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HARNACK INEQUALITIES FOR FUNCTIONS IN THE GENERAL DE GIORGI PARABOLIC CLASS

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Abstract

In this paper the authors prove that functions in the general De Giorgi parabolic class satisfy Harnack inequalities and hence extend the results obtained in the 1960 by Moser, Trudinger and Ladyzenskaja et al. in a sense.

§1. Introduction

In this paper we prove that the funotions in the general De Giorgi parabolio s satisfy Harnack inequalities and hence extend the results of Moser in [1, 2] . Trudinger in [3] as well as the relevant results of Ladyzenskaja et al. in [4] , sense. It was only shown that "solutions of parabolic eqautions in divergence n satisfy Harnack inequalities" (in [1, 2, 3]) and "funotions in the general De rgi parabolic class are Hölder continuous" (in [4]). But as we shall see the tions of parabolic equations in divergence form belong to the general De Giorgi abolic class, and the fact that functions satisfying the Harnack inequalities are)matically Hölder continuous is well known;

This work is an extension of [5] where the Harnack inequalities for the general Giorgi elliptic class were proved by means of the De Giorgi interation technique a measure argument used in [6] and [7] respectively. Although the 'esponding De Giorgi iteration technique and the measure argument have ady been established for the parabolic case (cf. [4] and [8]), there is still some ntial difficulty in the extension as can be realized from [1, 2] and [9]. In fact, o the measure argument the method of comparison function was used to get a fusion lemma" (i. e. Lemma 1.3 in [8] and Lemma 3.1 in [10]), but this hod does not apply to the general De Giorgi parabolic class. In the paper the fusion lemmas" (i. e. Propositions 3.7 and 3.9) were obtained by combining the Giorgi iteration technique with a method of "double enlargement".

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Let Ω be an open set in R^n, $T>0$, $Q_T=\Omega\times(0, T)$. The pace $V_2^{1,0}(Q_T)=C([0, T], L_2(\Omega))\cap L_2((0, T), W_2^1(\Omega))$, the norm of a function $u(x, t)$ in $V_2^{1,0}(Q_T)$ is defined by

$$|u|_{Q_T}=\max_{0<t<T}\|u(x, t)\|_{2,\Omega}+\|D_x u\|_{2,Q_T}$$

where $\|.\|_{2,\Omega}$ and $\|.\|_{2,Q_T}$ denote the L_2 norms over Ω and Q_T respectively, $D_x u=(u_{x_1}, \cdots, u_{x_n})$.

Definition 1.1. *A function $u(x, t)$ is said to belong to the general De Giorgi parabolic class $DG(Q_T, N, \chi, \hat{k}, N_1)$, $N>0$, $\chi>0$, $\hat{k}>0$, $N_1>0$, if $u(x, t)\in V_2^{1,0}($ and for any $k\geqslant 0$ the function $w(x, t)=\pm u(x, t)$ satisfies the following inequalities*

$$|w^{(k)}|^2_{\hat{Q}(\rho-\sigma_1\rho, \tau-\sigma_2\tau)}\leqslant N\{[(\sigma_1\rho)^{-2}+(\sigma_2\tau)^{-1}]\|w^{(k)}\|^2_{2,\hat{Q}(\rho,\tau)}$$
$$+(\hat{k}^2+k^2\sigma^{-n\varkappa})\hat{\mu}^{\frac{2}{n+2}(1+\chi)}(k, \rho, \tau)\}, \qquad ($$

$$\max_{t_0<t<t_0+\tau}\|w^{(k)}(x, t)\|^2_{2,B_{\rho-\sigma_1\rho}}\leqslant N_1\|w^{(k)}(x, t_0)\|^2_{2,B_\rho}+N[(\sigma_1\rho)^{-2}\|w^{(k)}\|^2_{2,Q(\rho,\tau)}$$
$$+(\hat{k}^2+k^2\rho^{-n\chi})\mu^{\frac{2}{n+2}(1+\chi)}(k, \rho, \tau)], \qquad ($$

$$|w^{(k)}|^2_{Q(\rho-\sigma_1\rho, \tau-\sigma_2\tau)}\leqslant N\{[(\sigma_2\rho)^{-2}+(\sigma_2\tau)^{-1}]\|w^{(k)}\|^2_{2,Q(\rho,\tau)}$$
$$+(\hat{k}^2+k^2\rho^{-n\chi})\mu^{\frac{2}{n+2}(1+\chi)}(k, \rho, \tau)\}, \qquad ($$

where

$$w^{(k)}(x, t)=\begin{cases}u^{(k)}(x, t)=[u(x, t)-k]^+\triangleq\max[u(x. t)-k, 0], & \text{for } w=u,\\ \max[k-u(x, t), 0]\triangleq[k-u(x, t)]^+, & \text{for } w=-u,\end{cases} \qquad ($$

ρ and τ are arbitrary p0sitive numbers, $\hat{Q}(\rho, \tau)$ and *$Q(\rho, \tau)$ are arbitrary cyli belonging to Q_T, σ_1 and σ_2 are arbitrary numbers from $(0, 1)$, $\hat{Q}(\rho-\sigma_1\rho, \tau-$ $(Q(\rho_1-\sigma_1\rho, \tau-\sigma_2\tau)$ resp.,) is coaxial with and has a common vertex with $\hat{Q}(\rho$ $(Q(\rho, \tau)$ resp., of the form*

$$\hat{Q}(r, s)\triangleq Q(x_0, t_0, r, t)=B_r(x_0)\times(t_0-s, t_0),$$
$$(Q(r, s)\triangleq Q(x_0, t_0, r, s)=B_r(x_0)\times(t_0, t_0+s) \text{ resp.,})$$
$$B_r\triangleq B_r(x_0)=\{x=(x^1, \cdots, x^n);\ \Big[\sum_{i=1}^n(x^i-x_0^i)^2\Big]^{1/2}<r\},$$
$$\hat{\mu}(k, \rho, \tau)=|\{(x, t)\in\hat{Q}(\rho, \tau);\ w^{(k)}(x, t)>0\}|,$$
$$\mu(k, \rho, \tau)=|\{(x, t)\in Q(\rho, \tau);\ w^{(k)}(x, t)>0\}|.$$

Here and in the sequel $|S|$ and mes G denote the $(n+1)$–dimensional n–dimensional Lebesgue measure of S and G respectively.

The Harnack type inequalities that we will establish in this paper ar following theorems.

Theorem 1.2. *Let $u(x, t)\in DG(Q_T, N, \chi, \hat{k}, N_1)$ θ, $R>0$, $B_r(\bar{x})\times(\bar{t}, \bar{t}+$ $\subset Q_T$, $\sigma\in(0, 1)$. Then for any $p>0$ there exists a constant $C>0$ such that*

$$\sup_{B_{\sigma R}(\bar{x})\times(\bar{t}+(1-\sigma^2)\theta R^2, \bar{t}+\theta R^2)} u(x, t)\leqslant C\Big\{\Big(\int_{B_R(\bar{x})\times(\bar{t}, \bar{t}+\theta R^2)}[u^+]^p dx dt\Big)^{1/p}+\hat{k}R^{n\chi/2}\Big\},$$

where C depends only on p, n, N, χ, N_1, σ and θ.

We always set

$$\fint_S v\,dx\,dt=\frac{1}{|S|}\int_S v\,dx\,dt.$$

Theorem 1.3. *Let $u(x,t)\in DG(Q_T, N, \chi, \hat{k}, N_1)$, $u(x,t)\geqslant 0$, θ, $R>0$, $B_R(\bar{x})$ $(\bar{t}, \bar{t}+\theta R^2)\subset Q_T$, then for any $\sigma_1, \sigma_2\in(0,1)$, $0<\theta_1<\theta_2<\theta_3<\theta$, there exist positive stants p and C such that*

$$\{\hat{k}R^{n\chi/2}+\inf_{B_{\sigma_1 R}(\bar{x})\times(\bar{t}+\theta_2 R^2, \bar{t}+\theta_3 R^2)} u(x,t)\}\geqslant C\Big[\fint_{B_{\sigma_2 R}(\bar{x})\times(\bar{t},\bar{t}+\theta_1 R^2)} u^p dx dt\Big]^{1/p},$$

re p and C depend only on n, N, χ, N_1, θ_1, θ_2, θ_3, θ, σ_1 and σ_2.

Combining Theorem1.2 and Theorem 1.3 we have the following full Harnack quality.

Theorem 1.4. *Let $u(x,t)\in DG(Q_T, N, \chi, \hat{k}, N_1)$, $u(x,t)\geqslant 0$, R, $\theta>0$, $B_R(\bar{x})$ $(\bar{t}, \bar{t}+\theta R^2)\subset Q_T$, then for any $\sigma_1, \sigma_2\in(0,1)$, $0<\theta_1<\theta_2<\theta_3<\theta$, there exists a stant $C>0$ such that*

$$\sup_{B_{\sigma_1 R}(\bar{x})\times(\bar{t},\bar{t}+\theta_1 R^2)} u(x,t)\leqslant C\Big[\inf_{B_{\sigma_2 R}(\bar{x})\times(\bar{t}+\theta_2 R^2, \bar{t}+\theta_3 R^2)} u(x,t)+\hat{k}R^{n\chi/2}\Big],$$

re C depends only on n, N, χ, N_1, θ_1, θ_2, θ_3, θ, σ_1 and σ_2.

The paper consists of five sections. §2 is mainly devoted to the proof of Theorem . The "diffusion lemmas" were obtained in §3. Theorem 1.3 is derived in §4, the asuae lemma (Lemma 4.3) used there appeared originally in [11], its proof can found in [12].

This paper can be considered as an extended form of [12], it extends the results [1 2] from "homogeneous" case to the "general" case. In doing this the main iculty is the infimum estimate (corresponding to Lemma 2.4 in [12] and position 3.1 in [5]), since neither the procedure in [12] nor the one in [5] can used here. We get the estimate by a variant usage of the De Giorgi iteration hnique.

For convenience, we assume $N_1=1$ in §3 and § 4, and then consider the general in §5. In §5 we also derive a fact that the bounded solutions of the quasilinear abolic equations discussed in [3] belong to the De Giorgi class.

§ 2. The Maximum Principle For Functions in $DG(Q_T, N, \chi, \hat{k}, 1)$

Lemma 2.1. *Assume $u(x,t)\in V_2^{1,0}(Q_T)$. θ, $R>0$, $\hat{Q}(R,\theta R^2)\triangleq\hat{Q}(x_0, t_0, R, \theta R^2)$ $_T$. If the inequality (1.1) is valid for $w(x,t)=u(x,t)$, then for any $\mu\in(0,1)$, re exists a positive constant C such that*

$$\sup_{\hat{Q}(\mu R, \theta\mu^2 R^2)} u(x,t)\leqslant\frac{C}{(1-\mu)^{(2+n\chi)/2/s}}\Big[\Big(\fint_{\hat{Q}(R,\theta R^2)}(u^+(x,t))^2 dx dt\Big)^{1/2}\Big]+\hat{k}R^{n\chi/2},$$

where the constant C depends only on, n, N, χ, and θ,

$$\varepsilon=\min\left\{\frac{2}{n+2},\ \frac{n\chi}{n+2}\right\}.$$

This lemma can be proved in the same way as in Lemma 2.1 of [12] combining with the mathods used in proving Lemma 2.1 of [5]. The details are omitted here.

Proof of Theorem 1.2 By the Hölder inequality we only need to considei the case $p\in(0,2)$. Taking $x^0=\bar{x}$, $t^0=\bar{t}+\theta R^2$ and setting

$$f(s)=\sup_{B_{sR}(x^0)\times(t_0-\theta s^2R^2,t_0)} u(x,t),$$

by Lemma 2.1 we have, for any $t>s$ with $s,\ t\in[\mu,1]$,

$$f(s)\leqslant\frac{C}{\left(1-\frac{s}{t}\right)^{(2+n\chi)/2/\varepsilon}}\left[\int\!\!\!\int_{\hat{Q}(tR,\theta t^2R^2)}(u^+)^2dx\,dt\right]^{1/2}+\hat{k}R^{n\chi/2}t^{n\chi/2}.$$

Denote by ω_n the volume of the n-dimensional unit ball and note $[(1+n\chi)/2/\varepsilon$ $(n+2)/n$, then, for any $p\in(0,2)$, by the Young Inequality, we have

$$f(s)\geqslant\frac{C}{(t-s)^{(2+n\chi)/2/\varepsilon}}\sup_{\hat{Q}(tR+\theta t^2R^2)}(v^+)^{1-p/2}\left\{\int\!\!\!\int_{-(R,\theta R^2)}(u^+)^p\,dx\,et\right\}^{1/2}+\hat{k}R^{n\chi/2}$$

$$\leqslant\frac{1}{2}f(t)+(2C)^{2/p}\frac{1}{(t-s)^{(2+n\chi)/p/\varepsilon}}\left\{\int\!\!\!\int_{\hat{Q}(R+\theta R^2)}(u^+)^p\,dx\,dt\right\}^{1/p}+\hat{k}R^{n\chi/2}.$$

By Lemma 2.3 in [12] (page 10), we then have

$$\sup_{\hat{Q}(\eta R,\theta\mu^2R^2)}u(x,t)\leqslant\frac{\widetilde{C}}{(1-\mu)^{(2+n\chi)/p/\varepsilon}}\left\{\left[\int\!\!\!\int_{\hat{Q}(R,\theta R^2)}(u^+)^p\,dx\,dt\right]^{1/p}+\hat{k}R^{n\chi/2}\right\},$$

where $\widetilde{C}=\widetilde{C}(n,N,\chi,p,\theta)$, which proves the theorem.

§ 3. The Diffusion Properties for Functions in $DG(Q_T, N, \chi, \hat{k}, 1)$

Lemma 3.1. *Assume* $u\in V_2^{1,0}(Q_T)$, (1.2) *is valid for* $w(x,t)=-u(x,t)$, u; $\rho>0$. *If*

$$\operatorname{mes} B^-_{\zeta,\rho}(t_0)\leqslant\frac{1}{M}\operatorname{mes} B_\rho$$

where $M>1$, $\zeta\geqslant\hat{k}\rho^{n\chi/2}$, *then for any* $\xi\in(\sqrt{1/M},1)$, *there exist positive numbers* $\bar{\theta}$ $=\bar{\theta}(\xi,n,N,\chi,M)<1$, $b(\xi)=b(\xi,n,N,\chi,M)<1$ *such that*

$$\operatorname{mes} B_{(-\xi\zeta),\rho}(t)\leqslant b(\xi)\operatorname{mes} B_\rho,\quad \forall t\in[t_0,\ t_0+\theta\rho^2],\qquad (\text{:}$$

where $B^-_{k,\rho}(t)=\{x\in B_\rho,\ u(x,t)<k.\}$.

Proof Taking $k=\zeta$ in (1.2) we have, for $\sigma_1\in(0,1)$,

$$\max_{t_0<t<t_0+\bar{\theta}\rho^2}\int_{B^-_{\zeta,\rho-\sigma_1\rho}}(\zeta-u(x,t))^2dx\leqslant\int_{B^-_{\zeta,\rho}}(\zeta-u(x,t_0))^2dx$$

$$+N\left\{(\sigma_1\rho)^{-2}\int_{Q(\rho,\bar{\theta}\rho^2)}[(\zeta-u(x,t))^+]^2dx\,d\right.$$

$$\left.+(\hat{k}^2+\zeta^2\rho^{-n\chi})\mu^{\frac{n(\bar{\pi}+\chi)}{n+2}}(\zeta,\rho,\bar{\theta}\rho^2)\right\},$$

where $\bar{\theta}\in(0,1)$ is a constant to be determined later. Hence we have

$$\int_{B^-_{\zeta,\rho-\sigma_1\rho}} (\zeta-u(x,t))^2 dx \leqslant \zeta^2\left\{\frac{1}{M}+\left(\frac{M}{\sigma_1^2}+C_0N\right)\bar\theta^\delta\right\}\text{mes } B_\rho,$$

where C_0 depends only on n, $\delta=\min\left\{\frac{n(1+\chi)}{n+2}, 1\right\}$.

On the other hand, for any $\xi\in(0, 1)$,

$$(\xi\zeta)^2 \text{mes } B^-_{(1-\xi)\zeta,\rho\sigma-1\rho}(t)\leqslant\int_{B^-_{(1-\xi)\zeta,\rho-\sigma_1\rho}(t)}[\zeta-u(x,t)]dx.$$

Therefore

$$\text{mes } B^-_{(1-\xi)\zeta,\rho-\sigma_1\rho}(t)\leqslant\xi^{-2}\left\{\frac{1}{M}+\left(\frac{N}{\sigma_1^2}+C_0N\right)\bar\theta^\delta\right\}\text{mes } B_\rho.$$

For any $\xi\in(\sqrt{1/M}, 1)$ it is obviously possible to select positive numbers σ_1, $\bar\theta(\xi)$ and $b(\xi)$ so that

$$n\sigma_1+\xi^{-2}\left\{\frac{1}{M}+\left(\frac{N}{\sigma_1^2}+C_0N\right)\bar\theta^\delta\right\}\in(0, 1).$$

The lemma is proved

To derive Lemma 3.3 below we need the following lemma which can be proved in the same way as in the proof of Inequality (5.5) of Ch, II in [4].

Lemma 3.2. *Assume* $u\in W_1^1(B_\rho)$. *If* mes $B^-_{\eta,\rho}\leqslant b$ mes B_ρ, $b\in(0, 1)$. *Then for any* $h<k<\eta$ *we have*

$$(k-h)\text{ mes } B^-_{h,\rho}\leqslant\frac{\beta\rho}{1-b}\int_{B^-_{k,\rho}\setminus B^-_{h,\rho}}|D_xu|dx, \tag{3.2}$$

where β *is a constant depending only on* n.

Lemma 3.3. *Let* $u(x, t)\in V_2^{1,0}(Q_T)$, *inequalities* (1.2) *and* (1.3) *be valid for* $v(x, t)=-u(x, t)$. *Assume* $u(x, t)\geqslant0$, $\eta>0$, mes $B^-_{\eta,\rho}(t_0)\leqslant\frac{1}{M}$ mes B *and* $M>1$, *Hence for any* $\xi\in(\sqrt{1/M}, 1)$, $\bar\theta=\bar\theta(\xi)$ *can be determined by Lemma* 3.1. *Then for any* $\gamma>0$ *and any* $\theta\in(0, \bar\theta]$, *there exists* $s=s(\xi, \gamma, \theta)>0$ *such that, for* $(1-\xi)^{s+1}\eta>\rho^{n\chi/2}$,

$$|Q^-(\eta(1-\xi)^s, \rho, \theta\rho^2)|\leqslant\gamma|Q(\rho, \theta\rho^2)|, \tag{3.3}$$

where $Q^-(k, \rho, \theta\rho^2)=\{(x, t)\in Q(\rho, \theta\rho^2); u(x, t)<k\}$, *provided* $Q(2\rho, 4\theta\rho^2)\subset Q_T$.

Proof By the assumption mes $B^-_{\eta,\rho}(t_0)\leqslant\frac{1}{M}$ mes B_ρ, we have

$$\text{mes}^-_{(1-\xi)^i\eta,\rho}(t_0)\leqslant\frac{1}{M}\text{ mes } B_\rho,\ i=0, 1, \cdots, s-1,$$

(s to be determined). Therefore Lemma 3.1 is applicable to $u(x, t)$ in the $Q(\rho, \theta\rho^2)$ for the level $\zeta=(1-\xi)^i\eta$, it guarantees that

$$\text{mes } B^-_{(1-\xi)^{i+1}\eta,\rho}(t)\leqslant b(\xi)\text{ mes } B_\rho,$$

for

$$t\in[t_0, t_0+\theta\rho^2].$$

Let us apply inequality (3.2) to the function $u(x, t)$ and the levels $h=(1-\xi)^{i+1}\eta$, $k=(1-\xi)^i\eta$ for $t\in[t_0, t_0+\theta\rho^2]$. This gives

$$\xi(1-\xi)^i\eta \operatorname{mes} B^-_{(1-\xi)^{i+1}\eta,\rho}(t) \leqslant \frac{\beta\rho}{1-b(\xi)}\int_{\mathscr{D}_i(t)} |D_x u(x,t)\,dx,$$

where

$$\mathscr{D}_i(t) = B^-_{(1-\xi)^i\eta,\rho}(t) - B^-_{(1-\xi)^{i+1}\eta,\rho}(t).$$

We integrate both sides of this inequality with respect to t over $[t_0,\ t_0+\theta\rho^2]$; then we square both sides, after which we estimate the right hand side by the Cauchy inequality

$$\begin{aligned}&[\xi(1-\xi)^i\eta]^2|Q^-((1-\xi)^{i+1}\eta,\ \rho,\ \theta\rho^2)|^2\\ &\leqslant \frac{\beta^2\rho^2}{(1-b(\xi))^2}\left[\int_{t_0}^{t_0+\theta\rho^2}\int_{\mathscr{D}_i(t)} |D_x u|\,dx\,dt\right]^2\\ &\leqslant \frac{\beta^2\rho^2}{[1-b(\xi)]^2}\int_{t_0}^{t_0+\theta\rho^2}\int_{\mathscr{D}_i(t)} |D_x u|^2\,dx\,dt\int_{t_0}^{t_0+\theta\rho^2}\operatorname{mes}\mathscr{D}_i(t)\,dt. \qquad (3.\end{aligned}$$

For an estimate of the first integral on the right hand side we use inequali (1.3), choosing for $Q(\rho,\ \tau)$ the cylinder $Q(2\rho,\ 4\theta\rho^2)$, and for $Q(\rho-\sigma_1\rho,\ \tau-\sigma_2\tau)$ t cylinder $Q(\rho,\ \theta\rho^2)$. After obvious simplification it gives

$$\left.\begin{aligned}&\int_{t_0}^{t_0+\theta\rho^2}\int_{\mathscr{D}_i(t)} |D_x u|^2\,dx\,dt \leqslant C_0[(1-\xi)^{i+1}\eta]^2|Q(\rho,\ \theta\rho^2)|\\ &C_0 = N2^{n+2}\left[\frac{3\theta+1}{3\theta}\ \omega_n^{2/(n+2)} + (1+2^{n\varkappa})\,\omega_n^{n\varkappa/(n+2)}\right].\end{aligned}\right\} \qquad (3.$$

For $i=0,\ 1,\ \cdots,\ s-1$ from (3.4) and (3.5) it follows that

$$|Q^-[(1-\xi)^{i+1}\eta,\ \rho,\ \theta\rho^2]|^2 \leqslant \frac{C_0\beta^2\rho^2}{\xi^2[1-b(\xi)]^2} Q(\rho,\ \theta\rho^2)|^{n/(n-2)}\int_{t_0}^{t_0+\theta\rho^2}\operatorname{mes}\mathscr{D}_i(t)\,dt,$$

and furthermore

$$|Q^-[(1-\xi)^s\eta,\ \rho,\ \theta\rho^2]|^2 \leqslant \frac{C_0\rho^2\beta^2}{\xi^2[1-b(\xi)]^2}|Q(\rho,\ \theta\rho^2)|^{n/(n+2)}\int_{t_0}^{t_0+\theta\rho^2}\operatorname{mes}\mathscr{D}_i(t)\,dt.$$

Let us sum these inequalities with respect to i from 0 to $s-1$. This gives

$$s|Q^-[(1-\xi)^s\eta,\ \rho,\ \theta\rho^2]|^2 \leqslant \frac{\beta^2C_0(\theta\omega_n)^{-2/(n+2)}}{\xi^2[1-b(\xi)]^2}|Q(\rho,\ \theta\rho^2)|^2,$$

from which it is seen that for

$$s = s(\xi,\ \gamma,\ \theta,\ ) \geqslant \frac{1}{\gamma^2}\,\frac{C_0\beta^2(\theta\omega_n)^{-2/(n+2)}}{\xi^2[1-b(\xi)]^2},$$

we will have (3.3).

Lemma 3.4. *Assume* $u\in V_2^{1,0}(Q_T)$, $w(x,\ t) = -u(x,\ t)$ *satisfies* (1.2) *and* (1.: *If* $u(x,\ t)\geqslant 0$, *and*

$$\operatorname{mes} B^-_{\eta,\rho}(t_0) \leqslant \frac{1}{M}\operatorname{mes} B_\rho,$$

where $M>1$, $\eta>0$, *then for any* $\xi\in(\sqrt{1/M},\ 1)$, *there exists a constant* $\bar\theta=\bar\theta(\xi)$ *p sessingthe following property: for any* $\theta\in(0,\ \bar\theta]$ *one can find an* $s^*=s^*(\xi,\ \theta)$ *such tha*

$$\inf_{B_{\frac{1}{2}\rho}(x_0)\times(t_0,t_0+\frac{1}{4}\theta\rho^2)} u(x,\ t) \geqslant \lambda\eta - \hat k\rho^{n\varkappa/2},$$

where $\lambda=\lambda(\xi,\ \theta)=(1-\xi)^{s^*+1}$, *provided* $Q(2\rho,\ 4\theta\rho^2)\subset Q_T$.

Proof Without loss of genterality we may assume $\hat k\rho^{n\varkappa/2}\leqslant(1-\xi)^{s^*+1}\eta$, with s^*

to be determined later. (Otherwise the lemma holds automatically.) For any $\xi \in (\sqrt{1/M}, 1)$, we can determine $\bar{\theta}=\bar{\theta}(\xi, M)$ by Lemma 3.7. For any $\theta \in (0,\bar{\theta}]$, set $\rho \geqslant R > r > 0$, $\tau_R=\theta R^2$, $\tau_r=\theta r^2$ and denote

$$a(k, r)=\int_{t_0}^{t_0+\tau_r} \text{mes}\, B^-_{k,r}(t)\,dt,$$

$$b(k, r)=\int_{t_0}^{t_0+\tau_r}\int_{B^-_{k,r}(t)}(k-u)^2\,dx\,dt.$$

By an argument similar to that in Lemma 2.1, we can get, for any $h\geqslant 0$,

$$b(h, r)\leqslant C_1[a(h, R)]^{2/(n+2)}\left\{\frac{1}{(R-r)^2}\, b(h,R)+(\hat{k}^2+h^2R^{-n\chi})\,[a(h,R)]^{n(1+\chi)/(n+2)}\right\}, \tag{3.6}$$

where $C_1=4\beta^2[(N+4\theta N+16\theta)/\theta$. Noticng, for $h>k\geqslant 0$,

$$b(h, R)=\int_{t_0}^{t_0+\tau_R}\int_{B^-_{h,R}(t)}(h-u)^2dx\,dt\leqslant h^2a(h,R),$$

$$b(h, r)=\int_{\tau^0}^{t^0+t_r}\int_{B^-_{h,r}(t)}(h-u)^2\,dx\,dt\geqslant (h-k)^2a(k,r),$$

from (3.6) we have

$$(h-k)^2a(k, r)\leqslant C_1[a(h, R)^{2/(n+2)}\left\{\frac{h^2}{(R-r)^2}\,a(h, R)\right.$$
$$\left.+(\hat{k}^2R^{n\chi}+h^2)R^{-n\chi}[a(h, R)]^{n(1+\chi)/(n+2)}\right\}. \tag{3.7}$$

Setting $k_j=\{(1-\xi)^{s^*+1}+2^{-j}((1-\xi)^{s^*}-(1-\xi)^{s^*+1}]\}\eta$, $\rho_j=\rho/2+2^{-(j+1)}\rho$, $(j=0,1, \cdots,)$, taking in (3.7) $h=k_j$, $k=k_{j+1}$, $r=\rho_{j+1}$, and denoting $a_j=a(k_j, \rho_j)$, we have

$$(k_j-k_{j+1})^2a_{j+1}\leqslant C_1a_j^{2/(n+2)}\{(1-\xi)^{2s^*}\eta^2 2^{2(j+2)}a_j\rho^{-2}$$
$$+[(1-\xi)^{2(s^*+1)}\eta^2+(1-\xi])^{2s^*}\eta^2]2^{n\chi}\rho^{-n\chi}a_j^{n(1+\chi)/(n+2)}\},$$

i.e.

$$a_{j+1}\leqslant Cb^ja_j^{2/(n+2)}\left\{\frac{a_j}{|Q(\rho, \theta\rho^2)|^{2/(n+2)}}+\frac{a_j^{n(1+\chi(/(n+2)}}{|Q(\rho, \theta\rho^2)|^{n\chi/(n+2)}}\right\},$$

where $C=C_1 2^{n\chi+6}[(\theta\omega_n)^{2/(n+2)}+(\theta\omega_n)^{n\chi/(n+2)}]/(1-\xi)^2\xi^2$, $b=2^4$. Setting

$$Y\quad \frac{a_j}{|Q(\rho, \theta\rho^2)|}, \qquad s=\min\left[\frac{2}{n+1}, \frac{n\chi}{n+2}\right],$$

we have

$$Y_{j+1}\leqslant 2Bb^jY_j^{j+s}. \tag{3.8}$$

By Lemma 3.3, for $\hat{r}=(2C)^{-1/s}b^{-1/s^2}$, we can determine $s^*=s(\xi, \hat{r}, \theta)$ such that

$$|Q^-[(1-\xi)^{s^*}, \rho, \theta\rho^2]|\leqslant\hat{r}|Q(\rho, \theta\rho^2)|,$$

i.e.

$$Y_0\leqslant(2C)^{-1/s}b^{-/s^2}.$$

Then by (3.8) we have $Y_j\to 0$ as $j\to\infty$, which proves the lemma.

Remark 3.5. Let $w(x, t)=-u(x, t)$ satisfy (1.2) and (1.3), $u(x, t)\geqslant 0$. If $u(x, t)\geqslant\eta_0$, for any $(x, t)\in B_r\cdot\{t=\tau\}$, then for any $m\geqslant 1$, there exsits $\bar{\theta}_m=\bar{\theta}(m)$ possessing the following property: for any $\theta_m\in(0, \bar{\theta}_m]$ one can find a constant $\lambda_m=\lambda(m, \theta_m)$ such that

$$u(x,\ t)\geqslant\lambda_m\eta_0-\hat{k}(2mr)^{n\chi/2},\ \text{for}\ (x,\ t)\in B_{mr}\cdot(\tau,\ \tau+\theta_m(m)^2).$$

provided $Q(4mr,\ \theta_m(4mr)^2)\subset Q_T$.

Proof Without loss of generailty we may assume $\eta_0>0$. For any $m\geqslant1$, there exists $M(m)>1$ such that

$$\text{mes}\,B_{2mr}-\text{mes}\,B_r\leqslant\frac{1}{M(m)}\,\text{mes}\,B_{2mr},$$

hence

$$\text{mes}\,B_{\eta_0,2mr}(\tau)\leqslant\frac{1}{M(m)}\,\text{mes}\,B_{2mr}.$$

By Lemma 3.4 with $\xi(m)=\frac{1}{2}[1+\sqrt{1/M(m)}]$, there exists $\bar\theta_m=\bar\theta(m)$ such that f(any $\theta_m\in(0,\ \bar\theta_m]$ it is valid that

$$u(x,\ t)\geqslant\lambda_m\eta_0-\hat{k}(2mr)^{n\chi/2},\ \text{for}\ (x,\ t)\in B_{mr}\times(\tau,\ \tau+\theta_m(mr)^2),$$

where $\lambda_m=(1-\xi(m))^{s^*+1}=\lambda(m,\ \theta_m)$ with s^* determined by Lemma 3.4.

Lemma 3.6. *Let* $w(x,\ t)=-u(x,\ t)$ *satisfy* (1.2) *and* (1.3), $u(x,\ t)\geqslant\eta>0$, $\rho>0$. *Then there exist constants* $\delta=\delta(\theta)$, $\lambda_0=\lambda_0(\theta)$, *such that the inequality*

$$|Q^-(\eta,\ \rho,\ \theta\rho^2)|\leqslant\delta|Q(\rho,\ \theta\rho^2)| \qquad (3.$$

implies that

$$u(x,\ t)\geqslant\lambda_0\eta-\hat{k}\rho^{n\chi/2},\ \textit{for}\ (x,\ t)\in B_{\rho/2}\times(t_0+\bar\theta_0\rho^2/8,\ t_0+\bar\theta_0\rho^2/4),$$

where $\bar\theta_0=\min[\theta,\ \theta(1/2)]$, $\theta(1/2)$ *is the one in Lemma* 3.4 *with* $\xi=1/2$, $M=$! *provided* $Q(2\rho,\ 4\theta\rho^2+\bar\theta_0\rho^2/8)\subset Q_T$.

Proof Taking $\delta=\delta(\theta)=2^{-6}\bar\theta_0\theta^{-1}$, by (3.9) we have

$$|Q^-(x^0,\ t_0,\ \eta,\ \rho,\ \bar\theta_0\rho^2/8)\leqslant|Q^-(\eta,\ \rho,\ \theta\rho^2)|\leqslant\delta|Q(\rho,\ \theta\rho^2)|$$
$$=8\delta\theta\bar\theta_0^{-1}|Q(\rho,\ \bar\theta_0\rho^2/8)|\leqslant2^{-3}|Q(\rho,\ \bar\theta_0\rho^2/8)|.$$

Therefore there exists $\bar t\in(t_0,\ t_0+\bar\theta_0\rho^2/8)$ such that

$$\text{mes}\,B^-_{\eta,\rho}(\bar t)\leqslant\left(\frac{1}{2}\right)^3\text{mes}\,B_\rho.$$

By Lemma 3.4 with $\xi=2^{-1}$, $M=2^3$, we can find $\lambda_0=\lambda(\bar\theta_0)$ such that

$$u(x,\ t)\geqslant\lambda_0\eta-\hat{k}\rho^{\frac{1}{2}n\chi}\quad\forall(x,\ t)\in B_{\rho/2}\times(\bar t,\ \bar t+\bar\theta_0\rho^2/4),$$

which implies

$$u(x,\ t)\geqslant\lambda_0\eta-\hat{k}\rho^{n\chi/2},\quad\forall(x,\ t)\in B_{\rho/2}\times(t_0+\bar\theta_0\rho^2/8,\ t_0+\bar\theta_0\rho^2/4).$$

Proposition 3.7. *Let* $w(x,\ t)=-u(x,\ t)$ *satisfy* (1.2) *and* (1.3), $u(x,\ t)\geqslant\eta>0$, θ, $R>0$. *Then for any* $m_1\geqslant1$, $m_2>0$, *there exist positive constants* $\delta=\delta(\theta)$, λ $\lambda(m_1,\ m_2,\ \theta)$, $C_0=C_0(m_1,\ m_2,\ \theta)$, *such that the inequality*

$$|Q^-(\eta,\ R,\ \theta R^2)|\leqslant\delta|Q(R,\ \theta R^2)|$$

implies that

$$u(x,\ t)\geqslant\lambda\eta-C_0\hat{k}R^{n\chi/2},\ \forall(x,\ t)\in B_{m_1R}(x^0)\times(t_0+\theta R^2,\ t_0+\theta R^2+m_2\theta(m_1R)^2),$$

provided that $Q(4m_1R^2,\ \theta R^2+(m_2+1)\theta(4m_1R)^2)\subset Q_T$.

Proof By Lemma 3.6 there exist $\delta=\delta(\theta)$, $\lambda_0=\lambda_0(\theta)$ such that if

$$|Q^-(\eta, R, \theta R^2)|\leqslant\delta|Q(R, \theta R^2)|,$$

hen

$$u(x, t)\geqslant\lambda_0\eta-\hat{k}R^{n\chi/2},\ \forall(x, t)\in B_{R/2}(x^0)\times(t_0+\bar{\theta}^0\rho^2/8,\ t_0+\bar{\theta}_0\rho^2/4),$$

here $\bar{\theta}_0=\min\left[\theta, \theta\left(\frac{1}{\cdot}\right)\right]$. Hence for any $m_1\geqslant1$, we can find $\bar{\theta}_{m_1}=\bar{\theta}(2m_1)$ by using emark 3.5 with $m=2m_1$, $r=R/2$, $\tau=t\in(t_0+\bar{\theta}_0\rho^2/8,\ t_0+\bar{\theta}_0\rho^2/4)$, such that for $_{m_1}=\min[\bar{\theta}_{m_1}, \theta]$ there exists $\lambda_{m_1}=\lambda_{m_1}(m_1, \theta_{m_1})$, which guarantees that

$$u(x, t)\geqslant\lambda_{m_1}\lambda_0\eta-\lambda_{m_1}\hat{k}R^{n\chi/2}-\hat{k}(2m_1r)^{n\chi/2}$$
$$\geqslant\lambda_{m_1}\lambda_0\eta-(1+2^{n\chi/2}m_1^{n\chi/2})\hat{k}R^{n\chi/2},$$
$$\forall(x, t)\in B_{m_1R}(x^0)\times(\bar{t},\ \bar{t}+Q_{m_1}(m_1R)^2).$$

epeating the above procedure for at most m_0 times where

$$m_0=\text{the integer part of}\left[\frac{\theta R^2+m_2\theta(m_1R)^2}{\theta_{m_1}(m_1R)^2}\right]+2,$$

ne can obtain

$$u(x, t)\geqslant\lambda_{m_1}^{m_0}\lambda_0\eta-(1+m_02^{n\chi/2}m_1^{n\chi/2})\hat{k}R^{n\chi/2},$$
$$\forall(x, t)\in B_{m_1R}(x^0)\times(\bar{t},\ t_0+\theta R^2+m_2\theta(m_1R)^2),$$

hich leads to tha proposition by taking $\lambda=\lambda_{m_1}^{m_0}\lambda_0$, and $C=1+m_0(2m_1)^{n\chi/2}$.

Lemma 3.8. *Let $w(x, t)=-u(x, t)$ satisfy (1.3) and (1.2), $u(x, t)\geqslant0$, $\eta>0$, $>r_0>0$, $T_1>T_0=0$, and*

$$u(x, T_0)\geqslant\eta,\quad \forall x\in B_{r_0}(x^0).$$

Remark 3.5 with $m=2$, we denote $\bar{\theta}_2=\bar{\theta}(m)$, take $\theta_2\in(0,\ \bar{\theta}_2]$ and determine $\lambda_2=$ $2, \theta_2)$. If for some $k\in\mathbb{N}$ we have

$$R_0=2^kr_0,\quad T_1=T_0+\frac{4}{3}(4^k-1)\left(\frac{1}{2}\theta_2r_0^2\right), \tag{3.10}$$

en

$$u(x, t)\geqslant\eta\left(\frac{r_0}{R_0}\right)^s-\frac{4^\alpha}{2^\alpha-1}\hat{k}R_0^\alpha,\quad \alpha=n\chi/2,$$
$$\forall(x, t)\in B_{R_0}(x^0)\times(T_1-\theta_2R_0^2/2,\ T_1),$$

ere $s=s(\theta_2)=-\frac{l_n\lambda_2}{l_n2}>0$, provided $B_{4R_0}\times(T_0,\ T_1+15\theta_2R_0^2)\subset Q_T$.

Proof Set $r_i=2r_{i-1}$, $t_i=T_0+\sum_{j=1}^{i}\theta_2r_i^2$, $(i=1, 2, \cdots, k)$. Then one can obtain

$$\left.\begin{array}{l}u(x, t)\geqslant\lambda_2^k\eta-\hat{k}\left(\sum_{j=1}^{k}(2r_j)^\alpha\right),\\ \forall(x, t)\in B_{r_k}(x^0)\times(T_0,\ t_k).\end{array}\right\} \tag{3.11}$$

fact, taking $\tau=T_0$, $r=r_0$, $m=2$ in Remark 3.5 one has

$$u(x, t)\geqslant\lambda_2\eta-\hat{k}(4r_0)^\alpha,$$
$$\forall(x, t)\in B_{2r_0}(x^0)\times(T_0,\ T_0+\theta(2r_0)^2),$$

i. e.

$$u(x, t)\geqslant\lambda_2\eta-\hat{k}(2r_1)^\alpha,\quad \forall(x, t)\in B_{r_1}(x^0)\times(T_0,\ t_1+\theta r_1^2/2).$$

Then by using Remark 3.5 again with $r=r_1$, $m=2$, $\tau\in(t_1,\ t_1+\theta_2r_1^2/2)$, one has

$$u(x,\ t)\geqslant\lambda_2\eta-\hat{k}(2r_1)^\alpha-\hat{k}(2r_2)^\alpha,\qquad\forall(x,\ t)\in B_{r_2}(x^0)\times(T_0,\ t_2+\theta_2r_2^2/2).$$

Repeating the above procedure for k times one can obtain (3.11). But from (3.10) we have

$$k=-l_n\left(\frac{r_0}{R^0}\right)\Big/l_n2,\qquad T_1-\theta_2S^2/2\geqslant T_0.$$

Substituting these into (3.11) we conclude the proof of the lemma.

Proposition 3.9. *Let* $w(x,\ t)=-u(x,\ t)$ *satisfy* (1.2) *and* (1.3), $u(x,\ t)\geqslant0$. *If* $B\geqslant r_0>0$, $\alpha_2>\alpha_1>\alpha_0\geqslant0$, *and*

$$u(x,\ t)\geqslant\eta>0,\ \forall(x,\ t)\in B_{r_0}(x^0)\times\{t=t_0+\alpha_0R^2\},$$

then there exist constants

$$C=C(\alpha_2-\alpha_0,\ \alpha_1-\alpha_0)>0,$$
$$s=s(\alpha_1-\alpha_0)>0,$$
$$C_1=C_1(\alpha_2-\alpha_0,\ \alpha_1-\alpha_0)>0$$

such that, for $\alpha=n\chi/2$,

$$u(x,\ t)\geqslant C\left(\frac{r_0}{R}\right)^s\eta-C_1\hat{k}R^\alpha,\qquad\forall(x,\ t)\in B_R(x^0)\times(t_0+\alpha_1R^2,t_0+\alpha_2R^2),$$

provided $B_{8R}(x^0)\times(\alpha_0R^2,\ \alpha_2R^2+(8R)^2)\subset Q_T$.

Proof For $\bar{\theta}_2$ defined in Lemma 3.8, we take

$$\theta_2=\min[\bar{\theta}_2,\ 3(\alpha_1-\alpha_0)/64].$$

For given R and r_0, there exists a fixed $k\in\mathbb{N}$ such that

$$\left(\frac{1}{2}\right)^k\leqslant\frac{r_0}{R}\leqslant\left(\frac{1}{2}\right)^{k-1}.$$

Set $R_0=2^kr_0$, $T_0=t_0+\alpha_0R^2$, and $T_1=\sum_{i=1}^{k}\theta_2(2r_1)^2+T_0$. Then it is easy to see that

$$T_1=16\theta_2r^2(4^k-1)/3+T_0<\alpha_1R^2.$$

Now by Lemma 3.8 we have

$$u(x,\ t)\geqslant\left(\frac{r_0}{R_0}\right)^s\eta-\frac{4^\alpha}{2^\alpha-1}\hat{k}R_0^\alpha,\qquad\forall(x,\ t)\in B_{R_0}(x^0)\times(T_1-\theta_2R_0^2/2,\ T_1).$$

Hence

$$u(x,\ t)\geqslant\left(\frac{1}{2}\right)^s\left(\frac{r_0}{R}\right)^s\eta-2^\alpha\frac{4^\alpha}{2^\alpha-1}\hat{k}R^\alpha,\quad\forall(x,\ t)\in B_R(x^0)\times(T_1-\theta_2R^2/2,\ T_1),$$

where $s=s(\theta_2)=s(\alpha_1-\alpha_0)$. Then applications of Proposition 3.7 lead to conclusion.

§ 4. The Proof of Theorem 1.3

By means of the results in §3, Theorem 1.3 can be proved in the same wa in [12] by using the measure argument. To do this it is again needed to cons cubes instead of balls. Denoting

$$K_R(x_0)=\{x^1,\ \cdots,\ x^n);\ \max_{1\leqslant i\leqslant n}|x^i-x_0^i|=R\},$$

$$Q_{R,\theta}\triangleq Q_{R,\theta}(x_0,\ t_0)=K_R(x_0)\times(t_0,\ t_0+\theta R^2),$$

we may rewrite Proposition 3.7 and Proposition 3.9 into the following two lemmas respectively.

Lemma 4.1. *Assume $u(x,t)\in DG(Q_T, N, \chi, \hat{k}, 1)$, $u(x,t)\geqslant 0$, $\eta>0$, $r>0$, $\theta>0$. Then there exist positive constants μ, λ and C_0 depending only on n, χ, N and θ, such that*

$$|\{(x,t)\in Q_{r,\theta}(\bar{x},\bar{t});\ u(x,t)\geqslant\eta\}|\geqslant\mu|Q_{r,\theta}(\bar{x},\bar{t})|$$

implies

$$u(x,t)\geqslant\lambda\eta-C_0\hat{k}r^{n\chi/2},\quad \forall(x,t)\in\{K_{3r}(x)\times(t+\theta r^2,\ t+8\theta r^2)\}\cap Q_T,$$

provided $B_{12\sqrt{n}r}(x^0)\times(t_0,\ t_0+\theta r^2+2\theta(12\sqrt{n}\,r)^2)\subset Q_T$.

Lemma 4.2. *Assume $u(x,t)\in DG(Q_T, N, \chi, \hat{k}, 1)$, $u(x,t)\geqslant 0$, $\eta>0$, $\rho\geqslant\hat{r}>0$, $\alpha_2>\alpha_1>\alpha_0\geqslant 0$, $B_{8\sqrt{n}\rho}(x_0)\times(t_0+\alpha_0\rho^2,\ t_0+n(8\rho)^2+\alpha_2\rho^2)\subset Q_T$. If*

$$u(x,\ \alpha_0R^2)\geqslant\eta,\ \forall x\in K_R(x_0),$$

then there exist positive constants C, C_1 and s depending only on n, N, $\alpha_1-\alpha_0$, and $\alpha_2-\alpha_0$ and n, N, $\alpha_1-\alpha_0$ respectively such that

$$u(x,t)\geqslant C\left(\frac{\hat{r}}{\rho}\right)^s\eta-C_1k\rho^{n\chi/2},\quad \forall(x,t)\in K_\rho(x_0)\times(t_0+\alpha_1\rho^2,\ t_0+\alpha_2\rho^2).$$

(If $0<\rho<\hat{r}$, $q>1$ and $q\rho>\hat{r}$, then the lemma is also valid for C, C_1 and s depending also on q.)

Lemma 4.3. *Let $\Sigma\in Q_{\rho,\theta}(x,t)\triangleq Q_\rho$ be a given measurable set, $|\Sigma|\neq 0$, for fixed $\mu\in(0,1)$, denote $\tilde{\Sigma}=\Sigma\bigcup_{Q_{r,\theta}(x,t)\subset Q_\rho}\{K_{3r}(x)\times(t+\theta r^2,\ t+8\theta r^2)\cap Q_\rho;\ |\Sigma\cap Q_{r,\theta}(x,t)|\geqslant\mu|Q_{r,\theta}(x,t)|\}$. Then either for $\delta=\dfrac{12}{13-\mu}$ it holds that $|\tilde{\Sigma}|\geqslant\delta^{-1}|\Sigma|$, or there exists some $Q_{r_0,\theta}(x_0,t_0)\subset Q_\rho$ such that*

$$r_0\geqslant\frac{1}{4}\left\{\frac{|\Sigma|}{(2\rho)^n\theta}\right\}^{1/2},\quad |\Sigma\cap Q_{r_0,\theta}(x_0,t_0)|\geqslant\mu|Q_{r_0,\theta}(x_0,t_0)|.$$

By covering and changing arguments Theorem 1.3 can be derived from the following theorem.

Theorem 4.4. *Assume $u(x,t)\in DG(Q_T, N, \chi, \hat{k}, 1)$, $u(x,t)\geqslant 0$, R, $\theta_3>0$. Then for any σ_1, $\sigma_2\in(0,1)$, $0<\theta_1<\theta_2<\theta_3$, there exist constants p, $C>0$ such that if $\{B_{(12\sqrt{n}+2)\sigma_2R}(x_0)\times(t_0,\ t_0+\theta_1\sigma_1^{-2}+2\theta_1\sigma_1^{-2}(12\sqrt{n}\,R)^2)\cup B_{8\sqrt{n}\sigma R}(x_0)\times(t_0,\ t_0+\theta_3R^2+8\sigma R)^2)\}\subset Q_T$, $\sigma=\max\{\sigma_1,\ 3\sigma_2\}$, then*

$$\inf_{K_{\sigma_1R}(x_0)\times(t_0+\theta_2R^2,\,t_0+\theta_3R^2)}u(x,t)+\hat{k}R^{n\chi/2}\geqslant C\left(⨍_{K_{\sigma_2R}(x_0)\times(t_0,\,t_0+\theta_1^2)}u^p\,dx\,dt\right)^{1/p},$$

where p and C depend only on n, N, χ, σ_1, σ_2, θ_1, θ_2 and θ_3.

Proof Set $\sigma_2R=\rho$, $\theta_1\sigma_2^{-2}=\theta$. Denote $Q_\rho\triangleq Q_{\rho,\theta}(x_0,t_0)=K_{\sigma_2R}(x_0)\times(t_0,\ t_0+\theta_1R^2)$, $\Sigma_\eta=\{(x,t)\in Q_\rho;\ u(x,t)+(1-\lambda)^{-1}C_0\hat{k}R^\alpha\geqslant\eta\}$, $\forall\eta>0$, where λ and C_0 come from Lemma 4.1, $\alpha=n\chi/2$.

By the same usage of Lemmas 4.1, 4.2 and 4.3 as in the proof of Theorem 4.4

in [12], we may derive

$$u(x,t)\geqslant C^{-1}\eta\left[\frac{|\Sigma_\eta|}{|Q_\rho|}\right]^{1/\beta}-C\hat{k}R^\alpha,\quad \forall (x,t)\in K_{\sigma_1 R}(x_0)\times(t_0+\theta_2R^2,\ t_0+\theta_3R^2),\tag{4.1}$$

where $(1/\beta)\log\lambda(\log\delta)^{-1}+\delta/2>0$, and δ comes from Lemma 4.3 with μ being the same as in Lemma 4.1.

Denote

$$\xi=\inf_{K_{\sigma_1R}(x_0)\times(t_0+\theta_2R^2,\,t_0+\theta_3R^2)}[u(x,t)+C\hat{k}R^\alpha].\tag{4}$$

Then from (4.1), taking $p\in(0,\beta)$, we have

$$\begin{aligned}\int_{Q_\rho}u^p\,dx\,dt&\leqslant p\int_0^\xi \eta^{p-1}|\Sigma_\eta|\,d\eta+p\int_\xi^\infty\eta^{p-1}|\Sigma_\eta|\,d\eta\\&\leqslant C\left(\frac{1}{\beta-p}+1\right)\xi^p|Q_\rho|.\end{aligned}$$

Noting (4.2) we thus conclude the proof.

§ 5. Remarks

Remark 5.1. Here we consider the case for general N_1 in (1.2). Observi the greceding sections we can find that to establish the theorems in § 1 we on need to show the validity of Lemma 3.1, Lemma 3.3, Proposition 3.4 and Rema 3.5 in the present case.

By similar arguments as in Lemma 3.1 we can obtain the following lemma.

Lamma 3.1. *Assume* $u(x,t)\in V_2^{1,0}(Q_T)$, (1.2) *is valid for* $w(x,t)=-u(x,$ $u(x,t)\geqslant 0$, $\rho>0$. *If*

$$\operatorname{mes} B^-_{\zeta,\rho}(t_0)\leqslant\frac{1}{M}\operatorname{mes}B_\rho,$$

where $\zeta\geqslant\hat{k}\rho^{n\chi/2}$, $M>N_1$, *then for any* $\xi\in\left(\sqrt{\frac{N_1}{M}},\ 1\right)$ *there exist positive n numbers* $\bar\theta$ $\bar\theta(\xi,n,N,\chi,M,N_1)<1$, $b(\xi)=b(\xi,n,N,\chi,M,N_1)<1$, *such that*

$$\operatorname{mes}B^-_{(1-\xi)\zeta,\rho}(t)\leqslant b(\xi)\operatorname{mes}B_\rho,\qquad\forall t\in[t_0,\ t_0+\bar\theta\rho^2],$$

where $B^-_{k,\rho}(t)=\{x\in B_\rho;\ u(x,t)<k\}$.

From Lemma 3.1′ follow the results corresponding to Lemma 3.3, and Lemı 3.4 automatically.

Lemma 3.4′. *Assume* $u(x,t)\in V_2^{1,0}(Q_T)$, $w(x,t)=-u(x,t)$ *satisfies* (1.2) *a* (1.3). *If* $\alpha\in(0.1)$, $u(x,t)\geqslant0$ *and*

$$\operatorname{mes}B^-_{\eta,\rho}(t_0)\leqslant\frac{1}{M}\operatorname{mes}B_\rho,$$

where $M>1$, $\eta>0$, *then for any* $\xi\in\left(\sqrt{\frac{N_1}{M}},\ 0\right)$, *there exists* $\bar\theta=\bar\theta(\xi,n,N,\chi,M,N$ <1 *possessing the following property: for any* $\theta\in(0,\ \bar\theta]$, *one can find an* $s^*=s^*(\xi,n,N,\chi,M,\theta,N_1,\alpha)$ *such that*

$$\inf_{B_{\alpha\rho}(x_0)\times(t_0,t_0+\theta(\alpha\rho)^2)} u(x,\ t)\geqslant\lambda\eta-\hat{k}\rho^{n\chi/2},$$

here $\lambda=\lambda(\xi,\ n,\ N,\ \chi,\ M,\ \theta,\ N_1,\ \alpha)=(1-\xi)^{s^*+1}$.

Remark 3.5′. Let $w(x,t)=-u(x,\ t)$ satisfy (1.2) and (1.3), $u(x,t)\geqslant 0$. If

$$u(x,\ t)\geqslant\eta_0,\quad \forall(x,\ t)\in B_r\times\{t=\tau\},$$

ıen there exists $\bar\theta=\bar\theta(n,\ N,\ \chi,\ N_1)$ such that for any $m\geqslant 1$, $\theta\in(0,\ \bar\theta]$, one can find $_n=\lambda_m(m,\ N,\ n,\ \chi,\ \theta,\ N_1)$, $c_0=c_0(m,\ n,\ N,\ \chi,\ \theta,\ N_1)$ with the property:

$$u(x,\ t)\geqslant\lambda_m\eta_0-c_0\hat{k}r^{n\chi/2},\quad \forall(x,t)\in B_{mr}\times(\tau,\ \tau+(mr)^2).$$

Proof Note that for any $\beta>1$ there is a $\mu(\beta)<1$ such that

$$\text{mes } B_{\beta r}-\text{mes } B_r=\mu(\beta)\,\text{mes } B_{\beta r},$$

ıoose $\beta=\beta(N_1)>1$ so that $\mu(\beta)N_1=1/4$, and set $M(\beta)=\mu^{-1}(\beta)$, then $[N_1/M(\beta)]$ $=1/4$, hence

$$\text{mes }[B^-_{\eta_0,\beta r}(\tau)]\leqslant\frac{1}{M(\beta)}\text{ mes } B_{\beta r}.$$

'ake $\xi=\left[1+\sqrt{\frac{1}{4}}\right]\Big/2=3/4$, $\alpha=(1+\beta)^{-1}/2<1$. Then by Lemma 3.4′, there kists $\bar\theta=\bar\theta(N,\ M,\ \chi,\ N_1)$ such that for any $\theta\in(0,\ \bar\theta]$, one can find $s^*=s^*(n,\ \chi,\ n,\ \theta,$ $N_1)$ such that

$$u(x,\ t)\geqslant\lambda\eta-\hat{k}r^{n\chi/2},\ \forall(x,\ t)\in B_{\alpha\beta r}(x_0)\times(\tau,\ \tau+\theta(\alpha\beta r)^2),$$

'here $\lambda=(1-\xi)^{s^*+1}=\lambda(N,\ \chi,\ n,\ \theta,\ N_1)$. By a consideration similar to that in roposition 3.7, repeating the above process for certain times (depending only on ι, $\alpha\beta$) we have

$$u(x,\ t)\geqslant\lambda_m\eta_0-c_0\hat{k}r^{n\chi/2},\quad \forall(x,\ t)\in B_{mr}(x_0)\times(\tau,\ \tau+\theta(mr)^2),$$

'here $\lambda_m=\lambda_m(m,\ N,\ n,\ \chi,\ \theta,\ N_1)$, $c_0=c_0(m,\ N,\ n,\ \chi,\ \theta,\ N_1)$, which completes the roof of the remark.

Remark 5.2. If $|u|\leqslant M$ (M is a constant) and $w(x,\ t)=\pm u(x,\ t)$ satisfy 1.1), (1.2) and (1.3) for $M\geqslant k\geqslant 0$, then Theorem 1.2, Theorem 1.3 and Theorem .4 are all valid.

Obviously the proof of Theorem 1.2 does not need any change.

Observing the proofs in § 3 we can find that in the two diffusion lemmas Lemma 4.1 and Lemma 4.2) if we replace the condition $\eta>0$ by $M\geqslant\eta>0$, then he conclusion is also true. Thus in the proof of Theorem 4.4, the inequality (4.5) s valid for $M\geqslant\eta>0$. Note that $|\Gamma_\eta|=0$ for $M<\eta$, therefore (4.5) holds for any η, nd Theorem 4.4 follows.

Remark 5.3. Now we sketch out the proof of the fact that the bounded olution discussed in [3] belong to the De Giorgi class mentioned in Remark 5.2 with $\chi=2/n$.

Assume $u(x,\ t)$ is the weak solution of the equation

$$\text{div } A(x,\ t,\ u,\ D_xu)+B(x,\ t,\ u,\ D_xu)-u_t=0,\ \text{in } Q_T,$$

where $A(x, t, z, p)$ and $B(x, t, z, p)$ satisfy the structure conditions indicated in (1.2) on page 207 of [3]. As pointed out in [3], we may assume $u(x, t)\in W^{1,1}_{2,loc}(Q_T)$. Then for any bounded function $\varphi(x, t)\geqslant 0$, $\varphi\in W^1_2(\Omega\times\{t=\hat{t}\})$, $(0<\hat{t}<T)$, $u(x, t)$ satisfies

$$\int_{Q_T}\{\varphi u_t+\varphi_x A(x, t, u, D_x u)-\varphi B(x, t, u, D_x u)\}dx\,dt=0.$$

If $|u(x, t)|\leqslant M$, then taking $\varphi(x, t)=\eta^2(x, t)e^{b_0 u}(u-k)^+$ with $\eta\in C_0^\infty(B(x_0)\times\{t=\hat{t}\})$, $0<t_0-\tau<\hat{t}<t_0<T$, $0\leqslant\eta\leqslant 1$, we can derive

$$\begin{aligned}\frac{1}{4}e^{-b_0 M}\Big[\int_{t_0-\tau}^{t_0}\int_{B^+_{k,\rho}}\eta^2|(u-k)^+_x|^2\,dx\,dt+\int_{B_\rho}\eta^2(x, t_0)\,[(u(x, t_0)-k)^+]^2dx\Big\}\\ \leqslant C\int_{t_0-\tau}^{t_0}\int_{B^+_{k,\rho}}\{[(u-k)^+]^2[1+|\eta_x|+|\eta_t|+|\eta_x|^2]+(1+k^2+k^2|\eta^2|\eta_x|^2)\}dx\,d\\ +\int_{B_\rho}\eta^2(x, t_0-\tau)\,[(u(x, t_0-\tau)-k)^+]^2dx, \qquad (5.\end{aligned}$$

where $B^+_{k,\rho}=\{x\in B_\rho;\ u(x, t)>k\}$. Similarly we have, for$B^-_{k,\rho}=\{x\in B_\rho;\ u(x, t)<k\}$

$$\begin{aligned}\int_{t_0}^{t_0+\tau}\int_{B^-_{k,\rho}}\eta^2|(k-u)^+_x|^2\,dx\,dt+\int_{B_\rho}\eta^2(x, t_0+\tau)\{(k-u(x, t_0+\tau))^+\}^2dx\\ \leqslant C_1\int_{B_\rho}\eta^2(x, t_0)\{(k-u(x, t_0))^+\}^2dx\\ +C\int_{t_0}^{t_0+\tau}\int_{B^-_{k,\rho}}\{[(k-u)^+]^2[1+|\eta_x|+|\eta_t|+|\eta_x|^2]\\ +(1+k^2+k^2|\eta_x|^2)\}dx\,dt, \qquad (5.\end{aligned}$$

where $B_\rho=B_\rho(x_0)$, $B_\rho(x_0)\times(t_0, t_0+\tau)\subset Q_T$, C_1 and C are positive consta depending only on μ, a_0, b_0 (from (1.2) of [3]) and M.

Choosing suitably the function $\eta(x, t)$ in (5.1) and (5.2) we can then pr thatu(x, t) belongs to the De Giorgi class.

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