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EXISTENCE THEOREM AND FINITE ELEMENT METHOD FOR STATIC PROBLEMS OF A CLASS OF NONLINEAR HYPERELASTIC SHELLS

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Abstract

In this paper, various boundary value problems of hyperelastic shells are considered. It is assumed that the storede-nergy function $W(x, F)$ of the material, of which the shell is made, satisfies polyconvex conditions proposed by Ball[2]. Existence of minimum points of the total energy of the shell in suitably chosen function spaces, and in suitably chosen finite element spaces is proved. Convergence of the finite element solutions is proved under certain regular conditions on the minimum points and some additional assumptions on $W(x, F)$. A Gradient type computing scheme for solving the finite element solutions is given, and global convergent result is obtained.

§ 1. Notes and Basic Hypotheses about Elastic Shells

Assume ω, the middle surface of a shell ω^* (see [1]), be orientable connect compact regular two-dimensional Riemannian manifold with or without bounda of which the Riemannian metric is induced from R^3. For the sake of simplici assume that the thickness of ω^* is constant before deformatioh occurs, $2l$ denotes thickness.

Suppose the deformation of shells, when exterior forces are imposed up satisfies one of following basic hypotheses.

Hypothesis A: The line segments of the normals to the reference surface remain to be straight line segments after deformation and the change of leng along each line is uniform.

Hypothesis B: The normals to the reference surface remain normal to t deformed middle surface after deformation and there is no change of length alo these normals.

Hypothesis B is conventionally called Love–Kirchhoff hypothesis. We may a assume that the normals to the reference surface ω become quadric, cubic, quar

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or in general n-th curve after deformation. But for the sake of simplicity, we only discuss the case when Hypothesis A or Hypothesis B is satisfied. The other cases can be discussed in the same way.

Assume there are finite compatible local coordinate systems $\{\omega_i, R_i^{-1}\}_1^N$, where ω_i are connected open sets of ω, $\bigcup_{i=1}^N \omega_i=\omega$ and R_i^{-1}: $\bar{\omega}_i\to\bar{\Omega}_i\subset R^2$ are diffeomorphism, $\bar{\omega}_i$, $\bar{\Omega}_i$ are closures of ω_i, Ω_i respectively. We denote R_i: $\bar{\Omega}_i\to\omega_i$ the inverse of R_i^{-1}, and let $\Omega=\sum_{i=1}^N \Omega_i$, where $\cup$ denotes disjoint union. By the compatibility of local coordinate systems we mean that they induce the same orientation on ω.

The position vector in R^3 of a point on a shell can be expressed as

$$R^*(p)=R(p')+\xi A_3(p'), \tag{1.1}$$

where R^*: $\bar{\Omega}\times[-l, l]\to{}^*,\omega$ $p=(\theta_1, \theta_2, \xi)$, $p'=(\theta_1, \theta_2)$ is the injection of p on $\bar{\Omega}$. $R^*(\theta_1, \theta_2, \xi)=R_i(\theta_1, \theta_2)+\xi A_3^i(\theta_1, \theta_2)$, when $(\theta_1, \theta_2)\in\Omega_i$. $A_3^i(\theta_1, \theta_2)$ is the unit normal vector of ω at the point $R_i(\theta_1, \theta_2)$, determined by

$$A_3^i=\frac{\dfrac{\partial R_i}{\partial\theta_1}\times\dfrac{\partial R_i}{\partial\theta_2}}{\left\|\dfrac{\partial R_i}{\partial\theta_1}\times\dfrac{\partial R_i}{\partial\theta_2}\right\|}.$$

It follows from the compatibility that

$$A_3^i(p')=A_3^j(q')$$

provided that $p'\in\bar{\Omega}_i$, $q'\in\bar{\Omega}_j$ and $R_i(p')=R_j(q')$.

The position vector of deformed shell under the Hypothesis A or the hypothesis B can be generally expressed as

$$\begin{aligned} r^*(\theta_1, \theta_2, \xi)&=R^*(\theta_1, \theta_2, \xi)+u^0(\theta_1, \theta_2)+\xi u^1(\theta_1, \theta_2)\\ &=r(\theta_1, \theta_2)+\xi c(\theta_1, \theta_2). \end{aligned} \tag{1.2}$$

The position vector of deformed shell under the Hypothesis B can also be expressed simpiy as

$$r^*(\theta_1, \theta_2, \xi)=r(\theta_1, \theta_2)+\xi a_3(\theta_1, \theta_2), \tag{1.3}$$

where $a_3(\theta_1, \theta_2)$ is the exterior unit normal of the deformed middle surface at the point $r(\theta_1, \theta_2)$.

The vector $u^0(\theta_1, \theta_2)+\xi u^1(\theta_1, \theta_2)$ is called the displacement vector at point $R^*(\theta_1, \theta_2, \xi)$.

§2. The Existence of Solutions for Static Problem of a Class of Hyperelastic Shells

We say that an elastic material is hyperelastic (see Ball [2], [3], or Ciarlet [4]) if the stress tensor of the three-dimensinal body made of such material can be expressed in the reference domain as

$$\hat{T}_R(x)=\frac{\partial W(F)}{\partial F}, \tag{2.1}$$

where W is the stored-energy function, $F=I+\nabla u$, and u is displacement vector. We only discuss the case when the shell is made of isotropic hyperelastic materials. We call the shells made of hyperelastic materials hyperelastic shells.

Assume the reference domain is the naturalistate of the shell, i. e. stress tensor vanishes in the case when there are no exterior forces acting upon the shell and the shell is undeformed.

Let ω be a manifold with boundary. Let $\partial\omega=\partial\omega_0\cup\partial\omega_1$. Denote

$$\left.\begin{array}{l}\{(\theta_1,\ \theta_2)\in\bar{\Omega}\,|\,R(\theta_1,\ \theta_2)\in\partial\omega_0\}=\partial\Omega_0,\\ \{(\theta_1,\ \theta_2)\in\bar{\Omega}\,|\,R(\theta_1,\ \theta_2)\in\partial\omega_1\}=\partial\Omega_1.\end{array}\right\} \quad (2$$

Assume meas $(\partial\Omega_0)\neq 0$.

We introduce first the following boundary value problem of shells: Supp hat the deformation of the shell satisfies the Hypothesis A. For simplicity, supp hat the density of exterior forces (measured in $\Omega\times[-l,\ l]$) is $f^0(\theta_1,\ \theta_2)+\xi f^1(\theta_1,\ \theta$ Suppose clamped edge condition is given on $\partial\Omega_0\times[-l,\ l]$, i. e. $r^*(\theta_1,\ \theta_2,\ \xi)$ $r_0^*(\theta_1,\ \theta_2,\ \xi)=R^*(\theta_1,\ \theta_2,\ \xi)+u_0(\theta_1,\ \theta_2)+\xi u_1(\theta_1,\ \theta_2)$, $\forall(\theta_1,\ \theta_2,\ \xi)\in\partial\Omega_0\times[-l,$ where u_0, u_1 are given vector functions on $\partial\Omega_0$. And natural boundary condition given on $\partial\Omega_1\times[-l,\ l]$, the density (measured in $\partial\Omega_1\times[-l,\ l]$) of exterior for acting upon the edge corresponding to $\partial\Omega_1\times[-l,\ l]$ is $g^0(\theta_1,\ \theta_2)+\xi g^1(\theta_1,\ \theta_2)$. The how does the shell deform?

We now investigate this problem by means of the total energy functional the shell, which can be expressed in the present case as

$$E(u^0,\ u^1)\equiv\int_{\omega^*}W(F)\,dx\,dy\,dz-\int_{\Omega}\int_{-l}^{l}(\bar{f}^0+\xi\,\bar{f}^1)\cdot(u^0+\xi u^1)\,d\xi\,d\theta^1\,d\theta^2$$
$$-\int_{\partial\Omega_1}\int_{-l}^{l}(\bar{g}^0+\xi\bar{g}^1)\cdot(u^0+\xi u^1)\,d\xi\,d\sigma, \quad (2.$$

where $u^0+\xi u^1$ is displacement vector with $(u^0-\xi u^1)|_{\partial\Omega_0}=u_0+\xi u_1$, $\forall\xi\in[-l,\ l]$ a $F=I+\nabla u, \bar{f}^0=\sum_{i=1}^{N}(\varphi_i\circ R_i)f^0$, $\bar{g}^0=\sum_{i=1}^{N}g^0(\varphi_i\circ R_i)$, $\bar{g}^1=\sum_{i=1}^{N}g^1(\varphi_i\circ R_i)$, $\{\varphi_i\}_1^N$ are the pa tition of unity subordinate to $\{\omega_i\}_1^N$.

It follows from simple caculation that

$$\nabla u=\nabla[(u^0+\xi u^1)\circ R^{*-1}]=\frac{\partial(u^0+\xi u^1)}{\partial(\theta_1,\ \theta_2,\ \xi)}\left(\frac{\partial R^*(\theta_1,\ \theta_2,\ \xi)}{\partial(\theta_1,\ \theta_2,\ \xi)}\right)^{-1}, \quad (2.$$

$$I+\nabla u=\frac{\partial r^*(\theta_1,\ \theta_2,\ \xi)}{\partial(\theta_1,\ \theta_2,\ \xi)}\left(\frac{\partial R^*(\theta_1,\ \theta_2,\ \xi)}{\partial(\theta_1,\ \theta_2,\ \xi)}\right)^{-1}, \quad (2.$$

$$\int_{\omega^*}W(F)\,dx\,dy\,dz=\sum_{i=1}^{N}\int_{\omega^*}\varphi_i W(F)\,dx\,dy\,dz$$
$$=\sum_{i=1}^{N}\int_{\Omega_i\times[-l,l]}\varphi_i W(F)\left|\frac{\partial R^*(\theta_1,\ \theta_2,\ \xi)}{\partial(\theta_1,\ \theta_2,\ \xi)}\right|d\xi\,d\theta^1\,d\theta^2$$

$$=\int_{\Omega}\sum_{i=1}^{N}\int_{-l}^{l}\varphi_i W(F)(1-2\xi H+\xi^2 K)\cdot\left\|\frac{\partial R}{\partial\theta_1}\times\frac{\partial R}{\partial\theta_2}\right\|d\xi\,d\theta^1\,d\theta^2, \tag{2.6}$$

where H is the mean curvature of ω, and K is the Gauss curvature of ω. Let

$$\overline{W}(\theta_1, \theta_2, u^0(\theta_1, \theta_2), u^1(\theta_1, \theta_2)) =\sum_{i=1}^{N}\int_{-l}^{l}\varphi_i(\theta_1, \theta_2)W(F)(1-2\xi H+\xi^2 K)\left\|\frac{\partial R}{\partial\theta_1}\times\frac{\partial R}{\partial\theta_2}\right\|d\xi. \tag{2.7}$$

Then, under the Hypothesis A, the total energy functional (2.3) transforms to

$$E(u^0, u^1)\equiv\int_{\Omega}\overline{W}(\theta_1, \theta_2, u^0, u^1)d\theta^1 d\theta^2-\int_{\Omega}(\tilde{f}^0\cdot u^0+\tilde{f}^1\cdot u^1)d\theta^1 d\theta^2 -\int_{\partial\Omega_1}(\tilde{g}^0\cdot u^0+\tilde{g}^1\cdot u^1)d\sigma, \tag{2.8}$$

where

$$\tilde{f}^1=\frac{2}{3}l^3\bar{f}^1,\quad \tilde{f}^0=2l\bar{f}^0,\quad \tilde{g}^1=\frac{2}{3}l^3\bar{g}^1,\quad \tilde{g}^0=2l\bar{g}^0.$$

We are now going to show the existence of minimum point of functional (2.8) in a certain set for a class of stored-energy function.

Suppose that the stored-energy function $W(F)$ satisfies the following hypotheses (see Ball [2], [3] for reference):

Hypothesis (I): There exists a continuous differentiable convex function G: $M_+^3\times M_+^3\times R_+^1\to R_+^1$ such that

$$W(F)=G(F, \operatorname{adj}F, \det F),\quad \forall F\in M_+^3. \tag{2.9}$$

Hypothesis (II): The corresponding function G satisfies

$$\lim_{n\to\infty}G(F_n, H_n, \delta_n)=0 \tag{2.10}$$

provided that $F_n\to F$ in M_+^3, $H_g\to H$ in M_+^3, and $\delta_n\to 0^+$.

Hypothesis (III) (coerceiveness): There exist $a\in R^1$, $b>0$, $p\geqslant 2$, $q>1$, $\frac{1}{p}+\frac{1}{q}\leqslant 1$, $r>1$, such that the function G in Hypothesis (I) satisfies

$$G(F, H, \delta)\geqslant a+b(\|F\|^p+\|H\|^q+\delta^r). \tag{2.11}$$

Here M^3 is the linear space of real matrices of third order, $M_+^3=\{A\in M^3\,|\,\det A>0\}$, $\operatorname{adj}A$ denotes the adjoint matrix of A, and $\det A$ the determinent of A.

Theorem 2.1. *suppose $W(F)$ satisfies Hypotheses(I)—(III). $\overline{W}(\theta_1, \theta_2, u^0, u^1)$ is defined by (2.7). Then the energy functional of the shell defined by (2.8) reaches its minimum in the set*

$$D=\Big\{(u^0, u^1)\in(\mathbb{W}^{1,p}(\Omega))^2\,\Big|\,\frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}\ \mathbb{L}^p(\Omega\times[-l, l]),$$
$$\operatorname{adj}\frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}\in\mathbb{L}^q(\Omega\times[-l, l]),$$
$$\det\frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}\in L^r(\Omega\times[-l, l]),$$

$$\left.\det \frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}>0, \text{ a.e. in } \Omega\times[-l, l],\ u^0|_{\partial\Omega_0}=u_0,\ u^1|_{\partial\Omega_0}=u_1\right\}$$

provided that $D\neq\emptyset$ *and there exists* $(v^0, v^1)\in D$, *such that*

$$E(v^0, v^1)<+\infty.$$

Note 1 $u^0|_{\partial\Omega_0}=u_0$, $u^1|_{\partial\Omega_0}=u_1$ is clamped edge condition.

Proof of the theorem Let $(u_n^0, u_n^1)\in D$ be the minmizing sequence, and

$$E(u_n^0, u_n^1)\leqslant E(v^0, v^1)=M<+\infty, \quad \forall n>0.$$

We have

$$\left.\begin{aligned}\int_\Omega(\bar f^0\cdot u_n^0+\bar f^1\cdot u_n^1)d\theta^1 d\theta^2&=\int_\Omega\int_{-l}(\bar f^0+\xi\bar f^1)\cdot(u_n^0+\xi u_n^1)d\xi\, d\theta^1 d\theta^2\\&=\int_\Omega\int_{-l}^{l}(\bar f^0+\xi\bar f^1)\cdot r_n^* d\xi\, d\theta^1 d\theta^2\\&\quad-\int_\Omega\int_{-l}^{l}(\bar f^0+\xi\bar f^1)\cdot R^* d\xi\, d\theta^1 d\theta^2,\\\int_{\partial\Omega_1}(\tilde g^0\cdot u_n^0+\tilde g^1\cdot u_n^1)d\sigma&=\int_{\partial\Omega_1}\int_{-l}^{l}(\tilde g^0+\xi\tilde g^1)\cdot r_n^* d\xi\, d\sigma\\&\quad-\int_{\partial\Omega_1}\int_{-l}^{l}(\tilde g^0+\xi\tilde g^1)\cdot R^* d\xi\, d\sigma.\end{aligned}\right\}\quad($$

Hence there are constants $K_1\in R^1$, $K_2>0$, which only depends on Ω, R, f, g, su that

$$\int_\Omega(\bar f^0\cdot u_n^0+\bar f^1\cdot u_n^1)d\theta^1 d\theta^2+\int_{\partial\Omega_1}(\tilde g^0\cdot u_n^0+\tilde g^1\cdot u_n^1)d\sigma\leqslant K_1+K_2\|r_n^*\|_{p,\omega^*}. \quad (2.1$$

On the other hand, we have

$$\int_\Omega \overline W(\theta_1, \theta_2, u_n^0(\theta_1, \theta_2), u_n^n(\theta_1, \theta_2))d\theta^1 d\theta^2$$

$$=\int_{\omega^*}W(F_n)dx\,dy\,dz=\int_{\omega^*}G(F_n, \operatorname{adj}F_n, \det F_n)dx\,dy\,dz. \quad (2.1$$

By Hypothesis (III), there are constants $\bar a\in R^1$, $\bar b>0$, such that

$$\begin{aligned}\int_\Omega \overline W(\theta_1, \theta_2, u_n^0, u_n^1)d\theta^1 d\theta^2&\geqslant\bar a+\bar b(\|F_n\|_{p,\omega^*}^p+\|\operatorname{adj}F_n\|_{q,\omega^*}^q+\|\det F_n\|_{r,\omega^*}^r\\&\geqslant\bar a+\bar b\Bigg(\left\|\frac{\partial r_n^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}\right\|_{p,\Omega\times[-l,l]}^p+\left\|\operatorname{adj}\frac{\partial r_n^*}{\partial(\theta_1, \theta_2, \xi)}\right\|_{q,\Omega\times[-l,l]}^q\\&\quad+\left\|\det\frac{\partial r_n^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)}\right\|_{r,\Omega\times[-l,l]}^r\Bigg),\end{aligned} \quad (2.1$$

where

$$F_n=I+\nabla u_n=\frac{\partial r_n^*}{\partial(\theta_1, \theta_2, \xi)}\left(\frac{\partial R^*}{\partial(\theta_1, \theta_2, \xi)}\right)^{-1}.$$

From the fact that $r_n^*|_{\partial\omega_0^*}=r_0^*|_{\partial\omega_0^*}=R^*|_{\partial\omega_0^*}+u_0+\xi u^1$, and

$$\|r_n^*\|_{1,p,\omega^*}^p\leqslant\widetilde K(\|r_n^*\|_{p,\partial\omega_0^*}^p+|r_n^*|_{1,p,\omega^*}^p) \quad (2.1$$

(see Morrey [5]), where $\widetilde K$ is a constant, it follows that

$$\int_\Omega \overline W(\theta_1, \theta_2, u_n^0, u_n^1)d\theta^1 d\theta^2\geqslant\tilde a+\tilde b(\|r_n^*\|_{1,p,\omega^*}^p+\|\operatorname{adj}F_n\|_{q,\omega^*}^q+\|\det F_n\|_{r,\omega^*}^r). \quad (2.17)$$

By (2.13), (2.17) we get

$$E(u_n^0, u_n^1) \geqslant \overline{K}_1 + \overline{K}_2(\|r_n^*\|_{1,p,\omega^*}^p + \|\operatorname{adj} F_n\|_{q,\omega^*}^q + \|\det F_n\|_{r,\omega^*}^r). \qquad (2.18)$$

Here $\overline{K}_1 \in R^1$, $\overline{K}_2 > 0$ are constants.

Because $E(u_n^0, u_n^1) \leqslant M < +\infty$, there is a subsequence, we may assume it is the original sequence without loss of generality, such that

$$\left.\begin{aligned} r_n^* &\rightharpoonup r^* \text{ in } \mathbb{W}^{1,p}(\omega^*), \\ \operatorname{adj} F_n &\rightharpoonup H \text{ in } \mathbb{L}^q(\omega^*), \\ \det F_n &\rightharpoonup \delta \text{ in } L^r(\omega^*). \end{aligned}\right. \qquad (2.19)$$

In addition we have (see Ball [2], [3], or Ciarlet [4]):

$$H = \operatorname{adj} F, \quad \delta = \det F, \quad F = I + \nabla u, \quad u = r^* - R^*. \qquad (2.20)$$

Because ω_i^* are diffeomorphism to $\Omega_i \times [-l, l]$, $i = 1, \cdots, N$, we also have $r_n^* \rightharpoonup r^*$ in $\mathbb{W}^{1,p}(\Omega_i \times [-l, l])$, $i = 1, 2, \cdots, N$; or

$$r_n^* \rightharpoonup r^* \text{ in } \mathbb{W}^{1,p}(\Omega \times [-l, l]). \qquad (2.21)$$

But $r_n^* = R^*(\theta_1, \theta_2, \xi) + u_n^0(\theta_1, \theta_2) + \xi u^1(\theta_1, \theta_2)$, so

$$u_n^0 + \xi u_n^1 \xrightarrow{\mathbb{W}^{1,p}(\Omega \times [-l, l])} r^*(\theta_1, \theta_2, \xi) - R^*(\theta_1, \theta_2, \xi) \equiv U(\theta, \theta_2, \xi). \qquad (2.22)$$

Hence $\int_{-l}^{l} (u_n^0 + \xi u_n^1) d\xi \in \mathbb{W}^{1,p}(\Omega)$, $n = 1, 2, \cdots$, are bounded in $\mathbb{W}^{1,p}(\Omega)$. Thus there is a sequence, which may be assumed to be the original sequence, convergent to $2lU^0(\theta_1, \theta_2)$ in $\mathbb{W}^{1,p}(\Omega)$ with weak topology, i. e.

$$u_n^0 \xrightarrow{\mathbb{W}^{1,p}(\Omega)} U^0(\theta_1, \theta_2). \qquad (2.23)$$

From (2.22) and (2.23), it is obvious that $u_n^1(\theta_1, \theta_2)$ also weakly converges in $\mathbb{W}^{1,p}(\Omega)$. Suppose that

$$u_n^1 \xrightarrow{\mathbb{W}^{1,p}(\Omega)} U^1(\theta, \theta_2). \qquad (2.24)$$

It follows from (2.22), (2.23), (2.24) that

$$U(\theta_1, \theta_2, \xi) = U^0(\theta_1, \theta_2) + \xi U^1(\theta_1, \theta_2). \qquad (2.25)$$

Let $u^0 = U^0$, $u^1 = U^1$. It is obvious that $u^0|_{\partial\Omega_0} = u_0$, $u^1|_{\partial\Omega_0} = u_1$, and $(u^0, u^1) \in (\mathbb{W}^{1,p}(\Omega))^2$. From (2.19), (2.20) and (2.4), we get

$$\left.\begin{aligned} &\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)} \in \mathbb{L}^p(\Omega \times [-l, l]), \\ \operatorname{adj} &\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)} \in \mathbb{L}^q(\Omega \times [-l, l]), \\ \det &\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)} \in \mathbb{L}^r(\Omega \times [-l, l]). \end{aligned}\right\} \qquad (2.26)$$

By Hypothesis (I), it is easy to show that $E(u^0, u^1)$ is convex with respect to $(F, \operatorname{adj} F, \det F)$. Hence $E(u^0, u^1)$ is lower semi-continuous for $(F, \operatorname{adj} F, \det F)$ (see Ball [2], [3], or ciarlet [4]). This fact together with (2.19) and (2.20) shows that

$$E(u^0, u^1) \leqslant \underline{\lim}_{n \to \infty} E(u_n^0, u_n^1) = \inf_{(v, v_1^*) \in D} E(v^0, v^1). \qquad (2.27)$$

By Hypothesis (II) and (2.27), we have

$$\det \frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)} > 0, \text{ a. e. in } \Omega \times [-l, l]. \tag{2.28}$$

(2.26), (2.28) together with the fact that $u^0|_{\partial\Omega_0} = u_0$, $u^1|_{\partial\Omega_0} = u_1$ ensure that

$$(u^0, u^1) \in D.$$

Hence $E\ u^0, u^1) \geq \inf_{(v^0, v^1) \in D} E(v^0, v^1)$. This and (3.27) imply

$$E(u^0, u^1) = \inf_{(v^0, v^1) \in D} E(v^0, v^1). \tag{2.29}$$

This proves the theorem.

We are now going on discussing the boundary value problem for shells of whi the deformation satisfies Hypothesis B. The only independent variable in this case $u^0(\theta_1, \theta_2)$, i.e. the displacement vector of the middle surface of the shell. (1.3) giv the displacement vector of the deformed shell. Define

$$\overline{W}(\theta_1, \theta_2, u^0(\theta_1, \theta_2)) = \sum_{i=1}^{N} \int_{-l}^{l} \varphi_i(\theta_1, \theta_2) W(F)(1 - 2\xi H + \xi^2 K) \left\| \frac{\partial R_i}{\partial \theta_1} \times \frac{\partial R_i}{\partial \theta_2} \right\| d\xi, \tag{2.3}$$

where $F = I + \nabla u$ is given by (2.4). Remembering

$$r^*(\theta_1, \theta_2, \xi) = R(\theta_1, \theta_2) + u^0(\theta_1, \theta_2) + \xi a_3(\theta_1, \theta_2),$$

the energy functional of the shell can now be expressed as

$$\begin{aligned} E(u^0) = & \int_\Omega \overline{W}(\theta_1, \theta_2, u^0(\theta_1, \theta_2)) d\theta^1 d\theta^2 \\ & - \int_\Omega \int_{-l}^{l} (\bar{f}^0 + \xi \bar{f}^1) \cdot (r^* - R^*) d\xi\, d\theta^1 d\theta^2 \\ & - \int_{\partial\Omega_1} \int_{-l}^{l} (\bar{g}^0 + \xi \bar{g}^1) \cdot (r^* - R^*) d\xi\, d\sigma. \end{aligned} \tag{2.3}$$

Theorem 2.2. *suppose that $W(F)$ satisfies Hypotheses* (I)—(III), $\overline{W}(\theta_1, \theta_2, u$ *is defined by* (2.30). *Then the energy functional $E(u^0)$ defined by* (2.31) *reach its minimum in set*

$$\begin{aligned} D_1 = \Big\{ & u^0 \in \mathbb{W}^{1,p}(\Omega) \mid a_3 \in \mathbb{W}^{1,p}(\Omega), \frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)} \in \mathbb{L}^p(\Omega \times [-l, l]), \\ & \operatorname{adj} \frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)} \in \mathbb{L}^q(\Omega \times [-l, l]), \\ & \det \frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)} \in L^r(\Omega \times [-l, l]), \\ & \det \frac{\partial r^*(\theta_1, \theta_2, \xi)}{\partial(\theta_1, \theta_2, \xi)} > 0, \text{ a. e. in } \Omega \times [-l, l], u^0|_{\partial\Omega_0} = u_0 \Big\} \end{aligned}$$

provided that $D_1 \neq \phi$ and there is a $v^0 \in D_1$, such that

$$E(v^0) < +\infty.$$

Note 2 $u^0|_{\partial\Omega_0} = eu_0$ corresponds to hinge type condition with zero exteri moment.

The proof of the theorem Let $\{u_n^0\}$ be a minimizing sequence. There are constants $K_1 \in R^1$, $K_2 > 0$ such that

$$\int_{\Omega}\int_{-l}^{l}(\bar f^0+\xi\bar f^1)\cdot(r^*-R^*)d\xi\,d\theta^1 d\theta^2+\int_{\partial\Omega_1}\int_{-l}^{l}(\bar g^0+\xi\bar g^1)\cdot(r_n^*-R^*)d\xi\,d\sigma$$
$$\leqslant K_1+K_2\|r_n^*\|_{p,\omega^*}. \tag{2.32}$$

There are also constants $\bar a\in R^1$, $\bar b>0$, by Hypotheses (III), such that

$$\int_{\Omega}\overline W(\theta_1,\theta_2,u_n^0)d\theta^1 d\theta^2=\int_{\omega^*}W(F_n)dx\,dy\,dz$$
$$=\int_{\omega^*}G(F_n,\operatorname{adj}F_n,\det F_n)dx\,dy\,dz$$
$$\geqslant \bar a+\bar b(\|F_n\|_{p,\omega^*}^p+\|\operatorname{adj}F_n\|_{q,\omega^*}^q+\|\det F_n\|_{r,\omega^*}^r). \tag{2.33}$$

By definition we have $r_n^*=R(\theta_1,\theta_2)+u_n^0(\theta_1,\theta_2)+\xi a_{3n}(\theta_1,\theta_2)$. In particular, on $\Omega_0\times[-l,l]$, we have $r_n^*(\theta_1,\theta_2,\xi)=R(\theta_1,\theta_2)+x_0(\theta_1,\theta_2)+\xi a_{3n}(\theta_1,\theta_2)$, $(\theta_1,\theta_2)\in\Omega_0$. But $\|a_{3n}\|\equiv 1$. Hence

$$\|r_n^*\|_{p,\partial\omega_0^*}^p\leqslant\|R+u_0\|_{p,\partial\omega_0^*}^p+\beta\|a_{3n}\|_{p,\partial\omega_0^*}^p\leqslant\bar M, \tag{2.34}$$

where $\bar M$ is a constant. We conclude immediately from (2.34) and (2.16) that

$$\|r_n^*\|_{1,p,\omega^*}^p\leqslant\bar K(\|F_n\|_{p,\omega^*}^p+\bar\beta), \tag{2.35}$$

$\bar K$, $\bar\beta$ are constants here. (2.32), (2.33) and (2.35) imply that

$$E(u_n^0)\geqslant\tilde a+\tilde K(\|r_n^*\|_{1,p,\omega^*}^p+\|\operatorname{adj}F_n\|_{q,\omega^*}^q+\|\det F_n\|_{r,\omega^*}^r), \tag{2.36}$$

$\tilde a$, $\tilde K$ are constants. It may be assumed that $E(u_n^0)\leqslant E(v^0)<+\infty$. From this and (2.36) it follows that there is a subsequence of $\{u_n^0\}$, which may be assumed to be the original sequence itself without lossing generality, weakly convergent in $W^{1,p}(\Omega\times[-l,l])$ (see the proof of Theorem 2.1), i. e.

$$r_n^*\rightharpoonup r^* \text{ in } W^{1,p}(\Omega\times[-l,l]). \tag{2.37}$$

We have also (see the proof of Theorem 2.1)

$$\begin{cases}\operatorname{adj}F_n\xrightarrow{L^q(\Omega\times[-l,l])}\operatorname{adj}\left(\dfrac{\partial r^*}{\partial(\theta_1,\theta_2,\xi)}\left(\dfrac{\partial R^*}{\partial(\theta_1,\theta_2,\xi)}\right)^{-1}\right),\\ \det F_n\xrightarrow{L^r(\Omega\times[-l,l])}\det\left(\dfrac{\partial r^*}{\partial(\theta_1,\theta_2,\xi)}\left(\dfrac{\partial R^*}{\partial(\theta_1,\theta_2,\xi)}\right)^{-1}\right).\end{cases} \tag{2.38}$$

The boundedness of $r_n^*(\theta_1,\theta_2,\xi)=r_n(\theta_1,\theta_2)+\xi a_{3n}(\theta_1,\theta_2)$ in $W^{1,p}(\Omega\times[-l,l])$ implies the boundedness of $r_n(\theta_1,\theta_2)=\dfrac{1}{2l}\displaystyle\int_{-l}^{l}r_n^*(\theta_1,\theta_2,\xi)\,d\xi$ and of $a_{3n}(\theta_1,\theta_2)$ in $W^{1,p}(\Omega)$, so we may assume that

$$\left.\begin{array}{l} r_n\xrightarrow{W^{1,p}(\Omega)}F^0(\theta_1,\theta_2).\\ a_3\xrightarrow{W^{1,p}(\Omega)}F^1(\theta_1,\theta_2).\end{array}\right\} \tag{2.39}$$

From $\|a_{3n}(\theta_1),\theta_2\|=1$, $\forall(\theta_1,\theta_2)\in\Omega$, it follows that $\|F^1(\theta_1,\theta_2)\|=1$, a. e. in Ω. Let

$$r(\theta_1,\theta_2)\equiv F^0(\theta_1,\theta_2),\quad a_3(\theta_1,\theta_2)\equiv F^1(\theta_1,\theta_2). \tag{2.40}$$

We then have

$$r^*(\theta_1,\theta_2,\xi)=r(\theta_1,\theta_2)+\xi a_3(\theta_1,\theta_2). \tag{2.41}$$

Again $a_{3n}\xrightarrow{W^{1,p}(\Omega)}a_3$, $p\geqslant 2$ implies $a_{3n}\to a_3$ in $L^\infty(\Omega)$. Noting also

$$\frac{\partial r_n}{\partial \theta_i} \rightharpoonup \frac{\partial r}{\partial \theta_i} \quad \text{in } \mathbb{L}^p(\Omega),\ i=1,\ 2,$$

We have

$$\frac{\partial r_n}{\partial \theta_i} \cdot a_{3n} \rightharpoonup \frac{\partial r}{\partial \theta_i} \cdot a_3 \text{ in } \mathbb{L}^1(\Omega),\ i=1,\ 2.$$

But we already have

$$\frac{\partial r_n}{\partial \theta_i} \cdot a_{3n} \equiv 0.$$

We conclude that

$$\frac{\partial r}{\partial \theta_i} \cdot a_3 = 0, \text{ a.e. in } \Omega.$$

Hence a_3 is the unit normal vector of the deformed middle furface.

$$\det \frac{\partial r^*}{\partial(\theta_1,\ \theta_2,\ \xi)} \geqslant 0, \text{ a. e. in } \Omega \times [-l,\ l]$$

implies that a_3 preserves the orientation, i. e. a_3 is the exterior unit normal of deformed middle surface. By Hypothesis (I), we conclude

$$E(u^0) \leqslant \inf_{v \in D_1} E(v). \quad (2$$

(2.42) and Hypothesis (II) imply that

$$\det \frac{\partial r^*}{\partial(\theta_1,\ \theta_2,\ \xi)} > 0, \text{ a.e. in } \Omega \times [-l,\ l]. \quad (2.$$

Hence $u^0 \in D_1$. Thus it follows from (2.42) that

$$E(u^0) = \inf_{v \in D_1} E(v). \quad (2$$

This proves the theorem.

Corollary 1. *The corresponding conclusion of Theorem* 2.2 *holds if* D *substituted by* $D_1' = \{u^0 \in D_1 \mid a_3|_{\partial\Omega_0} = a_3^0\}$.

Here D_1' corresponds to the clamped edge condition.

Corollary 2. *Define*

$$E_1(u^0) = E(u^0) - \int_{\partial\Omega_0 \times [-l,l]} \xi \bar{e}^1 \cdot (r^* - R^*) \, d\xi d\sigma. \quad (2.$$

The concludion of Theorem 2.2 *still holds, if* E *is substituted by* E_1.

Here the term

$$\int_{\partial\Omega_0 \times [-l,l]} \xi \bar{e}^1 \cdot (r^* - R^*) \, d\xi \, d\sigma$$

corresponds to the given moment in the hinge type condition.

Note 3 Corollary 2 is the general form of Theorem 2.2 in which $\bar{e}^1 = 0$.

Corollary 3. *Define*

$$E_1(u^0,\ u^1) = E(u^0,\ u^1) - \int_{\partial\Omega_0} \tilde{e}^1 \cdot u^1 d\sigma. \quad (2.46)$$

$$D'=\Big\{(u^0, u^1)\in(\mathbb{W}^{1,p}(\Omega))^2\,\Big|\,\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}\in\mathbb{L}^p(\Omega\times[-l, l]),$$
$$\operatorname{adj}\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}\in\mathbb{L}^q(\Omega\times[-l, l],\ \det\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}\in L^r(\Omega\times[-l, l]),$$
$$\det\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}>0,\ \text{a. e. in } \Omega\times[-l, l],\ u^0|_{\partial\Omega_0}=u^0|_{\partial\Omega_0}=u_0\Big\}. \qquad (2.47)$$

We assert that the corresponding conclusion of Theorem 2.1 remains true if $E(u^0, u^1)$ is substituted by $E_1(u^0, u^1)$ and D is substituted by D', provided that $D'\neq\phi$ and there is a $(v^0, v^1)\in D'$ such that $E_1(v^0, v^1)<+\infty$.

Here the clamped edge condition in Theorem 2.1 is substituted by hinge type condition with $\tilde{e}_1$ acting as an exterior moment.

We have now proved the existence of minimum point on certain sets for the energy functional of a class of shells. We can also induce from the energy functional, according to the principle of virtual work, the partial differential equations, i. e. the Euler equation of the energy functional, that the displacement should satisfy. These equations are also called the balance equations of the shell. The problem is: Do those minimum points got above satisfy these equations in some sense? Before solving this problem, let us assume some further assumptions on $W(F)$. We will also require some regularity conditions for the minimum points.

Hypothesis (IV): There is an $s\geqslant\max\{p, q, r\}$ and continuous functions $C_1(d)$, $C_2(d)$: $R^1_+\to R^1_+$, such that $W(F)$ satisfies

$$\left\|\frac{\partial W(F)}{\partial F}\right\|\leqslant C_1(d)+C_2(d)\|F\|^s,\ \forall F\in\mathbb{M}^3_+,\ \det F\geqslant d>0. \qquad (2.49)$$

Theorem 2.3. *Suppose W satisfies Hypotheses* (I)—(IV). *Let $(u^0, u^1)\in D$ be a minimum point of $E(u^0, u^1)$ in D. Then for all*

$$(v^0, v^1)\in(\mathbb{W}^{1,\infty}(\Omega))^2,\ (v^0, v^1)|_{\partial\Omega_0}=0,$$
$$\int_{\omega^*}\frac{\partial W(F)}{\partial F}:\nabla(v^0+\xi v^1)\,dx\,dy\,dz-\int_\Omega\int_{-l}^{l}(\bar{f}^0+\xi\bar{f}^1)\cdot(v^0+\xi v^1)\,d\xi\,d\theta^1\,d\theta^2$$
$$-\int_{\partial\Omega_1}\int_{-l}^{l}(\bar{g}^0+\xi\bar{g}^1)\cdot(v^0+\xi v^1)\,d\xi\,d\sigma=0 \qquad (2.50)$$

provided that $(u^0, u^1)\in(\mathbb{W}^{1,\infty}(\Omega))^2$ and there is a $d>0$, such that

$$\det\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}\geqslant d>0,\ \text{a. e. } in\ \Omega\times[-l, l].$$

In (2.50), $A:B=\operatorname{tr}(AB^T)$, $\forall A, B\in\mathbb{M}^3$. Hence (u^0, u^1) satisfies in weak sense the Euler equation of the energy functional $E(u^0, u^1)$ defined by (2.2), i. e. (u^0, u^1) is a generalized solution of the balance equations of the shell.

Proof Because $(u^0, u^1)\in(\mathbb{W}^{1,\infty}(\Omega))^2$,

$$\det\frac{\partial r^*}{\partial(\theta_1, \theta_2, \xi)}\geqslant d>0,\ \text{a. e. in } \Omega\times[-l, l].$$

For any fixed $(v^0, v^1)\in(\mathbb{W}^{1,\infty}(\Omega))^2$, $(v^0, v^1)|_{\partial\Omega_0}=0$, we have

$$(u^0+tv^0,\ u^1+tv^1)\in D,\ \det\frac{\partial r^*_{tv}}{\partial(\theta_1,\ \theta_2,\ \xi)}\geqslant\frac{d}{2}>0,\ \text{a. e. in } \Omega\times[-l,\ l],$$

if t is sufficiently small. Here

$$r^*_{tv}=r^*+tv^0+\xi tv^1.$$

The fact that $(u^0,\ u^1)$ is a minimum point implies that

$$E(u^0+tv^0,\ u^1+tv^1)-E(u^0,\ u^1)\geqslant 0. \tag{2.51}$$

Divide both sides of (2.51) by t. Let $t\to 0^{\pm}$. By Hypothesis (IV), we get (2.50).

Corollary 4. *The corresponding conclusion remains true if* $E(u^0,\ u^1$ *substituted by* $E_1(u^0,\ u^1)$.

For shells which satisfy Hypothesis B, i. e. the Love-Kirchhoff hypothesis have

Theorem 2.4. *Suppose that* W *satisfies Hypotheses* (I)—(IV). *Let* $u^0\in D_1$ *minimum point of* $E(u)$ *in* D_1. *Then, if*

$$u^0\in W^{1,\infty}(\Omega),\ \text{and}\ \det\frac{\partial r^*}{\partial(\theta_1,\ \theta_2,\ \xi)}\geqslant d>0,\ \text{a. e. in } \Omega\times[-l,\ l]$$

for a constant $d>0$, (2.50) *will still hold for all* $v^0\in W^{1,\infty}(\Omega)$, $v^0\|_{\partial\Omega_0}=0$, *with*

$$v^1=\frac{r_1\times v_2^0}{\|r_1\times r_2\|}+\frac{v_1^0\times r_2}{\|r_1\times r_2\|}+\left[\frac{r_1\times r_2}{\|r_1\times r_2\|}\cdot\left(\frac{r_1\times v_2^0}{\|r_1\times r_2\|}+\frac{v_1^0\times r_2}{\|r_1\times r_2\|}\right)\right]\frac{r_1\times r_2}{\|r_1\times r_2\|},$$

where
$$r_i=\frac{\partial r}{\partial\theta_i},\ v_i^0=\frac{\partial v^0}{\partial\theta_i},\ i=1,\ 2.$$

This implies that u^0 is a generalized solution of the balance equations of shell, i. e. the Euler equation of the energy functional.

Corollary 5. *The corresponding conclusion will still hold if* $E(u)$ *and* D_1 *substituted by* $E_1(u)$ *and* D_1' *respectively.*

Note 4 The above arguments are all in the case where ω is a manifold \ boundary and the measure of $\partial\Omega_0$ is positive. We can also get similar results cases when meas $(\partial\Omega_0)=0$, or ω is a manifold without boundary. Of course, s additional conditions, such as the sum of exterior forces and moments vanish, sh(be taken into account. And boundary conditions should be substituted by some o constrains, for example $u^0|_{\partial\Omega_0}=u_0$ whould be changed into the form

$$\int_\omega u^0\,ds=0,\ \int_\omega u^0\times r\,ds=0.$$

The arguments are also similar. Take Corollary 3 as an example, where only need to substitute (2.48) by

$$\begin{aligned}\|r_n^*\|^p_{1,p,\omega^*}&\leqslant\bar{K}\left(\left|\int_{\omega^*}r_n^*\,dx\,dy\,dz\right|^p+|r_n^*|^p_{1,p,\omega^*}\right)\\&\leqslant\bar{K}\left(\left|\int_{\omega^*}r_0^*\,dx\,dy\,dz\right|^p+\int_{\omega^*}c_n\,dx\,dy\,dz\right|^p+|r_n^*|^p_{1,p,\omega^*}\right)\\&\leqslant\bar{K}|r_n^*|^p_{1,p,\omega^*}+\bar{K}_1.\end{aligned}$$

› rest part of proof is the same.

§ 3. Existence and Convergence of Finite Element Solutions

For simplicity, we only discuss the case when ω, the middle surface of a shell, iffeomorphism to a bounded connected open set of $\mathbb{R}^2$. The other cases can be ›ussed in the same way only if the corresponding finite element spaces are structed.

Let R: $\overline{\Omega}\subset R^2\to\overline{\omega}$ be diffeomorphism. Assume that $\partial\Omega$, consists of finite number losed segments of continuous curves. Let $\mathscr{T}_h$, $h>0$ be regular triangulation of see Ciarlet [6]). Denote

$$\Omega_h=\bigcup_{K\in\mathscr{T}_h}K,\ \Omega_h\subset\Omega.\ \text{And}\ K\in\mathscr{T}_h$$

closed triangular elements of $\mathscr{T}_h$. $\partial\Omega_h$ denotes the boundary of Ω_h, N_h the set of the nodes of $\mathscr{T}_h$. We demand that the triangulations $\mathscr{T}_h$ in the present paper sfy the following condition: for $j_1, j_2\in N_h\cap\partial\Omega_h$, where j_1, j_2 are adjacent nodes $\mathscr{T}_h$ on $\partial\Omega$, the points of $\partial\Omega$ between j_1 and j_2 are all in $\partial\Omega_0$, or otherwise all in . Denote $\partial\Omega_{0h}$ as the part of $\partial\Omega_h$ which corresponds to $\partial\Omega_0$, and denote $\partial\Omega_{1h}$ as the t of $\partial\Omega_h$ which corresponds to $\partial\Omega_1$.

ine

$$V_h=\{u\in\mathbb{C}(\Omega_h)\,|\,u|_K\in\mathbb{P}_1(K),\ \forall K\in\mathscr{T}_h\},$$

$$V_h^0=\{u\in V_h\,|\,u(j)=0,\ \forall j\in N_h\cap\partial\Omega_{0h}\},$$

$\overline{V}_h^0=\{u\in\mathbb{C}(\overline{\Omega})\,|\,u|_{\Omega_h}\in V_h^0,\ u(x)=0$ in the region bounded by $\partial\Omega_0$ and $\partial\Omega_{0h}$; u is a linear continuation of $u|_{\Omega_h}$ in the region bounded by $\partial\Omega_1$ and $\partial\Omega_{1h}\}$.

nember

$$r^*(u_h^0, u_h^1)=R(\theta_1, \theta_2)+\xi A_3(\theta_1, \theta_2)+u_h^0(\theta_1, \theta_2)+\xi u_h^1(\theta_1, \theta_2).$$

: any $\alpha_h\in\partial\Omega_{0h}$, there is an $\alpha\in\partial\Omega_0$, such that

$$\operatorname{dist}\{\alpha_h, \partial\Omega_0\}=\operatorname{dist}\{\alpha_h, \alpha\}.$$

α is unique if h is sufficiently small. This relation is written as

$$\alpha=\varphi(\alpha_h).$$

ine

$$\begin{aligned}\overline{V}_h(u_0, u_1)=\Big\{&(u_h^0, u_h^1)\in(\mathbb{C}(\overline{\Omega}))^2\,|\,(u_h^0, u_h^1)|_{\Omega_h}\in V_h\times V_h,\ (u_h^0(j), u_h^1(j))\\&=(u_0(\varphi(j)), u_1(\varphi(j))),\ \forall j\in N_h\cap\partial\Omega_{0h},\ (u_h^0, u_h^1)\\&\text{takes linear interpolation between } \alpha_h \text{ and } \varphi(\alpha_h),\\&\text{for all } \alpha_h\in\partial\Omega_{0h},\ (u_h^0, u_h^1) \text{ is a linear continuation}\\&\text{of } (u_h^0, u_h^1)|_{\Omega_h} \text{ in the region bounded by } \partial\Omega_1 \text{ and } \partial\Omega_{1h},\\&\det\frac{\partial r^*(u_h^0, u_h^1)}{\partial(\theta_1, \theta_2, \xi)}>0,\ \text{a. e. in } \Omega\times[-l, l]\Big\}.\end{aligned}$$

$$\overline{V}'_h(u_0)=\Big\{(u_h^0,\ u_h^1)\in(\mathbb{C}(\overline{\Omega}))^2\,|\,(u_h^0,\ u_h^1)|_{\Omega_h}\in V_h\times V_h,\ u_h^0(j)=u_0(\varphi(j)),$$

$\forall j\in N_h\cap\partial\Omega_{0h}$, u_h^0 takes linear interpolation between α_h and $\varphi(\alpha_h)$ for all $\alpha_h\in\partial\Omega_{0h}$, u_h^0 is a linear continuation of $u_h^0|_{\Omega_h}$ in the region bounded by $\partial\Omega_{1h}$ and $\partial\Omega_1$, u_h^1 is a linear continuation of $u_h^1|_{\Omega_h}$ in the region bounded by $\partial\Omega_h$ and $\partial\Omega$, and

$$\det\frac{\partial r^*(u_h^0,\ u_h^1)}{\partial(\theta_1,\ \theta_2,\ \xi)}>0,\ \text{a. e. in } \Omega\times[-l,\ l]\Big\}.$$

It is easy to prove that $\overline{V}_h(u_0,\ u_1)\in D$, $\overline{V}'_h(u_0)\in D'$.

Theorem 3.1. *Suppose that $W(F)$ satisfies the Hypotheses* (I)—(III). *E, are defined by* (2.2), (2.46) *respectively. Then E, E_1 have minimum points in $\overline{V}_h$ $u_1)$, $\overline{V}'_h(u_0)$ respectively, provided that $\overline{V}_h(u_0,\ u_1)\neq\phi$, $\overline{V}'_h(u_0)\neq\phi$ and there exist $v_h^1)\in\overline{V}_h(u_0,\ u_1)$, $(w_h^0,\ w_h^1)\in\overline{V}'_h(u_0)$ such that*

$$E(v_h^0,\ v_h^1)<+\infty,\quad E_1(w_h^0,\ w_h^1)<+\infty$$

respectively.

Proof The proof is similar to that of Theorem 2.1, by noting that in $\overline{V}_h(u_0,$ and $\overline{V}'_h(u_0)$ the weak convergence is equivalent to the strong convergence, equivalent to the convergence in continuous function space, because $\overline{V}_h(u_0,\ u_1)$ $\overline{V}'_h(u_0)$ are sets in finite dimensional spaces. The proof is actually much simpler

Theorem 3.2. *Suppose that $W(F)$ satisfies Hypotheses* (I)—(IV).

(i) *Suppose $(u_h^0,\ u_h^1)\in\overline{V}_h(u_0,\ u_1)$ is a minimum point of E in $\overline{V}_h(u_0,\ u_1)$, an*

$$\det\frac{\partial r^*(u_h^0,\ u_h^1)}{\partial(\theta_1,\ \theta_2,\xi)}\geqslant d>0,\ \text{a.e. in } \overline{\Omega}\times[-l,\ l].$$

Then (2.50) *holds for all* $(v^0,\ v^1)\in\overline{V}_h^0\times\overline{V}_h^0=\overline{V}_h(0,\ 0)$.

(ii) *Suppose that $(u_h^0,\ u_h^1)\in\overline{V}'_h(u_0)$ is a minimum point of E_1 in $\overline{V}'_h(u_0)$, and*

$$\frac{\partial r^*(u_h^0,\ u_h^1)}{\partial(\theta_1,\ \theta_2,\ \xi)}\geqslant d>0,\ \text{a. e. in } \Omega\times[-l,\ l].$$

Then (2.50) *holds for all* $(v^0,\ v^1)\in\overline{V}'_h(0)$.

Proof Note we have here $r^*(u_h^0,\ u_h^1)\in\mathbb{W}^{1,\infty}(\Omega\times[-l,\ l])$ previously. The pr is similar to that of Theorem 2.3.

Next, we are going to investigate the convergence of the finite element soluti under some regularity conditions. For simplicity, we set $u_0=u_1=0$.

Theorem 3.3. *Suppose $W(F)$ satisfies Hypothese* (I)—(IV). *The clamped e condition is imposed on $\partial\Omega_0$ with $u_0=u_1=0$. Suppose that $(u^0,\ u^1)\in D$ is an isola local minimum point, i. e. there exist in D a $(\mathbb{W}^{1,p}(\Omega))^2$ weak open set U, such t $(u^0,\ u^1)\in D$, and*

$$E(v^0,\ v^1)>E(u^0,\ u^1),\ \forall(v^0,\ v^1)\in\overline{U}^W\cap D,\ (v^0,\ v^1)\neq(u^0,\ u^1),\tag{3.1}$$

where $\overline{U}^W$ is the $(\mathbb{W}^{1,p}(\Omega))^2$ weak closure of U.

Suppose also that

h $(u^0, u^1) \in \bigcup_{d>0} \overline{V(d)}^{s_1}$,

$$V(d)=\left\{(v_0, v^1)\in(\mathbb{C}_c^\infty(\Omega\cup\partial\Omega_1))^2 \det\frac{\partial r^*(v^0, v^1)}{\partial(\theta_1, \theta_2, \xi)}\geqslant d>0 \ in\ \Omega\times[-l, l]\right\}$$

nd $\overline{V(d)}^{s_1}$ *the closure of* $V(d)$ *in* $(\mathbb{W}^{1,s_1}(\Omega))^2$, $s_1=\max\{2s, 3r\}$.

'hen for h sufficiently small, there exists a local minimum point (u_h^0, u_h^1) *of E in*

$$\overline{V}_h(0, 0)=\overline{V}_h^0\times\overline{V}_h^0,$$

uch that $(u_h^0, u_h^1)\in U$, *and*

$$(u_h^0, u_h^1)\xrightarrow{(\mathbb{W}^{1,p}(\Omega))^2}(u^0, u^1), \tag{3.2}$$

$$E(u_h^0, u_h^1)\longrightarrow E(u^0, u^1) \tag{3.3}$$

is $h\to 0$.

Proof For any $(\mathbb{W}^{1,p}(\Omega))^2$ weak open neighborhood $U_0\subset\overline{U}_0^W\subset U$ of (u^0, u^1) in D. It is easy to prove that there exists a constant $\varepsilon(U_0)>0$, such that

$$E(v^0, v^1)\geqslant E(u^0, u^1)+\varepsilon(U_0),\ \forall(v^0, v^1)\in\overline{U}^W\backslash U_0. \tag{3.4}$$

From the fact that

$$(u^0, u^1)\in\bigcup_{d>0}\overline{V(d)}^{s_1},$$

We have a $d>0$ and a sequence $\{(u_n^0, u_n^1)\}\subset V(d)$, such that

$$(u_n^0, u_n^1)\xrightarrow{(\mathbb{W}^{1,s_1}(\Omega))^2}(u^0, u^1).$$

Hence, from the fact that $s_1\geqslant\max\{p, 2q, 3r\}$, $(u^0, u^1)\in D$, and

$$\det\frac{\partial r^*(u^0, u^1)}{\partial(\theta_1, \theta_2, \xi)}\geqslant d>0, \text{ a. e. in } \Omega\times[-l, l].$$

By Hypothesis (IV), for any $\varepsilon>0$, there is an $N(\varepsilon)>0$, such that

$$E(u^0, u^1)\leqslant E(u_n^0, u_n^1)<E(u^0, u^1)+\varepsilon. \tag{3.5}$$

Fix $n_1\geqslant N(\varepsilon)$. Let $(u_{n_1,h}^0, u_{n_1,h}^1)$ be the interpolation of $(u_{n_1}^0, u_{n_1}^1)$ in $\overline{V}_h(0, 0)$. Then (see Ciarlet [6])

$$\|u_{n_1,h}^0-u_{n_1}^0\|_{1,\infty}\leqslant ch\|u_{n_1}^0\|_{2,\infty}, \tag{3.6}$$

$$\|u_{n_1,h}^1-u_{n_1}^1\|_{1,\infty}\leqslant ch|u_{n_1}^1\|_{2,\infty}. \tag{3.7}$$

So there is an $\bar{h}_0>0$, such that for any $h<\bar{h}_0$, $(u_{n_1,h}^0, u_{n_1,h}^1)\in D$, and

$$\det\frac{\partial r_{n_1,h}^*}{\partial(\theta_1, \theta_2, \xi)}\geqslant\frac{d}{2}>0, \text{ a. e. in } \Omega\times[-l, l],$$

where we denote

$$r_{n_1,h}^*=R^*+u_{n_1,h}^0+\xi u_{n_1,h}^1.$$

By Hypothesis (IV), there is an $h_0(\varepsilon)<\bar{h}_0$, such that

$$E(u^0, u^1)\leqslant E(u_{n_1,h}^0, u_{n_1,h}^1)<E(u_{n_1}^0, u_{n_1}^1)+\varepsilon \tag{3.8}$$

for any $h<h_0(\varepsilon)$. Take $\varepsilon=\varepsilon(U_0)/2$, then

$$E(u_{n_1,h}^0, u_{n_1,h}^1)<E(u^0, u^1)+\varepsilon(U_0). \tag{3.9}$$

We may assume that $(u_{n_1,h}^0, u_{n_1,h}^1)\in U$, because $(u_{n_1,h}^0, u_{n_1,h}^1)$ convergent to (u^0, u^1) in $(\mathbb{W}^{1,s_1}(\Omega))^2$ as $n_1\to\infty$ and $h\to 0$. By (3.4), (3.9) it is easy to see that for any

$h<h_0(s)$, there is a $(u_h^0, u_h^1)\in \bar{V}_h(0, 0)$, such that $(u_h^0, u_h^1)\in U_0$, and

$$E(u^0, u^1)\leqslant E(u_h^0, u_h^1)\leqslant E(v_h^0, v_h^1),\ \forall (v_h^0, v_h^1)\in \bar{V}_h(0, 0)\cap \bar{U}^W, \tag{3.10}$$

i. e. (u_h^0, u_h^1) is a local minimum point of E in $\bar{V}_h(0, 0)$.

From (3.5), (3.8), (3.10) and the arbitrarity of s, it follows that

$$\lim_{h\to 0} E(u_h^0, u_h^1)=E(u^0, u^1).$$

This is just (3.3).

(u^0, u^1) is an isolated local minimum point of E in U, and $(u_h^0, u_h^1)\in l$ $h<h_0(s(U_0)/2)$. Hence we see from (3.3) that (u_h^0, u_h^1) is a local minim sequence. Using an argument simlar to that of Theorem 2.1, we get

$$(u_h^0, u_h^1)\xrightarrow{(\mathbb{W}^{1,p}(\Omega))^3}(u^0, u^1)$$

and also

$$\operatorname{adj}\frac{\partial r^*(u_h^0, u_h^1)}{\partial(\theta_1, \theta_2, \xi)}\xrightarrow{\mathbb{L}^q(\Omega\times[-l, l])}\operatorname{adj}\frac{\partial r^*(u^0, u^1)}{\partial(\theta_1, \theta_2, \xi)},$$

$$\det\frac{\partial r^*(u_h^0, u_h^1)}{\partial(\theta_1, \theta_2, \xi)}\xrightarrow{L^r(\Omega\times[-l, l])}\det\frac{\partial r^*(u^0, u^1)}{\partial(\theta_1, \theta_2, \xi)}.$$

Note. For shells, the deformation of which satisfies Hypothesis B, i. e Love-Kirchhoff hypothesis, the finite element solution should be discussed different finite element space. Generally speaking, for conforming method, element of the finite element space should be in H^2. Thus it will be much complicated and more time-consuming to deal with the problem by comp though the independent variables reduced to 3 from the original 6. We wil discuss this problem here because of the limitations of space.

§4. Computing M thod and Its Convergence for Calculating Finite Element Solutions Approximately

We begin with the calculation of approximate minimum point of E in $\bar{V}$ $u_1)$ from the energy functional of shells as the deformation satisfies Hypothesi The energy functional, as defined in (2.8), is

$$E(u^0, u^1)\equiv\int_\Omega \bar{W}(\theta_1, \theta_2, u^0, u^1)d\theta^1 d\theta^2-\int_\Omega(\bar{f}^0\cdot u^0+\bar{f}^1\cdot u^1)d\theta^1 d\theta^2$$
$$-\int_{\partial\Omega_1}(\tilde{g}^0\cdot u^0+\tilde{g}^1\cdot u^1)d\sigma.$$

For simplicity, we only investigate the case when the functional is $E(u^0,$ and the finite element space is $\bar{V}_h(u_0, u_1)$ (see § 3). The other cases may be discussed similarly.

In this seciton, we suppose that $W(F)$ satisfies (I)—(IV). Define

$$P_1=\{(u_h^0, u_h^1)\in \overline{V}_h(u_0, u_1)\,|\,(u_h^0, u_h^1) \text{ is a critical point of } E \text{ in } \overline{V}_h(u_0, u_1)\},$$

$$P_2=\left\{(u_h^0, u_h^1)\in \overline{V}_h(u_0, u_1)\,\Big|\inf_{(\theta_1,\theta_2,\xi)\in\Omega\times[-l,l]}\left(\det\frac{\partial r^*(u_h^0, u_h^1)}{\partial(\theta_1, \theta_2, \xi)}(\theta_1, \theta_2, \xi)=0\right\},\right.$$

$$P=P_1\cup P_2.$$

or fixed $(u_h^0, u_h^1)\in \overline{V}_h(u_0, u_1)\setminus P$, there is a $d>0$, such that

$$\det\frac{\partial r^*(u_h^0, u_h^1)}{\partial(\theta_1, \theta_2, \xi)}\geqslant d>0,\ \forall(\theta_1, \theta_2, \xi)\in\Omega\times[-l, l].$$

y the hypothesses for $W(F)$, we have for any $(v_h^0, v_h^1)\in\overline{V}_h(0, 0)$

$$\lim_{t\to0}\frac{1}{t}(E(u_h^0+tv_h^0, u_h^1+tv_h^1)-E(u_h^0, u_h^1))$$

$$=\int_\Omega\frac{\partial\overline{W}(u_h^0, u_h^1)}{\partial(\nabla_\Omega u_h^1)}:\nabla_\Omega v_h^0\,d\theta^1 d\theta^2+\int_\Omega\frac{\partial\overline{W}(u_h^0, u_h^1)}{\partial(\nabla_\Omega u_h^1)}:\nabla_\Omega v_h^1\,d\theta^1 d\theta^2$$

$$+\int_\Omega\frac{\partial\overline{W}(u_h^0, u_h^1)}{\partial u_h^1}\cdot v_h^1\,d\theta^1 d\theta^2-\int_\Omega(\tilde{f}^0\cdot v_h^0+\tilde{f}^1\cdot v_h^1)\,d\theta^1 d\theta^2$$

$$-\int_{\partial\Omega_1}(\tilde{g}^0\cdot v_h^0+\tilde{g}^1\cdot v_h^1)\,d\sigma,$$

here $A:B=\mathrm{tr}(AB^T)$, $\forall A, B\in M^3$; ∇_Ω is the gradient operator of functional space Ω. Denote the right hand side of above equality by $\mathscr{L}(u_h^0, u_h^1; v_h^0, v_h^1)$. We have

$$\frac{d}{dt}E(u_h^0+tv_h^0, u_h^1+tv_h^1)\,|_{t=t_0}=\mathscr{L}(u_h^0+t_0v_h^0, u_h^1+t_0v_h^0; v_h^0, v_h^1). \tag{4.1}$$

For fixed $(u_h^0, u_h^1)\in\overline{V}_h(u_0, u_1)\setminus P$, $\mathscr{L}(u_h^0, u_h^1; v_h^0, v_h^1)$ is a continuous linear nction fo $(V_h^0, V_h^1)\in\overline{V}_h(0, 0)$. Now we are going to find a minimum point of $\mathscr{L}$in

$$\{(v_h^0, v_h^1)\in\overline{V}_h(0, 0)\,|\,|v_h^0|_{1,2}^2+|v_h^1|_{1,2}^2=1\}.$$

his is equivalent to finding a saddle point for

$$L(v_h^0, v_h^1, \lambda)=\mathscr{L}(u_h^0, u_h^1; v_h^0, v_h^1)+\lambda(|v_h^0|_{1,2}^2+|v_h^1|_{1,2}^2-1) \tag{4.2}$$

e. finding $(v_h^0, v_h^1)\in\overline{V}_h(0, 0)$, $\lambda\in R^1$, such that

$$L(v_h^0, v_h^1, \alpha)\leqslant L(v_h^0, v_h^1, \lambda)\leqslant L(w_h^0, w_h^1, \lambda),\ \forall(w_h^0, w_h^1)\in\overline{V}_h(0, 0),\ \forall\alpha\in R^1. \tag{4.3}$$

Suppose $\{(v_h^0, v_h^1), \lambda\}$ is a saddle point of L, then

$$2\lambda\int_\Omega\nabla_\Omega v_h^0:\nabla_\Omega w_h^0\,d\theta^1 d\theta^2+\mathscr{L}(u_h^0, u_h^1; w_h^0, 0)=0,\ \forall w_h^0\in\overline{V}_h^0, \tag{4.4}$$

$$2\lambda\int_\Omega\nabla_\Omega v_h^1:\nabla_\Omega w_h^1\,d\theta^1 d\theta^2+\mathscr{L}(u_h^0, u_h^1; 0, w_h^1)=0,\ \forall w_h^1\in\overline{V}_h^0, \tag{4.5}$$

$$|v_h^0|_{1,2}^2+|v_h^1|_{1,2}^2=1. \tag{4.6}$$

It is easy to show that (4.4)—(4.6) have a unique solution such that $\lambda\geqslant0$, nd $\lambda>0$ if $\mathscr{L}(u_h^0, u_h^1; \cdot, \cdot)\neq0$. Let $w_h^0=v_h^0$, $w_h^1=v_h^1$ in (4.4), (4.5) respectively. hen adding (4.4) to (4.5), by (4.6) we have

$$\lambda=-\frac{1}{2}\mathscr{L}(u_h^0, u_h^1; v_h^0, v_h^1).$$

Suppose $((v_h^0, v_h^1), \lambda)$ is a solution of (4.4)—(4.6), and define

$$\bar{v}_h^0=2\lambda v_h^0,\quad \bar{v}_h^1=2\lambda v_h^1.$$

Then, $(\bar{v}_h^0, \bar{v}_h^1)\in\overline{V}_h(0, 0)$ satisfies

$$\left.\begin{aligned}-\int_\Omega \nabla_\Omega \bar{v}_h^0 : \nabla_\Omega \bar{w}_h^0\, d\theta^1 d\theta^2 = \mathscr{L}(u_h^0,\ u_h^1;\ \bar{w}_h^0,\ 0),\ \forall \bar{w}_h^0 \in \bar{V}_h^0,\\ -\int_\Omega \nabla_\Omega \bar{v}_h^1 : \nabla_\Omega \bar{w}_h^1\, d\theta^1 d\theta^2 = \mathscr{L}(u_h^0,\ u_h^1;\ 0,\ \bar{w}_h^1),\ \forall \bar{w}_h^1 \in \bar{V}_h^0.\end{aligned}\right\} \tag{4.7}$$

On the other hand, suppose $(\bar{v}_h^0,\ \bar{v}_h^1) \in \bar{V}_h^0 \times \bar{V}_h^0$ is a solution of (4.7) with

$$\mathscr{L}(u_h^0,\ u_h^1;\ \cdot,\ \cdot) \neq 0.$$

Define

$$\lambda^2 = -\frac{1}{4}\mathscr{L}(u_h^0,\ u_h^1;\ v_h^0,\ v_h^1),\ \lambda > 0,$$

$$v_h^0 = \frac{1}{2\lambda}\bar{v}_h^0,\quad v_h^1 = \frac{1}{2\lambda}\bar{v}_h^1.$$

Then $((v_h^0,\ v_h^1),\ \lambda)$ is a solution of (4.4)—(4.6) and $\lambda > 0$.

The above argument proves the following theorem.

Theorem 4.1. *The steepest descent direction and the descent speed of functional E at $(u_h^0,\ u_h^1) \in \bar{V}_h(u_0,\ u_1)\backslash P$ can be characterized by the solution of* (4. *The steepest descent direction is* $\dfrac{(\bar{v}_h^0,\ \bar{v}_h^1)}{(|\bar{v}_h^0|_{1,2}^2 = |\bar{v}_h^1|_{1,2}^2)^{1/2}}$, *the descent speed is* $(|\bar{v}_h^0|_{1,}^2$ $|\bar{v}_h^1|_{2,2}^2)$, i. e.

$$\frac{d}{dt}E(u_h^0 + t\bar{v}_h^0,\ u_h^1 + t\bar{v}_h^1)\,|_{t=0} = -(|\bar{v}_h^0|_{1,2}^2 + |\bar{v}_{h2}^2|_{1,}).$$

Considering the following system

$$\left.\begin{aligned}&-\int_\Omega \nabla_\Omega \frac{du_h^0(t)}{dt} : \nabla_\Omega w_h^0\, d\theta^1 d\theta^2 = \mathscr{L}(u_h^0(t),\ u_h^1(t);\ w_h^0,\ 0),\\ &-\int_\Omega \nabla_\Omega \frac{du_h^1(t)}{dt} : \nabla_\Omega w_h^1\, d\theta^1 d\theta^2 = \mathscr{L}(u_h^0(t),\ u_h^1(t);\ 0,\ w_h^1),\\ &(u_h^0(0),\ u_h^1(0)) = (u_h^0,\ u_h^1).\end{aligned}\right\} \tag{4}$$

(4.8) is an autonomous system with finite number (the dimension of $(\mathscr{T}_h)$ variables. From the above discussion and the theory of ordinary differen equations, it follows that for any

$$(u_h^0,\ u_h^1) \in \bar{V}_h(u_0,\ u_1)\backslash P,$$

there are $t_1 < 0 < t_2$ such that there is a unique function $(u_1^0(t),\ u_h^1(t))$ defined $(t_1,\ t_2)$ satisfying (4.8). In addition, $(u_h^0(t),\ u_h^1(t) \equiv (u_h^0,\ u_h^1)$, $-\infty < t < +\infty$, if $u_h^1) \in P_1$, i. e. P_1 consists of singular points of the system. For

$$(u_h^0,\ u_h^1) \in \bar{V}_h(u_0,\ u_1)\backslash P,$$

(4.8) implies that $E(u_h^0(t),\ u_h^1(t))$ is a strictly decreasing function of t. Let $(t_1,$ be the maximum interval on which $(u_h^0(t),\ u_h^1(t))$ can be defined. Then it foll from the theory of ordinary differential equations that

$$(u_h^0(t),\ u_h^1(t)) \to \text{a point in } P_2,\ \text{as } t \to t_1 \text{ or } t \to t_2;$$

$$\text{or } (u_h^0(t),\ u_h^1(t)) \to \text{a point in } P_1,\ \text{as } t \to t_1 \text{ or } t \to t_2,$$

and for the later case, $t_1 = -\infty$, $t_2 = +\infty$. It is easy to prove that there is an absorb eighborhood corresponding to each isolated local minimum point, i. e. the solution (4.8) with its initial point in this neighborhood convergents to the corresponding cal minimum point.

We are now going to define an iterative method for calculating the critical oint of E.

For fixed $x_0=(u_h^0, u_h^1)\in \overline{V}_h(u_0, u_1)\backslash P$, let

$$y(x_0)=(v_h^0(x_0),\ v_h^1(x_0))$$

the solution of (4.7) with initial point x_0. Denote $v_h^0=v_h^0(x_0)$, $v_h^1=v_h^1(x_0)$, then

$$\frac{d}{dt}E(u_h^0+tv_h^0),\ u_h^1+tv_h^1)\,|_{t=0}$$

$$=-\int_\Omega \nabla_\Omega v_h^0 : \nabla_\Omega v_h^0\, d\theta^1 d\theta^2-\int_\Omega \nabla_\Omega v_h^1 : \nabla_\Omega v_h^1\, d\theta^1 d\theta^2. \tag{4.9}$$

(4.9) implies that $E(u_h^0+tv_h^0, u_h^1+v_h^1)$ decreases at $t=0$ as t increases. On the other and, as t increases, either (i) there is a $\bar{t}>0$, such that

$$\det\frac{\partial r^*(u_h^0+tv_h^0,\ u_h^1+tv_h^1)}{\partial(\theta_1,\theta_2,\ \xi)}>0,\ \text{a. e. in } \Omega\times[-l, l] \tag{4.10}$$

longer holds for $t>\bar{t}$, i. e. $(u_x^0+tv_h^0, u_h^1+tv_h^1)\bar{\in} D$, if $t>\bar{t}$, while (4.10) holds for $\leq\bar{t}$,

or (ii) (4.10) holds for all $t>0$. But in this case

$$\lim_{t\to\infty}(\|u_h^0+tv_h^0\|_{1,h}+\|u_h^1+tv_h^1\|_{1,h})=+\infty.$$

By Hypothesis (III) and (2.18), it is easy th show there is a $t_0>0$, such that

$$(u_h^0+tv_h^0,\ u_h^1+tv_h^1)\in \overline{V}_h(0,\ 0)\backslash P,\ \forall t\in[0,\ t_1],$$

nd

$$E(u_h^0+t_0v_h^0,\ u_h^1+t_0v_h^1)=E(u_h^0,\ u_h^1).$$

ence there is a $t^*>0$, such that

$$\mathscr{L}(u_h^0+t^*v_h^0,\ v_h^1+t^*v_h^1;\ v_h^0,\ v_h^1)=0.$$

e define

$$t_1(x_0)=\min\{t>0\,|\,(u_h^0+tv_h^0,\ u_h^1+tv_h^1;\ v_h^0,\ v_h^1)=0\}.$$

For the case (i), $t_1(x_0)$ is defined as above provided that

$$\{t>0\,|\,\mathscr{L}(u_h^0+tv_h^0,\ u_h^1+tv_h^1;\ v_h^0, v_h^1)=0\}\neq\phi.$$

herwise, define

$$t_1(x_0)=\bar{t}.$$

Note also that

$$d(t,\ x_0)=\operatorname*{ess\ inf}_{(\theta_1,\theta_2,\xi)}\det\frac{\partial r^*(u_h^0+tv_h^0,\ u_h^1+tv_h^1)}{\partial(\theta_1,\ \theta_2,\ \xi)}$$

is a continuous function of t and $d(0,\ x_0)>0$. Define $t_2(x_0)$ the number satisfying

$$d(t,\ x_0)>0,\ \forall t\in[0,\ t_2(x_0)],$$

$$d(t_2(x_0),\ x_0)=0.$$

For fixed constants $0<\eta<1$, $M<0$, define

$$t(x_0)=\min\{M,\ \eta t_1(x_0),\ \eta t_2(x_0)\}, \tag{4.11}$$

$$x_1=x_0+t(x_0)y(x_0). \tag{4.12}$$

It is easy to verify that $x_1\bar{\in}P$ provided that $x_0\bar{\in}P$. Thus (4.12) defines a map

$$B:\ \overline{V}_h(u_0,\ u_1)\backslash P\longrightarrow\overline{V}_h(u_0,\ u_1)\backslash P.$$

(4.12) can also be writen as $x_1=B(x_0)$.

Define on $\overline{V}_h(u_0,\ u_1)\backslash P$ a sequence x_n by

$$x_n=B(x_{n-1})=B^n(x_0),\ n=1,\ 2,\ \cdots. \qquad (4.$$

From the definition of the map B, it follows easily that

$$E(x_n)<E(x_{n-1}),\ n=1,\ 2,\ \cdots,$$

i. e. for all $x\in\overline{V}_h(u_0,\ u_1)\backslash P$,

$$E(B^n(x))<E(B^{n-1}(x)),\ n\geqslant 1. \qquad (4.$$

E is continuous on $\overline{V}_h(u_0,\ u_1)\backslash P_2$. We also have the following lemma.

Lemma 4.1. *The function $t(x)$ defined by (4.11) is lower semi--continuous.*

Lemma 4.2. *Let E be the functional defined by (2.8) and B be the map defi by (4.12), then $E\circ B(x)$ is upper semi-continucus.*

Theorem 4.2. *For any fixed*

$$x_0=(u_h^0,\ u_0^1)\in\overline{V}_h(u_0,\ u_1)\backslash P,$$

the limit points of the sequence $\{x_n\}$ defined by (4.13) are all in P. i. e.

$$\overline{\{x_n\}}\backslash\{x_n\}\subset P.$$

proof Suppose that

$$x=\{w_h^0,\ +w_h^1\}\in\overline{\{x_n\}}\backslash\{x_n\},$$

but $x\bar{\in}P$. Then there is a subsequence $\{x_{n_i}\}$, $x_{n_i}\longrightarrow x$, and $x\in\overline{V}_h(u_0,\ u_1)\backslash P$. H $B(x)$ is well defined and $B(x)\neq x$, $E(B(x))<E(x)$ by the definition of B.

Let

$$E(x)-E(B(x))=\delta.$$

By the continuity of E, there is a neighborhood U_x of x, and a neighbor $U_{B(x)}$ of $B(x)$,

$$U_x=\{y\mid |E(y)-E(x)|<\delta/2\},$$

$$U_{O(x)}=\{y\mid |E(y)-E(B(x))|<\delta/2\}.$$

It is obvious that $U_x\subset U_{B(x)}=\phi$. On the other hand, from the upper s continuity of $E\circ B$, it follows that there is a neighborhood V_x of x, such that

$$E\circ B(y)<E\circ B(x)+\delta/2,\ \forall y\in V_x.$$

Let $W_x=V_x\cap U_x$, then

$$E\circ B(y)<\inf_{z\in W_x}E(z),\ \forall y\in W_x. \qquad (4$$

From $x_{n_i}\longrightarrow x$, there is an $N>0$, $x_{n_i}\in W_x$ *if* $i\geqslant N$. We may assume that N Thus from $x_{n_1}\in W_x$ and (4.15) we have

$$E\circ B(x_{n_1})<E(x_{n_2}). \tag{4.16}$$

But $x_{n_2}=B^{n_2-n_1}(x_{n_1})$, $n_2-n_1\geqslant 1$. Hence
$$E(x_{n_2})\leqslant E\circ B(x_{n_1}). \tag{4.17}$$
(4.17) contradicts (4.16). This proves the theorem.

Theorem 4.3. *For fixed*
$$x_0\in \overline{V}_h(u_0, u_1)\setminus P,$$
$\{x_n\}$ is defined by (4.13). *Suppose*
$$x\in \overline{\{x_n\}}\setminus\{x_n\}$$
is an isolated critical point of E. If $x\overline{\in} P_2$, and x is a local minimum point of E, then
$$\overline{\{x_n\}}\setminus\{x_n\}=\{x\},\ \text{i. e. } x_n\to x \text{ as } n\to\infty.$$

Proof Let U be an open neighbohood of x such that x is the unique critical point on $\overline{U}$, the closure of U. E'(y) denotes the differential operator of E at y. Denote
$$2\delta=\operatorname{dist}\{x, U^c\},\quad U^c=(\overline{V}_h(u_0, u_1)\setminus P)\setminus U.$$
Let U_0 be another neighborhood of x, such that $x\in U_0\subset\overline{U}_0\subset U\subset\overline{U}$, and
$$\|E'(y)\|<\delta/M,\ \forall y\in U_0. \tag{4.18}$$
We may assume $U_0\subset\{y\,|\operatorname{dist}\{x, y\}<\delta\}$. On the other hand, there is an $\mathscr{E}(U_0)<0$, such that
$$\mathrm{E}(y)\geqslant(\mathrm{E}(x)+\mathscr{E}(U_0)),\ \forall y\in\overline{U}\setminus U_0. \tag{4.19}$$
Suppose the subsequence $\{x_{n_i}\}$ converges to x. Hence
$$E(x_{n_i})\longrightarrow E(x),$$
$$E'(x_{n_i})\longrightarrow 0.$$
So there is an $N>0$, such that
$$E(x_{n_i})<E(x)+\mathscr{E}(U_0) \tag{4.20}$$
and $x_{n_i}\in U_0$, as $i\geqslant N$. We may assume $N=1$. Thus, by the definition of B and (4.18), we conclude
$$\|x_{n_1+1}-x_{n_1}\|=\|B(x_{n_1})-x_{n_1}\|\leqslant M\|E'(x_{n_1})\|<\delta.$$
Hence $x_{n_1+1}\in U$. But (4.20) implies
$$E(x_{n_1+1})<E(x_{n_1})<E(x)+\mathscr{E}(U_0).$$

So it follows from (4.19) that $x_{n_1+1}\in U_0$. Similar arguments will show that $x_n\in U_0$ for all $n\geqslant n_1$. But x is the only point of p in U_0.By Theorem 4.2, we conclude that
$$x_n\longrightarrow x \text{ as } n\longrightarrow\infty,$$
i. e.
$$\{x\}=\overline{\{x_n\}}\setminus\{x_n\}.$$

Corollary. *Let x be an isolated critical point of E in $\overline{V}_h(u_0, u_1)$, and be a local minimum point. Suppose $x\overline{\in} P_2$. Then, there exists an open set U_x in $\overline{V}_h(u_0, u_1)$, such that, for any $y\in U_x$,*
$$B^n(y)\longrightarrow x,\ as\ n\to\infty.$$

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Proof The conclusion follows from the proof of Theorem 4.3.

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