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SOME NONLINEAR EVOLUTION EUQATIONS WITH LAX PAIR OF 3×3 MATRICES**

TIAN CHOU (田 畴)*

Abstract

For some NEEs with Lax pairs of 3×3 matrices, the author presents the Bäcklund transformations (in Darboux form) and the corresponding Modified equations. The method deriving the Bäcklund transformations and the Modified equations can be considered as an extension and development of the mehtod in [1].

§ 1.

We discuss the Model equation for shallow water waves[2]:

$$u_t-u_{xxt}-3uu_t-3u_xw_t+u_x=0 \tag{1.1}$$

$$\left(w=D^{-1}u,\ D=\frac{d}{dx},\ D\circ D^{-1}=D^{-1}\circ D=I\right).$$

Theorem 1.1. *Suppose $\Omega=M\,dx+N\,dt$, where*

$$M=\begin{pmatrix}0&1&0\\0&0&1\\\lambda&-(3u-1)&0\end{pmatrix},$$

$$N=\begin{pmatrix}\frac{1}{3\lambda}(\lambda\mu-3u_{xt}-(3w_t-1)(3u-1))&\frac{u_t}{\lambda}&-\frac{1}{3\lambda}(3w_t-1)\\-\frac{1}{3}(3w_t-1)&\frac{\mu}{3}&0\\-u_t&-\frac{1}{3}(3w_t-1)&\frac{\mu}{3}\end{pmatrix}$$

(λ, μ *are arbitrary constants*). *Then*

$$d\Omega-\Omega\wedge\Omega=0\ (M_t-N_x+[M,N]=0)$$

if and only if (1.1) *is established.*

Proof $M_t-N_x+[M,N]$

$$=\begin{pmatrix}0&0&0\\0&0&0\\0&-3u_t&0\end{pmatrix}-\begin{pmatrix}\frac{1}{\lambda}(-u_{xxt}-(3w_tu)_x+u_t+u_x)&\frac{1}{\lambda}u_{tt}&-\frac{1}{\lambda}u_t\\-u_t&0&0\\-u_{xt}&-u_t&0\end{pmatrix}$$

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* Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, China.

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$$+\begin{pmatrix} -\frac{1}{3}(3w_t-1) & \frac{\mu}{3} & 0 \\ -u_t & -\frac{1}{3}(3w_t-1) & \frac{\mu}{3} \\ \frac{1}{3}(\lambda\mu-3u_{xt}) & u_t-\frac{\mu}{3}(3u-1) & -\frac{1}{3}(3w_t-1) \end{pmatrix}$$

$$-\begin{pmatrix} -\frac{1}{3}(3w_t-1) & \frac{1}{3\lambda}(\lambda\mu-3u_{xt}) & \frac{1}{\lambda}u_t \\ 0 & -\frac{1}{3}(3w_t-1) & \frac{\mu}{3} \\ \frac{\lambda\mu}{3} & -u_t-\frac{\mu}{3}(3u-1) & -\frac{1}{3}(3w_t-1) \end{pmatrix}$$

$$=\begin{pmatrix} \frac{1}{\lambda}(u_{xxt}+3uu_t+3u_xw_t-u_t-u_x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, (1.1) can be considered as the completely integrable condition the linear equations on

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$V_x=MV,\ V_t=NV, \qquad (1$$

i. e. $V_{xt}=V_{tx}$ if and only if (1.1) is established.

Suppose $\varphi=v_2/v_1$, $\psi=v_3/v_1$, then (1.2) are reduced to the following "Ricc form:

$$\varphi_x=\psi-\varphi^2, \qquad (1$$

$$\psi_x=\lambda-(3u-1)\varphi-\varphi\psi, \qquad (1$$

$$\varphi_t=-\frac{1}{3}(3w_t-1)+\Big(\frac{\mu}{3}-\frac{1}{3\lambda}(\lambda\mu-3u_{xt}-(3w_t-1)(3u-1)\Big)$$

$$-\frac{u_t}{\lambda}\varphi+\frac{1}{3\lambda}(3w_t-1)\varphi)\varphi, \qquad (1$$

$$\psi_t=u_t-\frac{1}{3}(3w_t-1)\varphi+\Big(\frac{\mu}{3}-\frac{1}{3\lambda}(\lambda\mu-3u_{xt}-(3w_t-1)(3u-1)\Big)$$

$$-\frac{u_t}{\lambda}\varphi+\frac{1}{3\lambda}(3w_t-1)\psi)\psi. \qquad (1$$

From (1.3), (1.5), (1.6) and (1.4), we have

$$\psi\varphi_t-\varphi\psi_t=u_t\varphi-\frac{1}{3}(3w_t-1)\varphi_x, \qquad ($$

$$u=\frac{1}{3}\Big(\frac{\lambda-\varphi_{xx}}{\varphi}-3\varphi_x-\varphi^2+1\Big), \qquad (\ldots)$$

$$u_t=\frac{1}{3}\Big(\Big(\frac{\lambda-\varphi_{xx}}{\varphi}\Big)_t-3\varphi_{xt}-2\varphi\varphi_t\Big), \qquad (1.9)$$

$$w_t=\frac{1}{3}\left(D^{-1}\left(\frac{\lambda-\varphi_{xx}}{\varphi}\right)_t-3\varphi_t-2D^{-1}(\varphi\varphi_t)\right). \tag{1.10}$$

Substituting (1.8)—(1.10) into (1.7), we obtain

$$\varphi_x+\varphi^2\varphi_t+\varphi\left(\frac{\lambda-\varphi_{xx}}{\varphi}\right)_t-\varphi_x D^{-1}\left(\frac{\lambda-\varphi_{xx}}{\varphi}\right)_t+2\varphi_x D^{-1}(\varphi\varphi_t)=0. \tag{1.11}$$

When we change (φ, λ) to $(-\varphi, -\lambda)$, (1.11) is invariant. Therefore, we change (φ, λ) to $(-\varphi, -\lambda)$ in (1.3)—(1.6), they are a Lax pair as well. Suppose the corresponding solution of (1.1) is $\bar{u}$, then (1.8) is changed to

$$\bar{u}=\frac{1}{3}\left(\frac{\lambda-\varphi_{xx}}{\varphi}+3\varphi_x-\varphi^2+1\right). \tag{1.12}$$

Subtracting (1.8) from (1.12), we obtain

$$\bar{u}=u+2\varphi_x. \tag{1.13}$$

Theorem 1.2. *If u satisfies* (1.1),

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

satisfies equations (1.2), $\varphi=v_2/v_1$, *then*

$$\bar{u}=u+2\varphi_x$$

is a solution of (1.1) *as well.*

(1.13) is a Bäcklund transformation (in Darboux form) of (1.1). When we want to obtain a new solution $\bar{u}$ from a known solution u, we just need to solve the equations (1.2), which are a system of completely integrable Pffafian equations.

Theorem 1.2 can be checked directly as well.

Letting $\lambda=0$ in (1.11) and (1.8), we obtain the following theorem

Theorem 1.3. *φ satisfies equation*

$$\varphi_x+\varphi^2\varphi_t-\varphi_{xxt}+\frac{\varphi_{xx}\varphi_t}{\varphi}+\varphi_x D^{-1}\left(\frac{\varphi_{xx}}{\varphi}+\varphi^2\right)_t=0, \tag{1.14}$$

and

$$u=-\frac{1}{3}\left(\frac{\varphi_{xx}}{\varphi}+3\varphi_x+\varphi^2-1\right) \tag{1.15}$$

transforms (1.1) *to* (1.14).

(1.14) is called the modified model equation for shallow water waves, (1.15) is similar to Miura transformation well known.

If we substitute (1.15) into (1.1), we can obtain

$$u_t-u_{xxt}-3uu_t-3u_x w_t+u_x$$
$$=-\frac{1}{3}\left(\frac{D^2}{\varphi}-\frac{\varphi_{xx}}{\varphi^2}+3D+2\varphi\right)\left(\varphi_x+\varphi^2\varphi_t-\varphi_{xxt}+\frac{\varphi_{xx}\varphi_t}{\varphi}+\varphi_x D^{-1}\left(\frac{\varphi_{xx}}{\varphi}+\varphi^2\right)_t\right).$$

§ 2.

For the Boussinesq equation

$$u_x+u_{xxx}+6uu_x+D^{-1}u_{tt}=0, \tag{2.1}$$

we have the following results.

Theorem 2.1. *Suppose $\Omega=M\,dx+N\,dt$, where*

$$M=\begin{pmatrix} 0 & 1 & 0 \\ \lambda-u & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x) & \frac{\sqrt{3}}{4}(2u+4\lambda+1) & 0 \end{pmatrix},$$

$$N=\begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x) & \frac{\sqrt{3}}{4}(2u+4\lambda+1) & 0 \\ \frac{3}{4}\left(u_{xx}-(2u+4\lambda+1)(\lambda-u)-\frac{1}{\sqrt{3}}u_t\right) & \frac{3}{4}(\sqrt{3}\,w_t-u_x) & \frac{\sqrt{3}}{4}(8\lambda-2u+1) \end{pmatrix}$$

(λ is an arbitrary constant, $w_x=u$), then $d\Omega-\Omega\wedge\Omega=0$ if and only if (2.1) i established.

Proof

$M_t-N_x+[M, N]$

$$=\begin{pmatrix} 0 & 0 & 0 \\ -u_t & 0 & 0 \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_{tt}+u_{xt}) & \frac{\sqrt{3}}{2}u_t & 0 \end{pmatrix}$$

$$-\begin{pmatrix} 0 & 0 & 0 \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,u_t+u_{xx}) & \frac{\sqrt{3}}{2}u_x & 0 \\ \frac{4}{3}\left(u_{xxx}-2u_x(\lambda-u)+(2u+4\lambda+1)u_x-\frac{1}{\sqrt{3}}u_{xt}\right) & \frac{3}{4}(\sqrt{3}\,u_t-u_x) & -\frac{\sqrt{3}}{3}u_x \end{pmatrix}$$

$$+\begin{pmatrix} 0 & 0 & 0 \\ -\frac{\sqrt{3}}{4}\left(u_{xx}-\frac{1}{\sqrt{3}}u_t\right) & \frac{\sqrt{3}}{2}u_x & 0 \\ \frac{3}{2}(\lambda-u)u_x & -\frac{3}{4}\left(u_{xx}-\frac{1}{\sqrt{3}}u_t\right) & -\frac{\sqrt{3}}{2}u_x \end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3}{4}(w_{tt}+u_{xxx}+6uu_x+u_x) & 0 & 0 \end{pmatrix}.$$

Therefore, Boussinesq equation can be considered as the completely integrabl condition of the linear equations on

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}:$$

$$V_x=MV,\ V_t=NV. \tag{2.2}$$

Suppose $\varphi=v_2/v_1$, $\psi=v_3/v_1$, then (2.2) are reduced to the following "Riccati" form:

$$\varphi_x=\lambda-u-\varphi^2-\frac{1}{\sqrt{3}}\psi, \tag{2.3}$$

$$\varphi_t=-\frac{\sqrt{3}}{4}(\sqrt{3}w_t+u_x)+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\varphi-\varphi\psi, \tag{2.4}$$

$$\psi_x=-\frac{\sqrt{3}}{4}(\sqrt{3}w_t+u_x)+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\varphi-\varphi\psi, \tag{2.5}$$

$$\psi_t=\frac{3}{4}\left(u_{xx}-(2u+4\lambda+1)(\lambda-u)-\frac{1}{\sqrt{3}}u_t\right)+\frac{3}{4}(\sqrt{3}w_t-u_x)\varphi$$
$$+\frac{\sqrt{3}}{4}(8\lambda-2u+1)\psi-\psi^2. \tag{2.6}$$

From (2.4), (2.5) and (2.6), we have

$$\psi=D^2\varphi_t, \tag{2.7}$$

$$u=\lambda-\varphi_x-\varphi^2-\frac{1}{\sqrt{3}}D^{-1}\varphi_t, \tag{2.8}$$

$$u_x=-\varphi_{xx}-2\varphi\varphi_x-\frac{1}{\sqrt{3}}\varphi_t, \tag{2.9}$$

$$u_{xx}=-\varphi_{xxx}-2\varphi\varphi_{xx}-2\varphi_x^2-\frac{1}{\sqrt{3}}\varphi_{xt}, \tag{2.10}$$

$$u_t=-\varphi_{xt}-2\varphi\varphi_t-\frac{1}{\sqrt{3}}D^{-1}\varphi_{tt}. \tag{2.11}$$

From (2.5) and (2.6)

$$\psi_t+\sqrt{3}\varphi\varphi_t=\frac{3}{4}\left(u_{xx}-(2u+4\lambda+1)\left(\lambda-u-\varphi^2-\frac{1}{\sqrt{3}}\psi\right)-\frac{1}{\sqrt{3}}u_t\right)$$
$$-\frac{3}{2}u_x\varphi-\sqrt{3}\psi\left(\lambda-u-\varphi^2-\frac{1}{\sqrt{3}}\psi\right). \tag{2.12}$$

Substituting (2.7)—(2.11) into (2.12), we obtain

$$\varphi_x+\varphi_{xxx}-6\varphi^2\varphi_x+6\lambda\varphi_x+D^{-1}\varphi_{tt}-2\sqrt{3}\varphi_xD^{-1}\varphi_t=0. \tag{2.13}$$

If we change (φ, t) to $(-\varphi, -t)$, (2.13) is invariant. Equation (2.1) is invariant as well when t is changed to $-t$. Therefore, (2.3)—(2.6) are a Lax pair of (2.1) when $(\varphi, t$ are changed to $(-\varphi, -t)$. Suppose the corresponding solution of (2.1) is $\bar{u}$, then (2.8) is changed to

$$\bar{u}=\lambda+\varphi_x-\varphi^2-\frac{1}{\sqrt{3}}D^{-1}\varphi_t. \tag{2.14}$$

Compairing (2.14) with (2.8), we have

$$\bar{u}=u+2\varphi_x. \tag{2.15}$$

Theorem 2.2. *If u is a solution of* (2.1),

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_2 \end{pmatrix}$$

satisfies equations (2.2), $\varphi=v_2/v_1$, *then*

$$\bar{u}=u+2\varphi_x$$

is a solution of (2.2) *as well.*

(2.15) is a Bäcklund transformation (in Darboux form) of Boussinesq equation (2.1). We can obtain some solutions of (2.1) different from [3] by using this Bäcklund transformation and some trivial solutions of (2.1).

Theorem 2.2 canbe checked directly as well.

Letting $\lambda=0$ in (2.8) and (2.13), we obtain the following theorem.

Theorem 2.3. *φ satisfies equation*

$$\varphi_x+\varphi_{xxx}-6\varphi^2\varphi_x+D^{-1}\varphi_{tt}-2\sqrt{3}\,\varphi_x D^{-1}\varphi_t=0 \qquad (2.1$$

and

$$u=-\left(\varphi_x+\varphi^2+\frac{1}{\sqrt{3}}D^{-1}\varphi_t\right) \qquad (2.1$$

transforms (2.1) *to* (2.16).

(2.16) is called the Modified Boussinesq equation, (1.17) can be considered the generalized Miura transformation.

If we substitute (2.17) into (2.1), we can obtain

$$u_x+u_{xxx}+6uu_x+D^{-1}u_{tt}$$
$$=-\left(D+2\varphi+\frac{1}{\sqrt{3}}D^{-1}D_t\right)(\varphi_x+\varphi_{xxx}-6\varphi^2\varphi_x+D^{-1}\varphi_{tt}-2\sqrt{3}\,\varphi_x D^{-1}\varphi_t).$$
$$\left(D_t=\frac{d}{dt}\right).$$

Theorem 2.4. *If*

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

is a solution of (2.2) *corresponding to* (u, λ),

$$V_0=\begin{pmatrix} (v_1)_0 \\ (v_2)_0 \\ (v_3)_0 \end{pmatrix}$$

is a solution of (2.2) *corresponding to* (u, λ_0), $\varphi_0=(v_2)_0/(v_1)_0$, *then*

$$\bar{v}_1=-\varphi_0 v_1+v_2 \qquad (2$$

is a solution of (2.2) *corresponding to* $\bar{u}=u+2(\varphi_0)_x$ *and* λ.

Proof From (2.2),

$$(v_1)_x=v_2,$$
$$(v_2)_x=(\lambda-u)v_1-\frac{1}{\sqrt{3}}v_3,$$
$$(v_3)_x=-\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x)\,v_1+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\,v_2,$$

we have

$$v_1=-\varphi_0 v_1+v_2,$$

$$\bar{v}_2=(\bar{v}_1)_x=(\lambda-u-(\varphi_0)_x)v_1-\varphi_0 v_2-\frac{1}{\sqrt{3}}v_3$$

$$=\Big(\lambda-\lambda_0+\varphi_0^2+\frac{1}{\sqrt{3}}D^{-1}(\varphi_0)_t\Big)v_1-\varphi_0 v_2-\frac{1}{\sqrt{3}}v_3,$$

$$\bar{v}_3=\sqrt{3}\,(\lambda-\bar{u})\bar{v}_1-\sqrt{3}\,(\bar{v}_2)_x$$

$$=-\Big((\varphi_0)_t+\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_{xx})\Big)v_1+\frac{\sqrt{3}}{4}(2u+4\lambda+1)v_2-(\varphi_0)v_3.$$

We need to prove that

$$(\bar{v}_3)_x=-\frac{\sqrt{3}}{4}(\sqrt{3}\,\bar{w}_t+\bar{u}_x)\bar{v}_1+\frac{\sqrt{3}}{4}(2\bar{u}+4\lambda+1)\bar{v}_2 \tag{2.19}$$

and

$$(V(\bar{u},\lambda))_t=N(\bar{u},\lambda)V(\bar{u},\lambda).$$

In fact, it suffices to prove (2.19) and $(\bar{v}_1)_t=\bar{v}_3$. It is easy to prove that $(\bar{v}_1)_t=\bar{v}_3$. In the following, we prove (2.19).

Since

$$(\varphi_0)_t=-\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x)+\frac{\sqrt{3}}{4}(2u+4\lambda_0+1)\varphi_0-\varphi_0 D^{-1}(\varphi_0)_t,$$

$$\bar{v}_3=-\frac{\sqrt{3}}{4}(2u+4\lambda_0+1)\varphi_0 v_1+\varphi_0 D^{-1}(\varphi_0)_t v_0+\frac{\sqrt{3}}{4}(2u+4\lambda+1)v_2-\varphi_0 v_3$$

$$=\frac{\sqrt{3}}{4}(2u+4\lambda+1)\bar{v}_1+\sqrt{3}\,(\lambda-\lambda_0)\varphi_0 v_1+\varphi_0 D^{-1}(\varphi_0)_t v_1-\varphi_0 v_3,$$

we have

$$\begin{aligned}(\bar{v}_3)_x=&\frac{\sqrt{3}}{2}u_x(-\varphi_0 v_1+v_2)+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\bar{v}_2+\sqrt{3}\,(\lambda-\lambda_0)\varphi_0\bar{v}_2\\&+\sqrt{3}\,(\lambda-\lambda_0)(\varphi_0)_x v_1+((\varphi_0)_x D^{-1}(\varphi_0)_t)v_1+\varphi_0(\varphi_0)_t v_1\\&+\varphi_0 D^{-1}(\varphi_0)_t v_2-(\varphi_0)_x v_3-\varphi_0\Big(-\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x)v_1\\&+\frac{\sqrt{3}}{4}(2u+4\lambda+1)v_2\Big),\end{aligned} \tag{2.20}$$

$$\begin{aligned}=&-\frac{\sqrt{3}}{4}(\sqrt{3}\,\bar{w}_t+\bar{u}_x)\bar{v}_1+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\bar{v}_2\\=&-\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+2\sqrt{3}\,\varphi_t+u_x+2\varphi_{xx})(-\varphi_0 v_1+v_2)\\&+\frac{\sqrt{3}}{4}(2u+4\lambda+1)\bar{v}_2\\&+\sqrt{3}\,(\varphi_0)_x\Big((\lambda-u-(\varphi_0)_x)v_1-\varphi_0 v_2-\frac{1}{\sqrt{3}}v_3\Big).\end{aligned} \tag{2.21}$$

We can see that (2.20) and (2.21) have the same coefficients of the terms: $\bar{v}_2$, v_1,

v_2 and v_3 (Noticing

$$(\varphi_0)_t = -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t + u_x) + \frac{\sqrt{3}}{4}(2u + 4\lambda_0 + 1)\varphi_0 - \varphi_0 D^{-1}(\varphi_0)_t$$

and

$$u_x = -(\varphi_0)_{xx} - 2\varphi_0(\varphi_0)_x - \frac{1}{\sqrt{3}}(\varphi_0)_t).$$

We complete the proof.

In fact, Theorem 2.2 can be considered as a corollary to Theorem 2.4.

Theorem 2.5. *If*

$$\varphi_0(u, \lambda_0) = (v_2)_0/(v_1)_0,\ \varphi(u, \lambda) = v_2/v_1,$$

$$V_0 = \begin{pmatrix} (v_1)_0 \\ (v_2)_0 \\ (v_3)_0 \end{pmatrix}$$

and

$$V = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are the solutions of (2.2) *corresponding to* (u, λ_0) *and* (u, λ) *respectively,*

$$\bar{\varphi} = \bar{v}_2/\bar{v}_1,\ \bar{V} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix}$$

is a solution of (2.2) *corresponding to* $\bar{u} = u + 2(\varphi_0)_x$ *and* λ, *then*

$$\bar{\varphi} = \frac{\lambda - \lambda_0}{\varphi - \varphi_0} - \varphi_0 - \frac{1}{\sqrt{3}}\,\frac{D^{-1}(\varphi - \varphi_0)_t}{\varphi - \varphi_0}. \tag{2.22}$$

Proof Substituting

$$\bar{v}_1 = -\varphi_0 v_1 + v_2$$

and

$$\bar{v}_2 = \left(\lambda - \lambda_0 + \varphi_0^2 + \frac{1}{\sqrt{3}}D^{-1}(\varphi_0)_t\right)v_1 - \varphi_0 v_2 - \frac{1}{\sqrt{3}}v_3$$

into $\bar{\varphi} = \bar{v}_2/\bar{v}_1$, we obtain (2.22).

We can obtain the recursion formulas by using Theorem 2.5:

$$\bar{u}(u, \lambda_0) = u + 2(\varphi_0)_x,$$

$$\begin{aligned}\bar{\bar{u}}(u, \lambda) &= \bar{u} + 2(\bar{\varphi})_x \\ &= u + 2\left(\frac{\lambda - \lambda_0}{\varphi - \varphi_0}\right)_x - \frac{2}{\sqrt{3}}\left(\frac{D^{-1}\varphi - \phi_0)_t}{\varphi - \varphi_0}\right)_x\end{aligned}$$

and so on.

§ 3.

We discuss the equation

$$u_t+u_{xxx}+6uu_x+D^{-1}u_{tt}=0. \tag{3.1}$$

The following Theorems 3.1, 3.2 and 3.3 can be considered as corollaries of the results in [4].

Theorem 3.1. *Suppose* $\Omega=Mdx+Ndt$, *where*

$$M=\begin{pmatrix} 0 & 1 & 0 \\ \lambda-u & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x) & \frac{\sqrt{3}}{4}(2u+4\lambda) & \frac{\sqrt{3}}{4} \end{pmatrix},$$

$$N=\begin{pmatrix} 0 & 0 & 1 \\ -\frac{\sqrt{3}}{4}(\sqrt{3}\,w_t+u_x) & \frac{\sqrt{3}}{4}(2u+4\lambda) & \frac{\sqrt{3}}{4} \\ N_{31} & N_{32} & \frac{\sqrt{3}}{4}(8\lambda-2u)-\frac{3}{16}\sqrt{3} \end{pmatrix},$$

$$N_{31}=\frac{3}{4}\Big(-(2u+4\lambda)(\lambda-u)+u_{xx}-\frac{1}{\sqrt{3}}u_t\Big)+\frac{3\sqrt{3}}{16}(\sqrt{3}\,w_t+u_x),$$

$$N_{32}=\frac{3}{4}(\sqrt{3}\,w_t-u_x)-\frac{3}{16}\sqrt{3}\,(2u+4\lambda)$$

(λ *is an arbitrary constant*). *Then* $d\Omega-\Omega\wedge\Omega=0$ *if and only if* (3.1) *is established.*

Theorem 3.2. *If* u *is a solution of* (3.1),

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

satisfies

$$V_x=MV,\ V_t=NV, \tag{3.2}$$

then

$$\bar{u}=u+2\varphi_x\quad(\varphi=v_2/v_1)$$

is a solution of (3.1) *as well.*

Theorem 3.3. φ *satisfies equation*

$$\varphi_t+\varphi_{xxx}-6\varphi^2\varphi_2+D^{-1}\varphi_{tt}-2\sqrt{3}\,\varphi_x D^{-1}\varphi_t=0, \tag{3.3}$$

and

$$u=-\Big(\varphi_2+\varphi^2+\frac{1}{\sqrt{3}}D^{-1}\varphi_t\Big)$$

transforms (3.1) *to* (3.3).

In a way similar to the proofs of Theorems 2.4 and 2.5, we can prove the following theorems.

Theorem 3.4. *If*

$$V=\begin{pmatrix} (v_1)_0 \\ (v_2)_0 \\ (v_3)_0 \end{pmatrix}$$

is a solution of (3.2) *corresponding to* (u, λ_0),

$$V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

is a solution of (3.2) *coresponding to* (u, λ), $\varphi_0=v_2/v_1$, *then*

$$\bar{v}_1=-\varphi_0 v_1+v_2$$

is a solution of (3.2) *corresponding to* $\bar{u}=u+2(\varphi_0)_x$ *and* λ.

Theorem 3.5. *If*

$$\varphi_0(u, \lambda_0)=(v_2)_0/(v_1)_0,\ \varphi(u, \lambda)=v_2/v_1,$$

$$V=\begin{pmatrix} (v_1)_0 \\ (v_2)_0 \\ (v_3)_0 \end{pmatrix} \quad \text{and} \quad V=\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

are the solutions of (3.2) *corresponding to* (u, λ_0) *and* (u, λ) *respectively,*

$$\bar{\varphi}=\bar{v}_2/\bar{v}_1,\ V=\begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix}$$

is a solution of (3.2) *corresponding to* $\bar{u}=u+2(\varphi_0)_x$ *and* λ, *then*

$$\bar{\varphi}=\frac{\lambda-\lambda_0}{\varphi-\varphi_0}-\varphi_0-\frac{1}{\sqrt{3}}\frac{D^{-1}(\varphi-\varphi_0)_t}{\varphi-\varphi_0}.$$

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