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SOME NEW CLASSES OF ALMOST PERIODIC FUNCTIONS

C.J.F. UPTON*

Abstract

In this paper classes of almost periodic functions are formed which lie between those established by W. Stepanoff and by H. Burkill.

Introduction

In this paper classes of almost periodic functions are formed which lie between the classes of almost periodic functions of Stepan off and Burkill [1, 3], and their properties are established. The Denjoy integrals of these functions form sub-classes of the classes of the ϕ^p-almost periodic functions defined in [12], to which paper the present paper forms a sequel.

Definitions and Notations

We consider measurable complex valued functions which are defined (at least almost everywhere) on the real line. If f is such a function we define the functions F and f_h by

$$F(x)=(D)\int_0^x f(t)\,dt \tag{1}$$

and

$$f_h(x)=h^{-1}(D)\int_0^h f(x+t)\,dt=h^{-1}[F(x+h)-F(x)], \tag{2}$$

for $0<h\leqslant 1$, whenever the integrals exist in the (special) Denjoy sense. Unless otherwise indicated, all integrals used throughout the paper will be assumed to be Denjoy integrals in the special sense.

The class V^p $(1\leqslant p<\infty)$ is defined in [8]; the class V^∞ in [12]. U ap, S^p ap and V^p ap functions $(1\leqslant p<\infty)$ are defined in [1] and [9], V^∞ ap and ϕ^p ap functions $(1\leqslant p\leqslant\infty)$ in [12] and D ap functions in [3].

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* University of Melbourne, Parkville, Victoria, 3052, Australia.

The D^p–Norm

We define the D^p–norm as follows. For $1\leqslant p<\infty$,

$$\|f\|_{D^p}=\sup_x \sup_\pi \left\{\sum_i |(D)\int_{x_{i-1}}^{x_i} f(t)\,dt|^p\right\}^{1/p},$$

where the supremum is taken firstly over all partitions

$$\pi:\ x=x_0<x_1<\cdots<x_k=x+1$$

of the interval $(x, x+1)$ and then over all x. It follows from the definitions that

$$\|f\|_{D^p}=\sup_x \sup_\pi \{\sum_i |F(x_i)-F(x_{i-1})|^p\}^{1/p}=\sup_x V_p(F;\ x,\ x+1)=\|F\|_{\phi^p}, \qquad (\text{:}$$

where F is given by (1), $V_p(F;\ x,\ x+1)$ is the Wiener generalized p–th variati of F over the interval $(x, x+1)$ [14], and $\|\cdot\|_{\phi^p}$ is the ϕ^p–norm [12].

For $p=\infty$, $\|f\|_{D^\infty}$ is defined to be

$$\sup_{\substack{x\\ 0<h<0}}\left|(D)\int_x^{x+h} f(t)\,dt\right|.$$

This norm has the same form as the D–norm defined in [3], and differs from it on in that the latter uses a general Denjoy intergral. It follows again from t definitions that

$$\begin{aligned}\|f\|_{D^\infty}&=\sup_x \sup_{x\leqslant y<y+h\leqslant x+1}\left|(D)\int_y^{y+h} f(t)\,dt\right|\\ &=\sup_x \sup_{x\leqslant y<y+h\leqslant x+1}|F(y+h)-F(y)|\\ &=\sup_x \text{osc.}\,(F;\ x,\ x+1)=\|F\|_{\phi^\infty}. \qquad (\end{aligned}$$

It is shown in Theorem 1 that $\|\cdot\|_{D^1}=\|\cdot\|_S$. From the behaviour of $\|\cdot\|_{\phi^p}$ then follows, using (3) and (4), that, as p increases from 1 to ∞, $\|\cdot\|_{D^p}$ decreas from $\|\cdot\|_S$ to $\|\cdot\|_{D^\infty}$.

It should be noted that the customary omission of the index when

$$p=1\ (\text{e. g. } \|\cdot\|_{\phi^1}=\|\cdot\|_\phi)$$

cannot be followed for the D^p–norm as $\|\cdot\|_{D^1}$ and $\|\cdot\|_D$ denote different norms.

Theorem 1. *For any (measurable) function* f, $\|f\|_{D^1}=\|f\|_S$.

Proof If both $\|f\|_{D^1}$ and $\|f\|_S$ are infinite, there is nothing to prove.

Suppose, then, that $\|f\|_S$ is finite and let F be defined by (1). Since f is (locall Lebesgue integrable and its Lebesgue and Denjoy integrals are consequently equa F is a Lebesgue integral of f and, by (3),

$$\|f\|_{D^1}=\sup_x V(F;\ x,\ x+1)=\sup_x \int_x^{x+1}|f(t)|\,dt=\|f\|_S.$$

Suppose, on the other hand, that $\|f\|_{D^1}$ is finite. Then $\|F\|_\phi$ is finite by (3) ar F therefore has bounded variation on every finite interval. Since F is also continuous, being a Denjoy integral, F' is (locally) Lebesgue integrable (see e. g.

[7, page 590]). Now $f=F'$ almost everywhere. Hence f is also Lebesgue integrable as well as Denjoy integrable, and the Lebesgue and Denjoy integrals of f must be equal. The conclusion then follows as before.

If $\|f\|_G$ is finite, where $\|\cdot\|_G$ denotes any of the norms used in this paper, then f is said to be *G–bounded*. If

$$\|f(x+h)-f(x)\|_G\to 0$$

as $h\to 0$, then f is said to be *G–continuous*. If there exists a function f such that $\|f_n-f\|_G\to 0$ as $n\to\infty$, then the sequence (f_n) is *G–convergent*. Clearly f is D^p–bounded (respectively D^p–ccontinuous) if and only if F is ϕ^p–bounded (respectively ϕ^p–continuous). Also the sequence (f_n) is D^p–convergent if and only if the sequence (F_n) is ϕ^p–convergent, where

$$F_n(x)=\int_0^x f_n(t)\,dt.$$

The D^p–Space

Since $\|f-g\|_{D^p}=0$ if and only if f and g are equal almost everywhere, we can form a function space, the *D^p–space*, whose elements are sets of measurable functions which differ at most on sets of zero measure and which are D^p–bounded. The norm fo the space is the D^p–norm.

The D^p–space is easily seen to be closed. For if (f_n) is a sequence of D^p–bounded functions and f is such that $\|f_n-f\|_{D^p}\to 0$ as $n\to\infty$,. then f is measurable and the inequality

$$\|f\|_{D^p}\leqslant\|f-f_n\|_{D^p}+\|f_n\|_{D^p}$$

shows that f is D^p–bounded.

The D^1–space is also complete. For the analogously formed S–space is complete[2], and the two spaces are identical. When $p>1$, however, it is proved in Theorem 4, Corollary, that the D^p–space is incomplete.

D^p–Almost Periodic Functions

For $1\leqslant p\leqslant\infty$, a Denjoy integrable function f is said to be *D^p–almost periodic* (D^pap) if its Denjoy integral is V^p and if, for every $\epsilon>0$, the set $\{\tau\}$ such that

$$\|f(x+\tau)-f(x)\|_{D^p}<\epsilon$$

is relatively dense.

The classes of D^1ap functions and Sap functions are identical. For $\|\cdot\|_{D^1}=\|\cdot\|_S$ by Theorem 1, while the class V^1, since it is simply the class of functions which are locally absolutely continuous, contains all integrals of Sap functions.

As p increases, the class of D^pap functions expands from the class of Sap functions to the class of D^∞ap functions, and the latter is a subclass of the class of Dap functions. Since the definitions of D^∞ap and Dap functions differ only in the nature of the Denjoy integrals involved, it can be seen that the properties of Dap functions which are established in [3] are also properties of D^∞ ap functions.

Theorem 2. *A function is D^pap if and only if its Denjoy integral is ϕ^pap.*

Proof The proof is immediate from (3), (4) and the definitions.

Two observations may be made. Firstly as a consequence of Theorem 2, mai of the properties of D^pap functions can be established immediately from analogo properties of ϕ^pap functions. Secondly, the subclass of the class of ϕ^pap functio which consists of Denjoy integrals of D^pap functions is a proper subclass since ϕ^pap function need not be differentiable anywhere.

Lemma 1. *If f is D^pap, then f is D^p–continuous and D^p–bounded.*

Proof Theorem 2 and [12, Lemma 7] show, in turn, that F is ϕ^pap, ϕ continuous and ϕ^p–bounded, where F is defined by (1). The lemma then follo immediately from (3) and (4).

Lemma 2. *If f and g are D^pap, so are $f\pm g$ and cf where c is any compl constant.*

Proof The proof follows immediately from Theorem 2 and [12, Lemma 8].

Lemma 3. *If f is D^p–continuous and f_h is defined by* (2), *then*

$$\|f_h-f\|_{D^p}\to 0 \text{ as } h\to 0.$$

Proof For $0<h\leqslant 1$, let

$$F_h(x)=h^{-1}\int_0^h F(x+t)\,dt,$$

where F is defined by (1). Then, by simple rearrangements of terms

$$\|f_h-f\|_{D^p}=\left\|\int_0^x f_h(t)\,dt-\int_0^x f(t)\,dt\right\|_{\phi^p}$$
$$=\|F_h(x)-F_h(0)-F(x)\|_{\phi^p}=\|F_h-F\|_{\phi^p}.$$

Since, by hypothesis, (3) and (4), F is ϕ^p–continuous, the right–hand side tends 0 with h by [12, Lemma 4]. The proof is thus complete.

Since $\|\cdot\|_{D^1}=\|\cdot\|_S$, the particular case when $p=1$ was established in [5].

Theorem 3. *Let (f_n) be a sequence of D^p ap functions which D^p–converges to f $n\to\infty$. Then f is D^pap.*

Proof The theorem may be proved directly following a standard argument f almost periodic functions. Alternatively, we may note that if

$$F_n(x)=\int_0^x f_n(t)\,dt \quad\text{and}\quad F(x)=\int_0^x f(t)\,dt,$$

then (F_n) is a sequence of ϕ^pap functions which ϕ^p–converges to F. Since F is ϕ^pap by [12, Theorem 4], it follows from Theorem 2 that f is D^pap.

The D^pap-Space

The *D^pap Space* is the space whose elements are sets of D^pap functions which are equal almost everywhere. Its norm is the D^p-norm. It is a subspace of the D^p-space and Theorem 3 shows that it is closed. The D^1ap space is also complete since the space of Sap functions is complete [2], and the two spaces are identical.

Theorem 4. *For $p>1$, the D^pap-space is incomplete.*

Proof We begin by considering any ϕ^pap function f. For any positive integer n, let

$$g_n(x)=n\{f(x+n^{-1})-f(x)\},$$

$$G_n(x)=\int_0^x g_n(t)\,dt \quad\text{and}\quad f_n(x)=n\int_0^{1/n} f(x+t)\,dt.$$

Then g_n is V^pap; by [12, Lemma 9], and it is therefore D^pap. Furthermore

$$\|g_n-g_m\|_{D^p}=\|G_n-G_m\|_{\phi^p}\leqslant\|G_n-f\|_{\phi^p}+\|f-G_m\|_{\phi^p}.$$

Now it is easily shown that

$$G_n(x)=f_n(x)-f_n(0).$$

Therefore

$$\|G_n-f\|_{\phi^p}=\|f_n-f\|_{\phi^p}$$

and tends to zero as $n\to\infty$ by [12, Theorem 3]. Hence (g_n) is a Cauchy sequence of D^pap functions.

Suppose that the D^pap-space is complete. Then there exists a D^pap function g such that $\|g_n-g\|_{D^p}\to 0$ as $n\to\infty$. Hence $\|G_n-G\|_{\phi^p}\to 0$ as $n\to\infty$, where

$$G(x)=\int_0^x g(t)\,dt.$$

Since

$$\|f-G\|_{\phi^p}\leqslant\|f-G_n\|_{\phi^p}+\|G_n-G\|_{\phi^p}$$

and both terms on the right-hand side tend to zero as $n\to\infty$, it follows that

$$\|f-G\|_{\phi^p}=0$$

and, therefore, that f and G differ only by a constant. Furthermore, since G is differentiable almost everywhere, it follows that the same must also be true of f.

That this conclusion is false can be seen by putting

$$f(x)=\sum_{\nu=0}^{\infty} c^{-\nu/q}\sin c^{\nu}x,$$

where $1<q<p$ and $c>1$. For f is differentiable nowhere [6], but it is shown in the proof of [12, Theorem 12] that f is ϕ^pap, as required. It follows that the D^pap space is incomplete.

Corollary. *For $p>1$, the D^p-space is incomplete.*

Proof We observe that the sequence (g_n) defined in the proof of Theorem 4 is a

Cauchy sequence of D^p–bounded functions. If it is then assumed that a D^p–bounded function g exists such that $\|g_n - g\|_{D^p} \to 0$ as $n \to \infty$, the argument of Theorem 4 can be used to show that this assumption leads to a contradiction. The D^p–space is therefore incomplete.

Lemma 4. *If f is D^p ap, then f_h is V^pap, where f_h is defined by* (2).

Proof Let F be defined by (1). Then F is ϕ^pap by Theorem 2, and

$$F(x+h) - F(x)$$

is V^p ap by [12, Lemma 9]. That f_h is V^pap then follows at once from (2).

Theorem 5. *The D^pap space is identical to the closure with respect to the D^p–no[r]m of the space of finite trigonometric polynomials.*

Proof Let A be the space of finite trigonometric polynomials and let $C_{D^p}\{$[A}] denote its closure with respect to the D^p–norm.

Let $f \in C_{D^p}\{A\}$. Then there exists a sequence (f_n) of finite trigonomet[ric] poynomials such that $\|f_n - f\|_{D^p} \to 0$ as $n \to \infty$. But each f_n is Uap and, therefore, D^pa[p]. Hence f is D^pap by Theorem 3.

Conversely, let f be D^pap and let $\varepsilon > 0$ be chosen arbitrarily. Since f [is] D^p–continuous by Lemma 1, it follows from Lemma 3 that we can choose $h > 0$ su[ch] that

$$\|f_h - f\|_{D^p} \leqslant \varepsilon/2,$$

where f_h is defined by (2). Now f_h is Uap by Lemma 4. Hence there exist[s a] trigonometric polynomial s such that

$$\|f_h - s\|_U < \varepsilon/2.$$

Since $\|\cdot\|_{D^p} \leqslant \|\cdot\|_U$ and $\|f - s\|_{D^p} \leqslant \|f - f_h\|_{D^p} + \|f_h - s\|_{D^p}$, it follows that

$$\|f - s\|_{D^p} < \varepsilon.$$

Hence $f \in C_{D^p}\{A\}$ and the proof is complete.

Fourier Series

If f is D^pap it is necessarily D^∞ap and, therefore, Dap. Hence, as is shown [3], the mean value $M\{f(x)e^{-i\lambda x}\}$ exists for all real λ, the set of values of λ f[or] which this mean value is non-zero is countable and f generates a unique Fouri[er] series

$$\sum_\lambda a_\lambda e^{i\lambda x}.$$

The (R, λ, k) summability of this Fourier series has been discussed by Burkill[3]. A form of strong summability is also possible and this is established in the followi[ng] theorem.

Theorem 6. *Let f be D^pap. Then there exists a sequence of trigonometric polynomials which D^p–converges to f and which formally converges to the Fourier series* of f.

Proof Let $f\sim\sum_\lambda a_\lambda e^{i\lambda x}$, let F be defined by (1) and let $h>0$. Then, by Lemma 1, $h^{-1}|F(x+h)-F(x)|$ is V^pap, and it therefore generates a Fourier series $\sum_\lambda b_\lambda(h)e^{i\lambda x}$, say. Since

$$h^{-1}[f(x+h)-f(x)]\sim\sum_\lambda h^{-1}(e^{i\lambda h}-1)a_\lambda e^{i\lambda x}$$

and

$$h^{-1}[F(x+h)-F(x)]$$

is bounded, it is easy to show that, for $\lambda\neq 0$,

$$b_\lambda(h)=(i\lambda h)^{-1}(e^{i\lambda h}-1)a_\lambda. \tag{5}$$

Now, by Theorem 2, F is ϕ^pap. Then the proof of [12, Theorem 9] shows that there exists a sequence (S_n) of integrals of Bochner–Fejér polynomials which ϕ^p–converges to F as $n\to\infty$, and that S_n has the form

$$S_n(x)=d_0^m b_0(h)x+\sum_{\lambda\neq 0}(i\lambda)^{-1}d_\lambda^m b_\lambda(h)e^{i\lambda x}.$$

Here m and h depend upon n, $m\to\infty$ and $h\to 0$ as $n\to\infty$, only finitely many of the set $\{d_\lambda^m\}$ are non–zero for any fixed m and $\lim\limits_{m\to\infty}d_\lambda^m=1$ for each fixed λ. It follows from the definitions that the sequence (S'_n) D^p–converges to f as $n\to\infty$, where, by (5),

$$S'_n(x)=d_0^m b_0(h)+\sum_{\lambda\neq 0}d_\lambda^m(i\lambda h)^{-1}(e^{i\lambda h}-1)a_\lambda e^{i\lambda x}.$$

Now for every fixed non–zero λ, $\lim\limits_{h\to 0}(i\lambda h)^{-1}(e^{i\lambda h}-1)=1$. Hence, as $n\to\infty$, the formal limit of $S'_n(x)$ can be written as

$$b_0(0+)+\sum_{\lambda\neq 0}a_\lambda e^{i\lambda x},$$

which is the Fourier series of f provided that $b_0(0+)=a_0$.

To prove the latter we observe first that, by [12, Lemma [12],

$$b_0(0+)=b_0(1)=M\{F(x+1)-F(x)\}.$$

For $T>0$, let

$$b_0(T,\ 1)=T^{-1}\int_0^T\{F(x+1)-F(x)\}dx$$

and let

$$a_0(T)=T^{-1}\int_0^T f(u)du.$$

Then

$$\begin{aligned}|b_0(T,\ 1)-a_0(T)|&=T^{-1}\left|\left[\int_0^{T+1}-\int_0^T\right]F(u)du-F(T)+F(0)\right|\\ &=T^{-1}\left|\int_0^1\{F(T+u)-F(u)-[F(T)-F(0)]\}du\right|\\ &\leqslant T^{-1}\sup_{0\leqslant u\leqslant 1}\{|F(T+u)-F(T)|+|F(u)-F(0)|\}\\ &\leqslant T^{-1}\{\text{osc.}\ (F;\ T,\ T+1)+\text{osc.}\ (F;\ 0,\ 1)\}\\ &\leqslant 2T^{-1}\|F\|_{\phi^p}=2T^{-1}\|f\|_{D^p}.\end{aligned}$$

Let $T\to\infty$. Then $|b_0(T,1)-a_0(T)|\to 0$ since f is D^∞ap and $\|f\|_{D^\infty}$ is therefore finite by Lemma 1. Also $b_0(T,1)\to b_0(1)$ and $a_0(T)\to a_0$. Hence $b_0(1)=a_0$ and the proof of the theorem is complete.

Integrals and Derivatives

Theorem 7. *If f is D^pap and F is D^∞–bounded, where F is defined by* (1), *t F is V^pap.*

Proof We observe that, by [12, Lemma 5],

$$\|F\|_U\leqslant\|F\|_{\phi^\infty}+\left\|\int_0^x F(t)\,dt\right\|_{\phi^\infty}=\|f\|_{D^\infty}+\|F\|_{D^\infty}.$$

Now F is D^∞–bounded by hypothesis and f is D^∞–bounded since it is D^∞ap. Henc is uniformly bounded. Since F is ϕ^p ap by Theorem 2 and all bounded ϕ functions are V^pap ([12, Theorem 10]), the proof is complete.

Theorem 8. *Let f be D^pap and let f' exist (finitely) everywhere and be continuous. Then f' is also D^pap.*

Proof Since f' exists finitely everywhere, it is Denjoy integrable and

$$f(x)-f(0)=\int_0^x f'(t)\,dt.$$

For $h>0$, put

$$g_h(x)=h^{-1}\int_0^h f'(x+t)\,dt=h^{-1}\{f(x+h)-f(x)\}.$$

Then g_h is D^pap since it is the difference between two D^pap functions. Furthermo since f' is D^p–continuous, Lemma 3 shows that g_h D^p–converges to f' as $h\to 0$. Her f' is D^pap by Theorem 3.

Products

That the product of two D^pap functions is not necessarily D^pap can be seen fr the function f which has period 1 and for which $f(x)=x^{-1/2}$ on $(0,1]$. For f clearly Sap and is thus D^pap for any $p\geqslant 1$. But f^2 is not Lebesgue integrable ar since it is one–signed, it cannot be Denjoy integrable. Hence f^2 is not D^pap. In t following theorem we consider sufficient conditions for a product fg to be D^pa given that f is D^pap.

Theorem 9. (i) *Let f be D^1ap and let g be Uap. Then the product fg is D^1ap.*

(ii) *Let f be D^pap and let g be Uap and locally absolutely continuous. Then sufficie condition s for the product fg to be D^pap are that*

(a) *g' is S^p–bounded when $1<p<\infty$,*

and (b) *g' is U–bounded when $p=\infty$.*

Proof Since, by Lemma 4, f_h is Uap, where f_h is defined by (2), the product g is Uap and, hence, it is D^pap. In order to prove that fg is also D^pap, it follows om Theorem 3 that it will suffice to prove that

$$\|(f_h-f)g\|_{D^p}\to 0 \text{ as } h\to 0.$$

Let $p=1$. Then, since $\|\cdot\|_{D^1}=\|\cdot\|_S$ by Theorem 1,

$$\begin{aligned}\|(f_h-f)g\|_{D^1}&=\sup_x\int_x^{x+1}|f_h(y)-f(y)|\cdot|g(y)|dy\\&\leqslant\|g\|_U\sup_x\int_a^{x+1}|f_h(y)-f(y)|dy\\&=\|g\|_U\|f_h-f\|_{D^1}.\end{aligned}$$

ow $\|g\|_U$ is finite by hypothesis and, since f is D^1-continuous by Lemma 1, $\|f_h-$ $\|_{D^1}\to 0$ as $h\to 0$ by Lemma 3. Hence the left-hand side tends to 0 with h and the roof of (i) is complete.

Let $p>1$. For F defined by (1), let

$$F_1(x)=\int_0^x F(t)\,dt.$$

et g' be locally Lebesgue integrable and let (a, b) be any finite interval. Then, y using integration by parts and Fubini's Theorem,

$$\begin{aligned}h\int_a^b f_h(y)g(y)\,dy&=\int_a^b[F(y+h)-F(y)]g(y)\,dy\\&=[\{F_1(y+h)-F_1(y)\}g(y)]_a^b\\&\quad-\int_a^b[F_1(y+h)-F_1(y)]g'(y)\,dy\\&=g(b)\int_b^{b+h}F(t)\,dt-g(a)\int_a^{a+h}F(t)\,dt\\&\quad-\int_a^b g'(y)\,dy\int_y^{y+h}F(t)\,dt\\&=\int_0^h[F(b+t)g(b)-F(a+t)g(x)]\,dt\\&\quad-\int_0^h dt\int_a^b F(y+t)g'(y)\,dy.\end{aligned}\tag{7}$$

urthermore, by applying the theorem on integration by parts for Denjoy integrals [7, page 711]),

$$\int_a^b f(y)g(y)\,dy=[F(y)g(y)]_a^b-\int_a^b F(y)g'(y)\,dy.\tag{8}$$

For $1<p<\infty$, let x be chosen arbitrarily and let π: $x=x_0<x_1<\cdots<x_k=x+1$ be ny partition of the interval $(x, x+1)$. Then

$$\left\{\sum_i\left|\int_{x_{i-1}}^{x_i}\{f_h(y)-f(y)\}g(y)\,dy\right|^p\right\}^{1/p}\tag{9}$$

$$\begin{aligned}&=\left\{\sum_i\left|h^{-1}\int_0^h\{[F(x_i+t)-F(x_i)]g(x_i)\right.\right.\\&\quad-[F(x_{i-1}+t)-F(x_{i-1})]g(x_{i-1})\}dt\end{aligned}$$

$$-h^{-1}\int_0^h dt\int_{x_{i-1}}^{x_i}[F(y+t)-F(y)]g'(y)dy\,\Big|^p\Big\}^{1/p} \tag{10}$$

$$\leqslant\Big\{\sum_i\Big|h^{-1}\int_0^h\{[F(x_i+t)-F(x_i)]-[F(x_{i-1}+t)-F(x_{i-1})]\}g(x_i)\,dt\Big|^p\Big\}^{1/p}$$

$$+\Big\{\sum_i\Big|h^{-1}\int_0^h[F(x_{i-1}+t)-F(x_{i-1})][g(x_i)-g(x_{i-1})]\,dt\Big|^p\Big\}^{1/p}$$

$$+\Big\{\sum_i\Big[h^{-1}\int_0^h\|F(y+t)-F(y)\|_U\,dt\int_{x_{i-1}}^{x_i}|g'(y)|\,dy\Big]^p\Big\}^{1/p} \tag{11}$$

$$\leqslant\|g\|_U\Big\{\sum_i h^{-1}\int_0^h|[F(x_i+t)-F(x_i)]-[F(x_{i-1}+t)-F(x_{i-1})]|^p\,dt\Big\}^{1/p}$$

$$+\Big\{\sum_i h^{-1}\int_0^h\|F(y+t)-F(y)\|_U^p\,|g(x_i)-g(x_{i-1})|^p\,dt\Big\}^{1/p}$$

$$+\Big\{Fh^{-1}\int_0^h\|F(y+t)-F(y)\|_U^p\,dt\sum_i\int_{x_{i-1}}^{x_i}|g'(y)|^p dy\,(x_i-x_{i-1})^{p-1}\Big\}^{1/p}$$

$$\leqslant\|g\|_U\Big\{h^{-1}\int_0^h V_p(F(y+t)-F(y);\ x\leqslant y\leqslant x+1)^p\,dt\Big\}^{1/p}$$

$$+\sup_{0<t<h}\|F(y+t)-F(y)\|_U\Big[V_p(g;\ x,\ x+1)+\Big\{\int_x^{x+1}|g'(y)|^p dy\Big\}^{1/p}\Big]$$

$$\leqslant\|g\|_U\sup_{0<t<h}\|F(y+t)-F(y)\|_{\phi^p}+\{\|g\|_{\phi^p}+\|g'\|_{S^p}\}\sup_{0<t<h}\|F(y+t)-F(y)\|_U.$$

In the above argument (10) follows from (7) and (8) in which $a=x_{i-1}$ and b and (11) and (12) follow from applications of Minkowski's and Hölder's inequal respectively. If the supremum is taken, firstly over all partitions π of $(x, x$ and then over all x, (9) may be replaced by $\|(f_h-f)g\|_{D^p}$.

Now $\|g\|_U$ and $\|g'\|_{S^p}$ are finite by hypothesis and $\|g\|_{\phi^p}$ is also finite since

$$\|g\|_{\phi^p}\leqslant\|g\|_\phi=\|g'\|_S\leqslant\|g'\|_{S^p}.$$

Furthermore, F is ϕ^pap by Theorem 2 and therefore it is both ϕ^p-continuous U-continuous ([12, Lemmas 6 and 7]). Hence, as $h\to 0$, the right-hand side of tends to zero and $\|(f_h-f)g\|_{D^p}\to 0$.

This completes the proof of (ii) (a).

Finally, for $p=\infty$, let x and k $(0<k\leqslant 1)$ be chosen arbitrarily. Then by pu $a=x$ and $b=x+k$ in (7) and (8) we see that

$$\Big|\int_x^{x+k}\{f_h(y)-f(y)\}g(y)\,dy\Big|$$

$$=\Big|h^{-1}\int_0^h\Big\{[[F(u+t)-F(u)]g(u)]_x^{x+k}-\int_x^{x+k}[F(y+t)-F(y)]g'(y)dy\Big\}dt\Big|$$

$$\leqslant h^{-1}\int_0^h 2\|F(u+t)-F(u)\|_U\|g\|_U\,dt+h^{-1}\int_0^h\|F(y+t)-F(y)\|_U\|g'\|_U\,dt$$

$$\leqslant\sup_{0<t<h}F(y+t)-F(y)\|_U\{2\|g\|_U+\|g'\|_U\}. \tag{14}$$

If the supremum is taken over all x and all permissible k, then the left side of 14) may be replaced by $\|(f_h-f)g\|_{D^\infty}$. Since g and g' are U-bounded by hypothesis nd F, as was observed previously, is U-continuous, it follows that

$$\|(f_h-f)g\|_{D^\infty}\to 0 \text{ as } h\to 0.$$

This completes the proof of (ii) (b) and of the whole theorem.

Corollary. *Let f be D^pap. Then fg is D^pap provided*

(i) *g is $S^p_{(1)}$ap when p is finite;*

(ii) *g is $U_{(1)}$ap when $p=\infty$. (For definitions of $S^p_{(1)}$ap and $U_{(1)}$ ap functions see* 11]).

Proof Since an $S^p_{(1)}$ap (respectively $U_{(1)}$ap) function is Uap and has an S^p-ounded (respectively U-bounded) derivative, the conditions of Theorem 9 are atisfied and the conclusions therefore follow immediately.

In [3], Eurkill proved that fg is Dap when f is Dap, g is Uap and g' is niformly continuous. As these conditions on g are shown in [13, Theorem 11] to e necessary and sufficient for g to be $U_{(1)}$ap, Burkill's argument applied to D^∞ap unctions provides an alternative proof to part (ii) of the above corollary.

Other Properties

Theorem 10. *If f is ϕ^∞ap and D^∞ap, then it is Uap.*

Proof By Lemma 1 and Lemma 3, $\|f_h-f\|_{D^\infty}\to 0$ as $h\to 0$, where f_h is defined by 2). Furthenmore, by [12, Theorem 3],

$$\|f_h-f\|_{\phi^\infty}\to 0$$

s $h\to 0$. Since it can be seen from (6) that

$$\|f_h-f\|_U\leqslant\|f_h-f\|_{\phi^\infty}+\|f_h-f\|_{D^\infty},$$

t follows that $\|f_h-f\|_U\to 0$ as $h\to 0$. But f_h is Uap, since it is V^∞ap by Lemma 4. 'herefore f is Uap.

Corollary. *If f is ϕ^pap and D^∞ap, then it is V^pap.*

Proof Since f is ϕ^∞ap, the above theorem shows that it is Uap, It is therefore ounded and, being ϕ^pap, it is also V^pap by [12, Theorem 10].

Theorem 11. *For $1\leqslant p<q$ let f be D^qap and D^p-continuous. Then f is D^pap.*

Proof Theorem 2 and the definitions show that F is ϕ^qap and ϕ^p-continuous, here F is defined by (1). Hence [12, Theorem 11] and Theorem 2, again, show, in ırn, that F is ϕ^pap and that f is D^pap.

Theorem 12. *For $1\leqslant p<q<r$, let f be D^rap and D^p-bounded. Then f is D^qap ut it is not necessarily D^pap.*

Proof Theorem 2 and the defin t ons show that F is ϕ^rap and ϕ^p-bounded, where F is defined by (1). Hence, by [12, Theorem 12], F is ϕ^qap and, by Theorem

2 again, f is D^qap.

That f is not necessarily D^pap cannot be established by means of the function which is used in the proof of [12, Theorem 12] to show that a ϕ^p–bounded ϕ^rap function need not be ϕ^pap. For the function used there is differentiable nowhere while the integral of the D^rap function which we need here must be differentiable almost everywhere.

The following contrary example which establishes the required result i constructed in three stages. Firstly we state a lemma.

Lemma 5. *Let $p\geqslant 1$, let $(\alpha_i)_{i=0}^{\infty}$ and $(\beta_i)_{i=0}^{\infty}$ be two strictly monotonic sequen each of which decreases to 0, and let f be the function defined on $[0, \alpha_0]$ such t $f(0)=0$, $f(\alpha_{2i})=0$ and $f(\alpha_{2i+1})=\beta_i$ for all i and, on each interval (α_{i+1}, α_i), the gra of f is linear. Then*

$$V_p(f;\ 0,\ \alpha_0)^p=2\sum_{i=0}^{\infty}\beta_i^p.$$

Proof The proof is elementary and is given in [10].

We next define a sequence of functions (g_n). Let $(u_i)_{i=0}^{\infty}$ be a stric monotonically decreasing sequence of real numbers with $u_0=1$ and $\lim_{i\to\infty} u_i=0$. T values of u_i for $i>0$ will be specified later. For $n=0, 1, 2, \cdots$, let $g_n(x)=0$ wh $x\leqslant 0$, when $x=2^{-n-i}$, $i=0, 1, 2, \cdots$, and when $2^{-n}<x$. On each interval $(2^{-n-i}$ $2^{-n-i})$, $i=0, 1, 2, \cdots$, the graph of g_n is formed by placing side by side 2^n simi isosceles triangles each with height $2^{-n/p}u_i$ and base of width $2^{-2n-i-1}$.

Then g_n is continuous and non–negative and has the following properties:

$$0\leqslant g_n(x)\leqslant 2^{-n/p} \text{ and } \max g_n(x)=2^{-n/p}; \qquad (1$$

$$g_n \text{ is absolutely continuous on } [\delta,\ 1] \text{ for every } \delta>0; \qquad (1$$

for $0<x\leqslant 1$,

$$(D)\int_0^x g_n'(t)\,dt=\lim_{\delta\to 0+}\int_\delta^x g_n'(t)\,dt=g_n(x)-\lim_{\delta\to 0+} g_n(\delta)$$
$$=g_n(x)-\lim_{i\to\infty} 2^{-n/p}u_i=g_n(x), \qquad (1$$

the Denjoy integral existing as the limit of a sequence of Lebesgue integra Furthermore, Lemma 5, with $\alpha_0=2^{-n}$, shows that

$$V_p(g_n;\ 0,\ 1)^p=V_p(g_n;\ 0,\ 2^{-n})^p=2\sum_{i=0}^{\infty}2^n(2^{-n/p}u_i)^p=2\sum_{i=0}^{\infty}u_i^p. \qquad (1$$

Since $g_n(x+2^{-n})=0$ for $0\leqslant x\leqslant 2^{-n}$, it follows from (18) that

$$V_p(g_n(x)-g_n(x+2^{-n});\ 0,\ 1)^p\geqslant V_p(g_n(x)-g_n(x+2^{-n});\ 0,\ 2^{-n})^p$$
$$=V_p(g_n;\ 0,\ 2^{-n})^p=2\sum_{i=0}^{\infty}u_i^p. \qquad (1$$

We now fix the values of u_i for $i>0$ by defining $u_1=1/2$ and $u_i=1/i^{1/p}(\log i)^2$ for $i\geqslant 2$. Then

$$\sum_{i=0}^{\infty} u_i^p = 1 + (1/2)^p + \sum_{i=2}^{\infty} 1/i(\log i)^{2p}, \tag{20}$$

and this is finite since $p \geqslant 1$. Denote it by $M/2$, say.

The resulting graph of g_n on te interval $[0, 1]$ is called an *n-graph*, and the final stage in our construction is to use these n-graphs and some ideas from [2] to define a sequence of functions (f_n) on the real line. To this end we use the letters μ, ν to denote integers.

f_0 is formed by placing a 0-graph on each interval $(3\nu+13\nu+2)$.,

f_1 is formed from f_0 by adding a 1-graph on each unoccupied interval $(\mu, \mu+1)$ within every interval $(3^2\nu+3, 3^2\nu+2.3)$.

f_2 is formed from f_1 by adding a 2-graph on each unoccupied interval $(\mu, \mu+1)$ within every interval $(3^3\nu+3^2, 3^3\nu+2.3^2)$.

....

f_n is formed from f_{n-1} by adding an n-graph on each unoccupied interval $(\mu, \mu+1)$ within every interval $(3^{n+1}\nu+3^n, 3^{n+1}\nu+2.3^n)$.

....

Then each f_n is a continuous periodic function and, for $|x| \leqslant 3^{n+1}$, $f_n(x) = f_{n+1}(x) = \cdots$. Therefore $f(x) = \lim_{n\to\infty} f_n(x)$ is defined for all x.

Now since $f - f_n$ consists only of j-gr aphs, where

$$j = n+1,\ n+2,\ n+3,\ \cdots,$$

we see from (15) that, for all x,

$$|f(x) - f_n(x)| \leqslant 2^{-(n+1)/p}.$$

Hence f_n converges to f uniformly in x as $n\to\infty$ and f is Uap. Thus f is ϕ^∞ap. Furthermore, (16) and (17) show, in turn, that f' exists almost everywhere and that

$$f(x) = \int_0^x f'(t)\,dt$$

for all x. It follows from Theorem (2) that f' is D^∞ap.

The function f' provides the required contrary example. For it is easily shown using (18) and (20) that

$$V_p(f;\, x, x+1) \leqslant 2M^{1/p},$$

for all x, so that f is ϕ^p-bounded and f' is D^p-bounded. Hence, from the first partof the present theorem, f' is D^qap.

On the other hand the above construction shows that, for any integer n, there exists an interval $(\nu, \nu+1)$ in which f is represented by an n-graph. Hence, by (19) and (20),

$$V_p(f(x) - f(x+2^{-n});\, \nu, \nu+1) \geqslant M^{1/p}.$$

It follows that $\|f(x) - f(x+2^{-n})\|_{\phi^p} \geqslant M^{1/p}$ and, since n may be chosen arbitrarily large, that f is not ϕ^p-continuous. Hence f cannot be ϕ^pap and, by Theorem 2, f' cannot be D^pap.

Remark. When $p=1$, the function f constructed in the above proof is ϕ^qap for $q>1$ and ϕ-bounded. But it is not ϕ-continuous and hence cannot be ϕap. Again, since it is Uap, it is also V^qap but it cannot be Vap. It therefore provides a counter example for part of [12, Theorem 12 and its corollary], where its existence was asserted without proof.

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