^*(c))=0$. This contradicts the hypothesis (9). We ive proved $(1)_n$ is oscillatory.

(B) To prove sufficient condition (10), notice that for all k_0, $k_0\in\{1, 2, \cdots, m\}$, e have

$$[\eta(t_{k_0}^+)-\eta(t_{k_0}^-)][-t_{k_0}/n]<\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)][-t_k/n]^n.$$

(C) To prove sufficient condition (11), notice that for $\zeta=t_1$, we have

$$\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-)](-\zeta/n)^n<\sum_{k=1}^{m}[\eta(t_k^+)-\eta(t_k^-][-t_k/n]^n.$$

(D) Now we prove sufficient condition (12). From (15) we obtain $\lambda_0>0$, $1-{}^{\lambda_0\tau}>0$, i. e., $\lambda_0\in(0, -\lg c/\tau)$. It is obvious that

$$[\eta(0)-\eta(-\tau^*)]\leqslant\lambda_0^n(e^{-\lambda_0\tau}-c)/c=[(\lambda_0\tau)^n(e^{-\lambda_0\tau}-c]/(c\tau^n).$$

t $\lambda_0\tau=z$, $G(z)=z^n(e^{-z}-c)$. We have

$$G^1(z)=z^{n-1}g(z)=z^{n-1}(n(e^{-z}-c)-ze^{-z})=0.$$

otice $g(0)=n(1-c)>0$, $g(-\lg c)=c\lg c<0$. So we can prove that there exists $(c)\in(0, -\lg c)$ such that $g(z_0^*(c))=G'(z_0^*(c))=0$, and

$$\widetilde{E}_n(c)=\max_z G(z)=[z_0^*(c)]^n[c\exp(-z_0^*(c))-c],$$

e., $[\eta(0)-\eta(-\tau^*)]<\tau^n\leqslant\widetilde{E}_n(c)$.

This contradicts the hypothesis (12). So $(1)_n$ is oscillatory.

(II) When n is even, the characteristic equation is changed into

$$f_n(\lambda)=\lambda^n(1-ce^{\lambda\tau})-\int_{-\tau^*}^{0}d\eta(\theta)=0.$$

We can prove there does not exista positive root of $f_n(\lambda)=0$, i. e., there does not exist bounded nonoscillatory solutions of $(1)_n$, i. e., all solutions bounded of $(1)_n$ are oscillatory. Notice $f_n(0)=-\int_{-\tau^*}^{0}d\eta(\theta)<0$. New we prove $f_n(\lambda)<0$ for all $\lambda\in\mathbb{R}^+$. Otherwise, if there exists $\lambda_0>0$ such that $f_n(\lambda_0)\geqslant 0$, then similarly, as in the proof of (I), we can obtain (15) and come to the conclusion of Theorem 2. Theorem 2 is proved.

Theorem 3. *Consider the equation*

$$\frac{d^n}{dt^n}[x(t)-cx(t-\tau)]+(-1)^{n-1}\int_{t_1}^{t_n}x(t+\theta)d\eta(\theta)=0. \qquad (\ )$$

For all positive integer n, the sufficient condition, under which there exists boun nonoscillatory solution of $(1)_n$, *is that*

$$\tau^n[\eta(t_1)-\eta(t_n)]\leqslant E_n^\mu(c), \qquad (\ $$

where

$$E_n^\mu(c)=\max_{z>0}[z^n(e^{-z}-c)e^{\mu z}], \quad \mu=(\tau+t_m)/\tau.$$

Proof Taking $e^{-\lambda t}$ in $(1)'_n$, we have a characteristic equation

$$F(\lambda)=(-\lambda)^n(1-ce^{\lambda\tau})+(-1)^{n-1}\int_{tm}^{t_1}e^{-\lambda\theta}d\eta(\theta)=0, \qquad (\ $$

$$F(0)=(-1)^{n-1}\int_{t_m}^{t_1}d\eta(\theta)=\begin{cases}>0, & \text{when } n \text{ is odd.}\\ <0, & \text{when } n \text{ is even.}\end{cases}$$

Suppose $z_\mu^*(c)$ satisfies

$$\max_{z>0}[z^n(e^{-z}-c)e^{\mu z}]=[z_\mu^*(c)]^n[\exp(-z_\mu^*(c))-c]\exp(\mu z_u^*(c))=E_n^\mu(c).$$

And we have

$$F(z_\mu^*(c)/\tau)=\{[-z_\mu^*(c)/\tau]^n[1-c\exp(z_\mu^*(c))]\exp[-(-\theta^*)z_\mu^*(c)/\tau]$$
$$+(-1)^{n-1}[\eta(t_1)-\eta(t_m)]\}\exp[(-\theta^*)z_\mu^*(c)/\tau],$$

where $\theta^*\in[t_m,t_1]$。Notice

$$G'(z)=z^{n-1}e^{\mu z}\{(e^{-z}-c)n+z[(\mu-1)e^{-z}-c\mu]\},$$
$$g(z)=n(e^{-z}-c)+z[(\mu-1)e^{-z}-c\mu],$$
$$g(0)=n(1-c)>0, \quad g(-\lg c)=c\lg c<0.$$

So there exists $z_\mu^*(c)\in(0,-\lg c)$ such that

$$g(z(^*_{\cdot}c))=G'(z_\mu^*(c))=0, \quad \text{i. e.,} \quad \max_z G(z)=G(z_\mu^*(c)).$$

When n is odd, we have

$$F(z_\mu^*(c)/\tau)=-\exp[(-\theta^*)z_\mu^*(c)/\tau]\{[z_\mu^*(c)]^n[1-c\exp(z_\mu^*(c))]$$
$$\cdot\exp[-(-\theta^*)z_\mu^*(c)/\tau]/\tau^n-[\eta(t_1)-\eta(t_m)]\}$$
$$<-\exp[(-\theta^*)z_\mu^*(c)/\tau]\{E_n^\mu(c)/\tau^n-[\eta(t_1)-\eta(t_m)]\}<0.$$

When n is even, we have

$$F(z_\mu^*(c)/\tau)=\{[z_\mu^*(c)/\tau]^n[1-c\exp(z_\mu^*(c))]\exp[-(-\theta^*)z_\mu^*(c)/\tau]$$
$$-[\eta(t_1)-\eta(t_m)]\}\exp[(-\theta^*)z_\mu^*(c)/\tau]$$
$$>\{E_n^\mu(c(/\tau^n-[\eta(t_1)-\eta(t_m)]\}\exp[(-\theta^*)z_\mu^*(c)/\tau>0.$$

So we can prove that there exists $\lambda_0\in(0,z_\mu^*(c)/\tau)$ such that $F(\lambda_0)=0$, i. e., there exists nonoscillatory solution $x(t)=e^{-\lambda_0 t}$ of $(1)'_n$. Theorem 3 is porved.

Theorem 4. *For any positive integer n, the sufficient condition, under which $(1)_n$ is oscillatory, is that*

$$\tau^n[\eta(t_1)-\eta(t_m)]>E_n^{\bar\mu}(c), \tag{18}$$

where $E_n^{\bar\mu}(c)=\max\limits_{z>0}\{z^n(e^{-z}-c)e^{\bar\mu z}\}$, $\bar\mu=(\tau+t_1)/\tau$.

Proof Taking $e^{-\lambda t}$ in $(1)'_n$, we obtain the characteristic equation (17).

(I) When n is odd, (17) is changed into

$$F(\lambda)=-\lambda^n(1-ce^{\lambda\tau})+\int_{t_m}^{t_1}e^{-\lambda\theta}\,d\eta(\theta)=0.$$

Now we prove $F(\lambda)>0$ for all $\lambda\in\mathbb{R}$. Otherwise, there then exists $\lambda_0\in\mathbb{R}$ such that $F(\lambda_0)\leqslant 0$, i.e.,

$$e^{\lambda_0(-t_1-\tau)}\tau^n[\eta(t_1)-\eta(t_m)]\leqslant\tau^n\int_{t_m}^{t_1}\frac{e^{-\lambda_0\theta}}{e^{\lambda_0\tau}}\,d\eta(\theta)\leqslant(\lambda_0\tau)^n(e^{-\lambda_0\tau}-c).$$

So we have

$$\tau^n[\eta(t_1)-\eta(t_m)]\leqslant(\lambda_0\tau)^n(e^{-\lambda_0\tau}-c)e^{-\lambda_0\tau\bar\mu},$$

where $\bar\mu=(\tau+t_1)/\tau^*$ i. e., $\tau^n[\eta(t_1)-\eta(t_m)]\leqslant E_n^{\bar\mu}(c)$.

This contradicts the hypothesis (18). So $(1)'_n$ is oscillatory.

(II) When n is even, (17) is changed into

$$F(\lambda)=\lambda^n(1-ce^{\lambda\tau})-\int_{t_m}^{t_1}e^{-\lambda\theta}\,d\eta(\theta)=0.$$

We prove $F(\lambda)<0$, for $\lambda\in\mathbb{R}^+$, i. e., all bounded solution of $(1)'_n$ are oscillatory. Otherwise, there then exists $\lambda_0\in\mathbb{R}^+$ such that $F(\lambda_0)\geqslant 0$, i. e.,

$$\lambda_0^n(1-ce^{\lambda_0\tau})\geqslant\int_{t_m}^{t_1}e^{\lambda_0\theta}\,d\eta(\theta),$$

$$\tau^n[\eta(t_1)-\eta(t_m)]\leqslant(\lambda_0\tau)^n(e^{-\lambda_0\tau}-c)e^{\lambda_0\tau\bar\mu}$$
$$\leqslant\max_{z>0}z^n(e^{-z}-c)e^{\bar\mu z}=E_n^{\bar\mu}(c).$$

This contradicts the hypothesis (18). So all bounded solutions of $(1)'_n$ are oscillatory. Tehorem 4 is proved.

Corollary 1. *If $(1)'_n$ is the simplest form*

$$\frac{d^n}{dt^n}[x(t)-cx(t-\tau)]+(-1)^{n-1}px(t-\tau_1)=0.\quad(\tau>0,\ p>0,\ \tau_1>0), \tag{19}$$

then for all positive integer n, (19) is oscillatory if and only if

$$\tau^n p>E_n^{\mu}(c). \tag{20}$$

Proof Notice that if $-\tau_1=t_1=t_2=\cdots=t_m<0$, then (19) coincides with $(1)'_n$. Because $\bar\mu=(\tau+t_1)/\tau$ in (18), $\mu=(\tau+t_m)/\tau$ in (16), we have $\bar\mu=\mu=(\tau-\tau_1)/\tau$ (if $\tau_1=\tau>0$, then $\bar\mu=\mu=0$) (16) and (18) are changed into (20). (if $\tau_1=\tau$, then (20) is changed into $\tau^n p>\tilde E_n(c)$). According to Theorem 3, Theorem 4, we can prove Corollary 1.

Corollary 2. *The sufficient condition, under which the equation*

$$\frac{d^n}{dt^n}[x(t)-cx(t-\tau)]+(-1)^{n-1}\sum_{i=1}^{m}p_i x(t-\tau_i)=0,$$
$$\tau_m>\tau_{m-1}>\cdots>\tau_1>0,\ \tau>0,\ p_i>0,\ i=1,2,\cdots,m, \quad (21)$$

is oscillatory, is that

$$\tau^n\left[\sum_{i=1}^{m}p_i\right]>E_n^{\bar{\mu}}(c). \quad (22)$$

The necessary condition is that

$$\tau^n\left[\sum_{i=1}^{m}p_i\right]>E_n^{\bar{\mu}}(c). \quad (2\ldots$$

§ 3. Some Examples

Examlpe 1.

$$\left[x(t)-\frac{1}{4e}x\left(t-\frac{3}{8}\right)\right]'+\int_{-2}^{-1}x(t+\theta)d\eta(\theta)=0, \quad (\ldots$$

where

$$\eta(\theta)=\begin{cases}3,\ \theta=-1,\\ \theta+2,\ -2<\theta<-1,\\ -1,\ \theta=-2,\end{cases}$$

i. e., $m=2,\ t_2=-2,\ t_1=-1,\ \tau=8/3,\ c=1/4e.$

It is obvious that $\mu=2$. The root of the equation $1-z-c\mu e^{\mu z}=0$ is

$$z_1^*(1/(4e))=1/2,\ E_1^*(1/(4e))=1/(8\sqrt{e}),$$
$$(-t_1)[\eta(t_1^+)-\eta(t_1^-)]+(-t_2)]\eta(t_2^+)-\eta(t_2^-)]$$
$$=1\times2+2\times1=4>1/(8\sqrt{e})=E_1^*(1/(4e)).$$

From the condition (2) or another condition of Theorem 1, we see that (24) oscillatory.

Example 2. $$\left[x(t)-\frac{1}{2e}x\left(t-\frac{3}{8}\right)\right]''-\int_{-2}^{-1}x(t+\theta)d\eta(\theta)=0 \quad (\ldots$$

The denition of $\eta(\theta)$ is similar to that in Example 1. Now the equation $n(1-$ $(1-ce^{\mu z})-c\mu ze^{\mu z}=0$ is changed into

$$2(1-z)(1-z^{2z}/(2e))-2ze^{2z}/(2e)=0.$$

The root of this equation is $z_2^*(1/2e))=1/2$. We have

$$E_n^*(c)=E_n^*\left(\frac{1}{2e}\right)=\left(\frac{1}{2}e^{-1/2}\right)^2\left(1-\frac{1}{2e}\cdot e\right)=\frac{1}{8e},$$
$$E_2\left(\frac{1}{2e}\right)=E_2^*\left(\frac{1}{2e}\right)\left(\frac{\mu}{\tau}\right)^{2-1}=\frac{3}{32e}.$$
$$(-t_1)^2[\eta(t_1^+)-\eta(t_1^-)]/4+(-t_2)^2[\eta(t_2^+)-\eta(t_2^-)]/4=3/2>E_2(1/(2e)).$$

According to Theorem 2, we see that (25) is oscillatory.

Example 3.

$$\left[x(t)-\frac{27}{29}e^{-1/5}x\left(t-\frac{20}{81}\right)\right]'''+\int_{-2}^{-1}x(t+\theta)d\eta(\theta)=0. \quad (26)$$

'he definition of $\eta(\theta)$ is the same as in Example 1.

$$\mu=\frac{20}{81}\times 3\Big/\left[\left(\frac{1}{3}\right)^3\times 2+\left(\frac{2}{3}\right)^3\times 1\right]=2$$

'he root of the equation

$$3(1-z)\left(1-\frac{27}{29}e^{-1/5}e^{2z}\right)-\frac{27}{29}e^{-1/5}2ze^{2z}=0$$

$z_3^*\left(\frac{27}{29}e^{-1/5}\right)=1/10$. And we have

$$E_3^*\left(\frac{27}{29}e^{-1/5}\right)=\frac{1}{1000}\times\frac{2}{29}\times e^{-3/10},$$

$$E_3\left(\frac{27}{29}e^{-1/5}\right)=\frac{2}{29}\times e^{-3/10}\times 10^{-3}\times\left(\frac{81}{10}\right)^3<\frac{2}{29},$$

$$\frac{(-t_1)^3}{3^3}[\eta(t_1^+)-\eta(t_1^-)]+\frac{(-t_2)^3}{3^3}[\eta(t_2^+)-\eta(t_2^-)]=\frac{10}{27}>\frac{2}{29}>E_3\left(\frac{27}{29}e^{-1/5}\right).$$

ccording to Theorem 2, we see that (26) is oscillatory.

Example 4.

$$\left[x(t)-\frac{1}{3e^2}x(t-2)\right]'''+\int_{-2}^{-1}x(t+\theta)d\eta(\theta)=0, \tag{27}$$

here $c=1/(3e^2)$, $\tau=2$, $\tau_1=-t_1=1$, $\tau_2=-t_2=2$, $\mu=0$. $\eta(\theta)$ are respectively 7/270, $(1+\theta)/270$, $-1/270$, when $\theta=-1$, $-1>\theta>-2$, $\theta=-2$. Now the equation $n(e^{-z}-c)$ $z[(\mu-1)e^{-z}-c\mu]=0$ is changed into $3(e^{-z}-1/(3e^2))-ze^{-z}=0$. Its root is $z_\mu^*(1/3e^2)$ 2. And we have

$$E_3^0\left(\frac{1}{3e^2}\right)=\frac{16}{3e^2},\qquad \tau^n[\eta(t_1)-\eta(t_2)]=\frac{32}{135}.$$

is obvious that $E_3^0(1/3e^2)>\tau^3[\tau(-1)-\eta(-2)]$. According to Theorem 3, we see ıat there exists nonoscillatory solution of (27).

Example 5.

$$\left[x(t)-\frac{1}{7e^3}x(t-4)''\right]-\int_{-2}^{-1}x(t+\theta)d\eta(\theta)=0, \tag{28}$$

here $c=1/7e^3$, $=4$, $t_1=-1$, $t_2=-2$, $\eta(\theta)$ are respectively 3/100, $(\theta+2)/100$, 1/100, when $\theta=-1$, $-2<\theta<-1$, $\theta=-2$. $\mu=1+\frac{t_2}{\tau}=\frac{1}{2}$. $z_{1/2}^*(1/7e^3)=3$ satisfies

$$n(e^{-z}-c)+z((\mu-1)e^{-z}-c\mu)=2(e^{-z}-1/7e^3)+z(-1/2e^z-1/14e^3)=0.$$

nd we have

$$E_n^\mu(c)=E_2^{1/2}(1/7e^3)=\frac{54}{7}e^{-3/2},$$

$$\tau^n[\eta(t_1)-\eta(t_2)]=\frac{16}{25}.$$

So we have $E_n^{1/2}(1/7e^3)>\tau^2[\eta(t_1)-\eta(t_2)]$. According to Theorem 3, we see that there exists bounded nonoscill atory solution of (28).

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