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ON THE GROWTH OF SOME RANDOM HYPERDIRICHLET SERIES*

LIU QUAN-SHENG (刘全升) **

Abstract

The paper considers the random L-Dirichlet series

$$f(s, \omega)=\sum_{n=1}^{\infty} P_n(s, \omega)\exp(-\lambda_n s)$$

and the random B-Dirichlet series

$$\varphi_{\tau_0}(s, \omega)=\sum_{n=1}^{\infty} P_n(\sigma+i\tau_0, \omega)\exp(-\lambda_n s),$$

where $\{\lambda_n\}$ is a sequence of positive numbers tending strictly monotonically to infinity, $\tau_0 \in R$ is a fixed real number, and

$$P_n(s, \omega)=\sum_{j=0}^{m_n} \varepsilon_{nj} a_{nj} s^j$$

a random complex polynomial of order m_n, with $\{\varepsilon_{nj}\}$ denoting a Rademacher sequence and $\{a_{nj}\}$ a sequence of complex constants. It is shown here that under certain very general conditions, almost all the random entire functions $f(s, \omega)$ and $\varphi_{\tau_0}(s, \omega)$ have, in every horizontal strip, the same order, given by

$$\rho=\limsup \frac{\lambda_n \log \lambda_n}{\log A_n^{-1}}$$

where

$$A_n=\max_{0\le j\le m_n} |a_{nj}|.$$

Similar results are given if the Rademacher sequence $\{\varepsilon_{nj}\}$ is replaced by a steinhaus seqence or a complex normal sequence.

§1. Introduction

Let $(\Omega, \mathscr{A}, P)$ be a probability space, where $\Omega=[0, 1]$, $\mathscr{A}$ is composed of all Lebesque measurable sets E and $P(E)$ is the Lebesque measure of E. Given a sequence of non-negative integers $\{m_n\}_{n=1}^{\infty}$, we denote by $\{\varepsilon_{nj}\}=\{\varepsilon_{10}, \varepsilon_{11}\cdots\varepsilon_{1m_1}, \varepsilon_{20}, \varepsilon_{21}, \cdots, \varepsilon_{2m_2}\cdots\}$ a Rademacher seqence and $\exp(2\pi i\theta_{nj})$ a steinhaus sequence on the probability space $(\Omega, \mathscr{A}, P)$, namely $\{\varepsilon_{nj}\}$ is a sequence of inde-pendent random variables taking the values $+1$ or -1 with the same probability 1/2, and $\{\theta_{nj}\}$ is a sequence of independent random variables equidistributed on $[0, 1]$. Suppose

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* Partially supported by the National Science Foundation of P. R. C.
** Department of Mathematics, Wuhan University Wuhan, Hubei, China.

moreover $\{Z_{nj}\}$ $(n=1, 2, \cdots, j=0, 1, \cdots, m_n)$ is a complex normal sequence, i. e., a sequence of independent standard complex gaussian variables (for the definition see [2], p. 168). We shall consider the following two types of random series:

$$f(s, \omega)=\sum_{n=1}^{\infty} P_n(s, \omega)\exp(-\lambda_n s) \tag{1.1}$$

and

$$\varphi_{\tau_0}(s, \omega)=\sum_{n=1}^{\infty} P_n(s+i\tau_0, \omega)\exp(-\lambda_n s) \tag{1.2}$$

with $P_n(s, \omega)=\sum_{j=0}^{m_n} \varepsilon_{nj}a_{nj}s^j$, or $\sum_{j=0}^{m_n}\exp(2\pi i\,\theta_{nj})a_{nj}s^j$ or $\sum_{j=0}^{m_n} Z_{nj}a_{nj}s^j$, where $\{a_{nj}\}$ is sequence of complex constants, $\tau_0\in R$ is a fixed real number and the λ_n are positi numbers tending strictly monotonicly to infinity. After Lepson and Blambert, t series (1.1) or (1.2) is called random L-Dirichlet series or random B-Dirich series respectively, both of them are called random hyperdirichlet series.

Let

$$A_n=\max_{0\le j\le m_n}|a_{nj}|. \tag{1.}$$

We shall consider the associated Dirichlet series

$$\{\psi\}:\ \sum_{n=1}^{\infty} A_n\exp(-\lambda_n s). \tag{1.}$$

In this paper, we suppose always that

$$L=\limsup \log n/\lambda_n<\infty, \tag{1.}$$

$$\beta^*=\limsup m_n/\lambda_n<\infty, \tag{1.}$$

and that

$$\sigma_c^\psi=\limsup(\log A_n)/\lambda_n=-\infty. \tag{1.}$$

For $\omega\in\Omega$, the series (1.1) and (1.2) represent a. s. entire functions (see [1]). study the growth property of (1.1) and (1.2) for $\omega\in\Omega$ such that (1.1) and (1. represent entire functions, M. Blambert and M. Berland [1] showed that the order them in each horizontal strip cannot exceed the (R) order of ψ. In [5] R. Parvatha proved that the order of (1.1) in a certain curvilinear strip (with very complicat hypotheses) is equal to that of ψ. In this paper we shall study the almost su growth property of (1.1) and (1.2) and we shall see that in this case the conclusi can be improved very much. Our results here are also generalizations of t corresponding theorems of Paley and Zygmund[4] and Yu Jia-Rong[6], where t case of random Tayolr series or random Dirichlet series was considered.

§ 2. Three Lemmas

We first generalize the Daley-Zygmund lemma ([4]).

Lemma 2.1. *Let $E\subset[0, 1]$ be any measurable set of points, $P(E)>0$, and $\{m_n\}$ is a sequence of non-negative integers. Then we can choose a number $N=N(E)$, such* hat for any $N'>N$ we have

$$\int_E \left|\sum_{n=N}^{N'}\sum_{j=0}^{m_n} c_{nj}s_{nj}(\omega)\right|^2 d\omega \geqslant \frac{1}{2}P(E)\sum_{n=N}^{N'}\left|\sum_{j=0}^{m_n} c_{nj}\right|^2\Big/(m_n+1) \tag{2.1}$$

nd

$$\int_E \left|\sum_{n=N}^{N'}\sum_{j=0}^{m_n} c_{nj}\exp(2\pi i\theta_{nj})\right|^2 d\omega \geqslant \frac{1}{2}P(E)\sum_{n=N}^{N'}\left|\sum_{j=0}^{m_n} c_{nj}\right|^2\Big/(m_n+1), \tag{2.2}$$

whatever the complex numbers c_{nj} may be.

Proof From the result of Paley and Zygmund[4], we can choose a number $N=N(E)$, such that for any $N'>N$

$$\int_E \left|\sum_{n=N}^{N'}\sum_{j=0}^{m_n} c_{nj}s_{nj}(\omega)\right|^2 d\omega \geqslant \frac{1}{2}P(E)\sum_{n=N}^{N'}\sum_{j=0}^{m_n}|c_{nj}|^2. \tag{2.3}$$

On the other hand, from the Schwartz inequality we know that

$$\left|\sum_{j=0}^{m_n} c_{nj}\right|^2 \leqslant (m_n+1)\sum_{j=0}^{m_n}|c_{nj}|^2. \tag{2.4}$$

So we get (2.1). The proof of (2.2) is similar.

Remark 2.1. For the normal sequence $\{Z_{nj}\}$ $(n=1, 2, \cdots, j=0, 1, \cdots, m_n)$ we have an analogous result, but in this case the number $\frac{1}{2}P(E)$ in (2.1) or (2.2) should be replaced by a constant $c=c(E)>0$. See [6].

In addition, we shall use the following two known results, which are due to B. Jepson[3] and M. Blambert and M. Berland[1] respectively.

Lemma 2.2.[3] *Suppose*

$$P(s)=\sum_{k=0}^{n} a_k s^k,$$

$$M=\max_{0\leqslant k\leqslant n}|a_k|.$$

Then

$$\max_{|s-s'|=R}|P(s)|\geqslant\frac{M}{(|s'|+1)^n}\min(R^n, 1).$$

Lemma 2.3.[1] *If the conditions* (1.5), (1.6) *and* (1.7) *are satisfied, the Ritt order of the B–Dirichlet series*

$$\varphi_{\tau_0}(s, \omega)=\sum_{n=1}^{\infty}P_n(\sigma+i\tau_0, \omega)\exp(-\lambda_n s)$$

is given by

$$\rho=\limsup_{n\to\infty}\frac{\lambda_n\log\lambda_n}{\log(A_n^{-1})},$$

where

$$P_n(s)=\sum_{j=0}^{m_n}a_{nj}s^j$$

and

$$A_n=\max_{0\leqslant j\leqslant m_n}|a_{nj}|.$$

§ 3. The Case of Random L–Dirichlet Series

Now let us consider the random L–Dirichlet series

$$f(s, \omega)=\sum_{n=1}^{\infty} P_n(s, \omega)\exp(-\lambda_n s), \tag{3.1}$$

where

$$P_n(s, \omega)=\sum_{j=0}^{m_n} \varepsilon_{nj}a_{nj}s^j \text{ or } \sum_{j=0}^{m_n} \exp(2\pi i\theta_{nj})a_{nj}s^j. \tag{3.2}$$

Given $\omega\in\Omega$, if (3.1) represents an entire function, it's Ritt order in a horizontal strip $\alpha\leqslant\operatorname{Im}s\leqslant\beta$ $(\alpha<\beta)$ is defined as

$$\rho=\limsup_{\sigma\to-\infty}\frac{\log\log\sup_{\alpha\leqslant\tau\leqslant\beta}|f(\sigma+i\tau, \omega)|}{-\sigma}. \tag{3.}$$

In addition, we denote by $\rho^{\psi}(\lambda^{\psi})$ the Ritt order (lower order) of ψ in $\mathbb{C}$.

Theorem 1. *Suppose the conditions* (1.5), (1.6) *and* (1.7) *are satisfied. Tl the entire functions defined by* (3.1) *have a. s., in every strip* $\alpha\leqslant\operatorname{Im}s\leqslant\beta(\alpha<\beta)$, *; same order, given by*

$$\rho=\rho^{\psi}=\limsup_{n\to\infty}\frac{\lambda_n\log\lambda_n}{\log A_n^{-1}}.$$

Proof It is well known that the order of $f(s)$ in every horizontal strip canı exceed ρ^{ψ}[1]. Suppose the theorem were false, there would exist two constants $\rho'<$ and $\sigma_0>0$, a horizontal strip $\alpha\leqslant\operatorname{Im}s\leqslant\beta(\alpha<\beta)$ and a set $E(P(E)>0)$ of ω, such tl

$$\left|\sum_{n=1}^{\infty}P_n(s, \omega)\exp(-\lambda_n s)\right|\leqslant\exp\exp(-\rho'\sigma) \tag{3.}$$

for $\sigma<-\sigma_0$, $\omega\in E$ and $\alpha\leqslant\operatorname{Im}s\leqslant\beta$, by an easy argument of reduction. Using Lemı 2.1 with

$$c_{nj}=a_{nj}s^j\exp(-\lambda_n s)$$

and supposing that the N in it is equal to 1 without loss of generality (otherwise consider the series

$$\sum_{n=N}^{\infty}P\ (s,\omega)\exp(-\lambda_n s)$$

instead of

$$\sum_{n=1}^{\infty}P_n(s, \omega)\exp(-\lambda_n s)),$$

we obtain, for $\sigma<-\sigma_0$ and $\alpha\leqslant\operatorname{Im}s\leqslant\beta$,

$$\sum_{n=1}^{\infty}\frac{1}{m_n+1}|P_n(s)|^2\exp(-2\lambda_n\sigma)\leqslant2\exp(2\exp(-\rho'\sigma)) \tag{3.}$$

where

$$P_n(s)=\sum_{j=0}^{m_n}a_{nj}s^j.$$

Consequently

$$\frac{1}{m_n+1}|P_n(s)|^2\exp(-2\sigma\lambda_n)\leqslant2\exp(2\exp(-\rho'\sigma)),$$

or namely

$$|P_n(s)|\leqslant(2(m_n+1))^{1/2}\exp(\sigma\lambda_n+\exp(-\rho'\sigma)) \tag{3.}$$

for $\sigma<-\sigma_0$ and $\alpha\leqslant\tau\leqslant\beta(s=\sigma+i\tau)$.

Now we use the Lemma 2.2 with

$$s'=\sigma+i\left(\alpha+\frac{\beta-\alpha}{2}\right),$$

where $\sigma<-\sigma_0-1$ and $0<R<\min(1,\ (\beta-\alpha)/2)$, and get

$$A_n\leqslant(2(m_n+1))^{1/2}((|\sigma+i(\alpha+(\beta-\alpha)/2)|+1)/R)^{m_n}\exp((\sigma+R)\lambda_n+\exp(-\rho'(\sigma-R))). \quad (3.7)$$

Since

$$\beta^*=\limsup_{n\to\infty}\frac{m_n}{\lambda_n}<\infty,$$

given $\varepsilon>0$, we can choose a number $\sigma_\varepsilon>0$ such that

$$(2(m_n+1))^{1/2}((|\sigma+i(\alpha+(\beta-\alpha)/2)|+1)/R)^{m_n}<\exp(-\varepsilon\sigma\lambda_n) \quad (3.8)$$

for $\sigma<-\sigma_\varepsilon$. Hence

$$A_n\leqslant\exp(((1-\varepsilon)\sigma+R)\lambda_n)+\exp(-\rho'(\sigma-R))) \quad (3.9)$$

for $\sigma<-\sigma_1=\min(-\sigma_0-1,\ -\sigma_\varepsilon)$ and $n\geqslant1$. Writing

$$\mu(\sigma)=\sup_{n\geqslant1}A_n\exp(-\lambda_n\sigma) \quad (3.10)$$

and

$$M(\sigma)=\sup_{\tau\in R}|\psi(\sigma+i\tau)|, \quad (3.11)$$

we can easily prove that

$$M(\sigma)\leqslant K(\varepsilon)\mu(\sigma-L-\varepsilon) \quad (3.12)$$

where

$$K(\varepsilon)=\sum_{n=1}^{\infty}\exp(-(L+\varepsilon)\lambda_n)<\infty.$$

Associating this with (3.9) we get

$$M((1-\varepsilon)\sigma+R+L+\varepsilon)<K(\varepsilon)\exp\exp(-\rho'(\sigma-R)), \quad (3.13)$$

from which we can easily deduce that $\rho^\psi\leqslant\rho'/(1-\varepsilon)$. Since $\varepsilon>0$ is arbitrary, we get $\rho^\psi\leqslant\rho'$. This contradication completes the proof.

§ 4. The Case of Random B-Dirichlet Series

Now let us consider the random B-Dirichlet series

$$\varphi_{\tau_0}(s,\ \omega)=\sum_{n=1}^{\infty}P_n(\sigma+i\tau_0,\ \omega)\exp(-\lambda_n s), \quad (4.1)$$

where

$$P_n(s,\ \omega)=\sum_{j=0}^{m_n}\varepsilon_{nj}(\omega)a_{nj}s^j \text{ or } \sum_{j=0}^{m_n}\exp(2\pi i\theta_{nj})a_{nj}s^j. \quad (4.2)$$

The Ritt order of (4.1) on a horizontal line $\tau=t$ are defined as

$$\rho_t(\varphi_{\tau_0})=\limsup_{\sigma\to-\infty}\frac{\log\log|\varphi_{\tau_0}(\sigma+it,\ \omega)|}{-\sigma}.$$

The function ψ is also defined by (1.4). We shall prove the following theorem.

Theorem 2. *If the conditions* (1.5), (1.6) *and* (1.7) *are satisfied, then, for* *each given $t\in R$, we have*

$$\rho_t(\varphi_{\tau_0})=\rho^{\psi}. \text{ a. s.} \tag{4.3}$$

Proof If the conclusion were false, there would exist two numbers $\rho'>\rho^{\psi}$, and $\sigma_0>0$ a line $\tau=t$, and a set $E(P(E)>0)$ of ω, such that

$$\left|\sum_{n=1}^{\infty} P_n(\sigma+i\tau_0,\ \omega)\exp(-\lambda_n s)\right|\leqslant \exp\exp(-\rho'\sigma), \tag{4.4}$$

for $\sigma<-\sigma_0$, $\omega\in E$ and $s=\sigma+it$. As in the proof of Theorem 1, using Lemma 2.1 we can get

$$\sum\frac{1}{m_n+1}|P_n(\sigma+i\tau_0)|^2\exp(-2\lambda_n\sigma)\leqslant 2\exp(2\exp(-\rho'\sigma), \tag{4.}$$

for $\sigma<-\sigma_0$.

On the other hand, by Lemma 2.3 we can easily prove that the B-Dirich series

$$\sum\frac{1}{m_n+1}(P_n(\sigma+i\tau_0))^2\exp(-2\lambda_n s) \tag{4.}$$

has the same order as ψ. So from (4.5) we get a contradiction that $\rho^{\psi}=\rho'$. Tl completes that proof of Theorem 2.

Arranging all the numbers of Q (the set of all rational numbers) as $\{t_n\}_1^{\infty}$, fr Theorem 2 we get immediately the following corollary.

Corollary 4.1. *Suppose the conditions* (1.5), (1.6) *and* (1.7) *are satisfied. Th we have a. s.* $\forall t_n\in Q$

$$\limsup_{\sigma\to\infty}\frac{\log\log|\varphi_{\tau_0}(\sigma+it_n,\ \omega)|}{-\sigma}=\rho^{\psi}.$$

Consequently the entire functions defined by (4.1) *have a. s. in each horizontal str the same order* (R) *as* ψ (*in* $\mathbb{C}$).

Since we say tnat a property holds almost everywhere almost surely equivalent to say that it holds almost surely almost everywhere, we can also get t following corollary.

Corollary 4.2. *Under the conditions* (1.5), (1.6) *and* (1.7), *the entire functic defined by* (4.1), *have a. s., on almost every horizontrl line, the same order* (R) *as* (*in* $\mathbb{C}$).

§ 5. The Case of Gaussian Hyperdirichlet Series

We finally deal with the Gaussian hyperdirichlet series

$$f(s,\ \omega)=\sum P_n(s,\ \omega)\exp(-\lambda_n s) \tag{5.}$$

and

$$\varphi_{\tau_0}(s,\ \omega)=\sum P_n(\sigma+i\tau_0,\ \omega)\exp(-\lambda_n s), \tag{5.}$$

where

$$P_n(s,\ \omega)=\sum_{j=0}^{m_n} Z_{nj}a_{nj}s^j. \tag{5.3}$$

rom J.-P. Kahane's argument ([2], p. 172, Prop. 3), we know that for almost all , there exist a constant $K=K(\omega)>0$, such that

$$|Z_{ni}|\leqslant K\Big(\log\sum_{j=1}^{n}(m_j+1)\Big)^{1/2}\quad (i=0,\ 1,\ \cdots,\ m_n). \tag{5.4}$$

nce $\beta^*=\limsup\limits_{n} m_n/\lambda_n<\infty$, it follows that

$$|Z_{ni}|\leqslant K_1\log(n\lambda_n)\ \text{a. s.}, \tag{5.5}$$

r sufficiently large n, say $n>n_\omega$, and a constant $K_1=K_1(\omega)>0$. Therefore if we ippose moreover $L=\limsup\limits_{n}\log n/\lambda_n<\infty$ and

$$\{\psi\}:\ \sum A_n\exp(-\lambda_n s), \tag{5.6}$$

here

$$A_n=\max_{0\leqslant i\leqslant m_n}|a_{ni}|,$$

represents an entire function of Ritt order ρ^ψ. From the well known expression f Ritt about the Ritt order and the formulae of Valiron about the abscissa of onvergence of Dirichlet series, we know easily that almost all the functions

$$\sum A_n(\omega)\exp(-\lambda_n s), \tag{5.7}$$

here

$$A_n(\omega)=\max_{0\leqslant i\leqslant m_n}|a_{ni}Z_{ni}(\omega)|,$$

re also entire and have the same order ρ^ψ. So the order in each horizontal strip of lmost all the entire functions defined by (5.1) cannot exceed ρ^ψ and almost all he functions defined by (5.2) have the same order as ρ^ψ. Hence in the same way as ve do in the proof of Theorem 1 and Theorem 2, we can prove the following results:

Theorem 3. *Suppose the conditions* (1.5), (1.6) *and* (1.7) *are satisfied. Then he entire functions defined by* (5.1) *have a. s. in each strip* $\alpha\leqslant\operatorname{Im}s\leqslant\beta\,(\alpha<\beta)$, *the ame order, given by* $\rho=\rho^\psi=\limsup\limits_{n\to\infty}\dfrac{\lambda_n\log\lambda_n}{\log A_n^{-1}}$.

Theorem 4. *Suppose the conditions* (1.5), (1.6) *and* (1.7) *are satisfied. Then, or each given* $t\in R$, *we have a. s.* $\rho_t(\varphi_{\tau_0})=\rho^\psi$, *where* $\varphi_{\tau_0}(s,\ \omega)$ *is defined by* (5.2).

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