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FREE DEFORMATION RETRACTION IN INJECTIVE METRIC SPACES

MAI JIEHUA(麦结华)* TANG YUN(唐 云)**

Abstract

In this paper the authors establish the concept of generalized ball-intersection (GBI) and prove that an injective metric space is freely contractible to its each GBI, which generalizes a result of Isbell from a point to a GBI.

Let X and Y be metric spaces. A mapping f: $X \to Y$ is a contraction if for an $x, x' \in X$, the distance

$$d_Y(f(x), f(x')) \leqslant d_X(x, x').$$

Y is called injective if every contraction from a subspace X_0 of any space X to] can be extended to a contraction over X. Injective metric spaces have many fin properties[1,3] due to the severe requirement of their definition. However, they ar neither special nor rare. For example, every metric space is a subspace of som injective metric space ([3, p. 71]); and every collapsible polyhedron (particularl a solid cone on any polyhedron) admit injective metrsic[4], etc. So it significant to study injective metric space, with which we may further explo the stаucture of general metric spaces and discover some of their new properties.

On the other hand, deformation retraction is an important and basic concep in topology. And intuitively, free deformation retraction is readily understoo Isbell[3] proved that an injective metric space is free contractible to each of i points. However, it seems to us that the proof of Theorem 1.1 in [3] is no complete.

In this paper we propose the concept of generelized ball-intersection (GBI) an prove that an injective metric space is freely contractible to each of its GAI, whic extends the Isbell's theorem from a point to a GBI.

Let X be a metric space with a distance function d. X is convex if any tw closed balls $B(x_1, r_1)$ and $B(x_2, r_2)$ in X such that $r_1+r_2 \geqslant d(x_1, x_2)$ have a commo point. A collection $\mathbb{E}$ of subsets of X has the binary intersecting property if an subcollection of $\mathbb{E}$ in pairwise intersection has a common point. Denote by $\mathbb{B}_0$ th

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* Department of Mathematics, Guangxi University, Nanning, Guangxi, China.
** Department of Applied Mathematics, Beijing Polytechnic University, Beijing, China.

collection of all closed balls in X. The following result is well-known.

Lemma 1.[5] *A convex metricspace is injective if and only if $\mathbb{B}_0$ has the binary intersecting property.*

Now we define a generalized ball-intersection (GBI). Let X be an injective metric spase. A connnected subset C of X is called a GBI set or a GBI if the collection of sets $\{C, \mathbb{B}_0\}$ has the binary intersecting property. Obviously, balls and their intersections are all GBI sets. Besides, the following example tells us that there are other GBI sets.

Example 1. Let $X=I^n$, $n\geqslant 2$, be an n-cube with a metric

$$d(x, y)=\max_i |x_i-y_i|,\ x=(x_1, \cdots, x_n),\ y=(y_1, \cdots, y_n)\in I^n,$$

where $I=[0, 1]$. (I^n, d) is then an injective metric space. Denote by $\mathbb{B}$ the collection of all closed balls in I^n and their intersections. Let

$$\mathbb{E}_1=\{[x_1, y_1]\times[x_2, y_2]\times\cdots\times[x_n, y_n]\,|\,x_i, y_i\in I\},$$

where $[x_i, y_i]=\varnothing$ if $x_i>y_i$,

$$\mathbb{K}=[I\times[x_2, y_2]\times\cdots\times[x, x_n]\,|\,x_i, y_i\in I\}$$

and

$$\mathbb{E}_2=\{C\,|\,C \text{ is an intersection of any subcollection of } \mathbb{B}\cup\mathbb{K}\}.$$

Then it is easy to check that $\mathbb{E}_1$, $\mathbb{E}_2$ and $\mathbb{B}$ are all collections of GBI sets in I^n, and

$$\mathbb{B}\subsetneqq\mathbb{E}_2\subsetneqq\mathbb{E}_1.$$

For a metric space X, a subset $C\subset X$ and a real number $r\geqslant 0$, the closed ball in X with the center C and the radius r is written as

$$B(C, r)=\{x\in X\,|\,d(x, C)\leqslant r\}.$$

Lemma 2. *Let C be a GBI in an injective metric space X. Then*

(i) *for every $x\in X$, $C\cap B(x, d(x, C))\neq\varnothing$;*

(ii) *for every $r\geqslant 0$, $B(C, r)$ is also a GBI.*

Proof (i) Since C is a GBI, we have

$$C\cap B(x, d(x, C))=C\cap(\cap\{B(x, d(x, C)+s)\,|\,s>0\})\neq\varnothing.$$

(ii) Take a collection of balls

$$\mathbb{E}=\{B(x_\alpha, r_\alpha)\,|\,\alpha\in\Lambda\}$$

in X so that the collection of sets

$$\mathbb{E}_1=\{\mathbb{E}, B(C, r)\}$$

meets pairwise. It suffices to show the members of $\mathbb{E}_1$ have a common point. Let

$$B'_\alpha=B(x_\alpha|r_\alpha+r).$$

From the property (i), it follows that the collection

$$\mathbb{E}'=\{B'_\alpha, \alpha\in\Lambda\}\cup\{C\}$$

meets pairwise. Since C is a GBI, by the definition, the members of $\mathbb{E}'$ have a common point y, and then

$$d(y, x_\alpha)\leqslant r_\alpha+r,\ \forall\alpha\in\Lambda.$$

Due to convexity of the space X, the collection $\mathbb{E}''=\{\mathbb{E}, B(y, r)\}$ meets pairwise. So there exists

$$z\in \cap\{W|W\in \mathbb{E}''\}\subset \cap\{W|W\in \mathbb{E}_1\},$$

which shows that $B(C, r)$ is a GBI.

Lemma 3. *Let X be an injective metric space and C a nonempty GBI set. Then there exists a contraction $f: X\to C$ such that for any x and $y\in X$,*

$$d(y, f(x))\leqslant \max\{d(y, C), d(y, x)\} \tag{1}$$

and

$$d(x, f(x))=d(x, C). \quad (\ $$

Proof Since (2) follows clearly from (1), it suffices to prove (1). By t Zermelo's theorem, the set $X-C$ can be well-ordered with an ordinal number α,

$$X-C=\{x_0, x_1, \cdots, x_\omega, \cdots\}.$$

For $0\leqslant\beta\leqslant\alpha+1$, let $X_\beta=C\cup\{x_\gamma|\gamma<\beta\}$. Then $X_0=C$ and $X_{\alpha+1}=X$. Use transfin induction ([2] p. 116) to construct f as follows:

i) Let $f(x)=x$ for $x\in X_0=C$;

ii) Suppose that for some $\beta\leqslant\alpha+1$, f has been defined on X_γ for every $\gamma<\beta$, a (1) holds for $x\in X_\gamma$ and $y\in X$. If the ordinal number β is not a successor, i. e. the is no last element for all ordinal number smaller thrn β, than f is automatical defined on X_β and (1) holds for every $x\in X_\beta$ and $y\in X$. If β is a successor, i. e. t ordinal number $\beta-1$ exists, set

$$C_\beta=C\cap B(x_{\beta-1}, d(x_{\beta-1}, C))\cap(\bigcap_{0<\gamma<\beta-1} B(f(x_\gamma), d(x_\gamma, x_{\beta-1}))$$

$$\cap(\bigcup_{y\in X} B(y, \max\{d(y, C), d(y, x_{\beta-1})\})). \quad (\ $$

Since (1) holds for each $x\in X_{\beta-1}$ and $y\in X$, all of the GBI sets in the right ha side of (3) meet pairwise. So their intersection $C_\beta\neq\varnothing$. Take a point $y_\beta\in C_\beta$ and $f(x_{\beta-1})=y_\beta$. Then f is defined on X_β. By the hypothesis of induction and (3), it easy to check that $f|_{X_\beta}$ is still a contraction and (1) holds for each $x\in X_\beta$ a $y\in X$.

According to the previous two steps of transfinite induction, we can final construct a contraction f on $X_{\alpha+1}=X$ satisfying the requirement of Lemma 3. T completes the proof.

Theorem 1. *Let X be an injective metric space, and let $Y=X\times[0, \infty)$ h metric*

$$d_Y((x, s), (y, t))=\max\{d(x, y), |s-t|\}, \ (x, s) \text{ and } (y, t)\in Y.$$

Then for any nonempty GBI set C in X, there exists a unique contraction ψ: $Y\to$ satisfying

i) $\psi(x, t)=x$ *for* $t\geqslant d(x, C)$;

ii) $\psi(x, 0)\in C$ *for* $x\in X$;

iii) $\psi(\psi(x, t), s) = \psi(x, s)$ *for* $x \in X$ *and* $0 \leqslant s \leqslant t$.

Proof For any two nonnegative integers m and n, let

$$r_{mn} = n/2^m \text{ and } C_{mn} = B(C, x_{mn}).$$

By Lemma 2 and Lemma 3, there exists a contraction $g_{mn}: X \to C_{mn}$ satisfying

$$d(y, g_{mn}(x)) \leqslant \max\{d(y, x), d(y, C_{mn})\}, \forall x \text{ and } y \text{ in } X. \tag{4}$$

Since $C_{m+1,2n} = C_{mn}$, we may take $g_{m+1,2n} = g_{mn}$, $\forall m, n \geqslant 0$. Let G_{mn} be the composition of the countable contractions

$$G_{mn} = g_{mn} g_{m,n+1} g_{m,n+2} \cdots : X \to C_{mn},$$

which is well defined on X due to

$$g_{mN}(x) = x \text{ and } G_{mn}(x) = g_{mn} g_{m,n+1} \cdots g_{mN}(x)$$

for every $x \in X$ and $N \geqslant \max\{n, 2^m d(x, C)\}$. By (4), for x and $y \in X$,

$$\begin{aligned} d(y, G_{mn}(x)) &= d(y, g_{mn} G_{m,n+1}(x)) \\ &\leqslant \max\{d(y, C_{mn}), d(y, G_{m,n+1}(x))\} \\ &\leqslant \max\{d(y, C_{mn}), d(y, C_{m,n+1}), d(y, G_{m,n+2}(x))\} \\ &\leqslant \cdots \leqslant \max_{k \geqslant n} \{d(y, x), d(y, C_{mk})\} \\ &= \max\{d(y, x), d(y, C_{mn})\}. \end{aligned} \tag{5}$$

Define maps $\psi_m: Y \to X$, $m = 0, 1, 2, \cdots$, by

$$\psi_m(x, t) = G_{mn}(x) \text{ for } (x, t) \in X \times [r_{mn}, r_{m,n+1}) \subset Y.$$

Claim 1.

$$d(\psi_{m+1}(x, t), \psi_m(x, t)) \leqslant 2^{-m}. \tag{6}$$

In fact, let $t \in [r_{mn}, r_{m,n+1})$. Then (6) follows from

$$\begin{aligned} & d(G_{m+1,2n}(x), G_{mn}(x)) \\ &= d(g_{m+1,2n} G_{m+1,2n+1}(x), g_{mn} G_{m,n+1}(x)) \\ &= d(g_{mn} G_{m+1,2n+1}(x), g_{mn} G_{m,n+1}(x)) \\ &\leqslant d(G_{m+1,2n+1}(x), G_{m,n+1}(x)) \\ &= d(g_{m+1,2n+1} G_{m+1,2n+2}(x), G_{m,n+1}(x)) \\ &\leqslant \max\{d(G_{m,n+1}(x), C_{m+1,2n+1}), d(G_{m,n+1}(x), G_{m+1,2n+2}(x))\} \\ &\leqslant \max\{2^{-m-1}, d(G_{m+1,2n+2}(x), G_{m,n+1}(x))\} \leqslant \cdots \\ &\leqslant \max\{2^{-m-1}, d(G_{m+1,2N}(x), G_{mN}(x))\} \ (N \text{ large enough}) = 2^{-m-1}, \end{aligned}$$

and

$$\begin{aligned} & d(G_{m+1,2n+1}(x), G_{mn}(x)) \\ &\leqslant d(G_{m+1,2n+1}(x), G_{m+1,2n}(x)) + d(G_{m+1,2n}(x), G_{mn}(x)) \\ &\leqslant d(G_{m+1,2n+1}(x), g_{m+1,2n} G_{m+1,2n+1}(x)) + 2^{-m-1} \\ &\leqslant d(G_{m+1,2n+1}(x), C_{m+1,2n}) + 2^{-m-1} \leqslant 2^{-m}, \ \forall x \in X. \end{aligned}$$

Since X is complete ([1]), by Claim 1, $\{\psi_m\}$ converges uniformly to a map ψ: $Y \to X$ as $m \to \infty$, and

$$d(\psi(x, t), \psi_m(x, t)) \leqslant 2^{-m+1}. \tag{7}$$

Let us check that the map ψ satisfies the requirement of Theorem 1. For every

$x, y\in X$ and $0\leqslant s\leqslant t$, let $s\in[r_{mi}, r_{m,i+1})$ and $t\in[r_{mj}, x_{m,j+1})$. Then $j\geqslant i$, and

$$G_{mi}G_{mj}=G_{mi}. \tag{8}$$

By (5),

$$\begin{aligned}&d(\psi_m(x, s), \psi_m(y, t))\\&=d(G_{mi}(x), G_{mj}(y))=d(G_{mi}G_{mj}(x), G_{mj}(y))\\&\leqslant\max\{d(G_{mj}(y), G_{mj}(x)), d(G_{mj}(y), O_{mi})\}\\&\leqslant\max\{d(x, y), r_{mj}-r_{mi}\}\leqslant\max\{d(x, y), t-s+2^{-m}\}.\end{aligned}$$

Using (7) and letting $m\to\infty$, one has

$$d(\psi(x, s), \psi(y, t))\leqslant d((x, s), (y, t)). \tag{9}$$

So ψ is a contraction.

For $x\in X$ and $0\leqslant s\leqslant t$, let $s\in[r_{mi}, r_{m,i+1})$, $t\in[r_{mj}, r_{m,j+1})$. Using (7) and (9) one has

$$\begin{aligned}&d(\psi(\psi(x, t), s), \psi_m(\psi_m(x, t), s))\\&\leqslant2^{-m+1}+d(\psi(\psi(x, t), s), \psi(\psi_m(x, t), s))\\&\leqslant2^{-m+1}+d(\psi(x, t), \psi_m(x, t))\leqslant2^{-m+2}.\end{aligned}$$

By (8),

$$\psi_m(\psi_m(x, t), s)=G_{mi}G_{mj}(x)=G_{mi}(x)=\psi_m(x, s).$$

Thus

$$d(\psi(\psi(x, t), s), \psi(x, s))\leqslant6.2^{-m}\to0 \text{ as } m\to\infty,$$

which implies

$$\psi(\psi(x, t), s)=\psi(x, s).$$

i) and ii) of this theorem easily follow from (7) and the definitions of ψ, ψ_m and $G_{m,n}$.

Now we prove the uniqueness of ψ. Suppose one has another map $\psi': Y\to X$ satisfying the requirement of Theorem 1. Write

$$\psi_t(x)=\psi(x, t) \quad\text{and}\quad \psi'_t(x)=\psi'(x, t).$$

Given an $\varepsilon>0$. For any x and $y\in X$ with

$$d(x, y)<\varepsilon,$$

and $t\geqslant0$, take an integer number N so that

$$\{x, y\}\subset B(O, t+N\varepsilon+\varepsilon).$$

By iii) of this theorem

$$\psi(x, t)=\psi_t\psi_{t+\varepsilon}\cdots\psi_{t+N\varepsilon}(x),\ \psi'(y, t)=\psi'_t\psi'_{t+\varepsilon}\cdots\psi'_{t+N\varepsilon}(y).$$

Since ψ' and ψ are contractions,

$$\begin{aligned}&d(\psi'_{t+N\varepsilon}(y), \psi_{t+N\varepsilon}(x))\\&=d(\psi'_{t+N\varepsilon}(y), \psi'_{t+N\varepsilon}\psi_{t+N\varepsilon}(x))\\&\leqslant d(y, \psi_{t+N\varepsilon}(x))\\&=d(\psi_{t+N\varepsilon+\varepsilon}(y), \psi_{t+N\varepsilon}(x))\leqslant\max\{d(x, y), \varepsilon\}\leqslant\varepsilon.\end{aligned}$$

By the same reason one has

$$d(\psi'_{t+Ns-s}\psi'_{t+Ns}(y),\ \psi_{t+Ns-s}\psi_{t+Ns}(x)) \leqslant \max\{d(\psi'_{t+Ns}(y),\ \psi_{t+Ns}(x)),\ s\} = s.$$

nd so on, lastly, one has

$$d(\psi'(y,\ t),\ \psi(x,\ t)) \leqslant s. \tag{10}$$

$y=x$, (10) holds for any $s>0$. Hence

$$\psi'(x,\ t)=\psi(x,\ t).$$

his completes the proof of the theorem.

According to [3, p. 66], a free deformation retraction of a topological space X on a subspace A is a homotopy $\{h_t,\ t\in I\}$ on X, which satisfies the condition at h_1: $X\to X$ is the identity, h_0: $X\to X$ is a retraction upon A, and every mposition $h_s h_t = h_t h_s = h_s$ for $0\leqslant s\leqslant t$. By Theorem 1, one has the following eorem.

Theorem 2. *For an injective metric space X and a nonempty GBI set C in X, ere exists a free deformation retraction $\{h_t,\ t\in I\}$ of X upon C such that every h_t: $\to X$ is a contraction.*

Proof Let ψ be as in Theorem 1. For any $x\in X$ and $t\in I$, let

$$h_t(x)=\begin{cases}\psi\left(x,\ \tan\dfrac{\pi t}{2}\right), & \text{if } 0\leqslant t<1;\\ x, & \text{if } t=1.\end{cases}$$

is easy to check that such a homotopy $\{h_t,\ t\in I\}$ satisfies the requirement of heorem 2.

Remark. Since every point of X is a GBI set, Theorem 1.1 of [3] is a special se of Theorem 2 of this paper.

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