Chin. Ann, of Math.
10B (2) 1989

ON THE ERROR FUNCTION OF THE SQUARE-FULL INTEGERS

ZHAN TAO (展 涛)*

Abstract

Let $L(x)$ denote the number of square-full integers not exceeding x. It is proved in [1] that

$$L(x)\sim\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}+\frac{\zeta(2/3)}{\zeta(2)}x^{1/3} \text{ as } x\to\infty,$$

where $\zeta(s)$ denotes the Riemann zeta function. Let $\Delta(x)$ denote the error function in the asymptotic formula for $L(x)$. It was shown by D. Suryanaryana[2] on the Riemann hypothesis (RH) that

$$\frac{1}{x}\int_1^x|\Delta(t)|\,dt=O(x^{1/10+\varepsilon})$$

for every $\varepsilon>0$. In this paper the author proves the following asymptotic formula for the mean-value of $\Delta(x)$ under the assumption of R. H.

$$\int_1^T\frac{\Delta^2(t)}{t^{6/5}}\,dt\sim c\log T,$$

where $c>0$ is a constant.

§1. Introduction

A positive integer n is called square-full if $p\mid n$ implies $p^2\mid n$, where p is a pri
Let $l(n)$ denote the character function of the square-full integers, i. e.,

$$l(n)=\begin{cases}1, & \text{if } n \text{ is square-full,}\\ 0, & \text{otherwise.}\end{cases}$$

The error function of the square-full integers is defined by

$$\Delta(x)=L(x)-\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}-\frac{\zeta(2/3)}{\zeta(2)}x^{1/3}.$$

It was shown by P. T. Bateman and E. Grosswald that

$$\Delta(x)=o(x^{1/6})$$

(see [1]). Some further results about the behavior of $\Delta(x)$ were obtained on assumption of the Riemann hypothesis (referred to simply as RH). Suryanarayana and R. Sita Rama Chandra Rao[2] proved on the assumption of : that

Manuscript received Nevember 2, 1986.

* Department of Mathematics, Shandong University, Jinan, Shandong, China.

$$\Delta(x)=O(x^{13/81+\varepsilon})$$

or every $\varepsilon>0$.

Recently D. Suryanarayana [3] proved a mean value result for $\Delta(x)$ on the ssumption of RH

$$\frac{1}{x}\int_1^x |\Delta(t)|\,dt=O(x^{1/10+\varepsilon})$$

or every $\varepsilon>0$.

In this note we establish the following asymptotic formula for the mean value f $\Delta(x)$ under the assumption of RH

$$\int_1^x \frac{\Delta^2(t)}{t^{6/5}}\,dt\sim c_1\log x$$

s $x\to\infty$, where $c_1>0$ is a constant.

In fact, our method also could be used to deal with the function $\Delta_m(x)$ ntroduced in [3], and obtain

$$\int_1^x \frac{\Delta_m^2(t)}{t^{2m+6/5}}\,dt\sim c_m'\log x \quad \text{as } x\to\infty,$$

here $\Delta_m(x)$ is defined as

$$\Delta_m(x)=\sum_{n\le x} l(n)\,n^m-\frac{\zeta(3/2)}{\zeta(3)}\,\frac{x^{m+1/2}}{2m+1}-\frac{\zeta(2/3)}{\zeta(2)}\,\frac{x^{m+1/3}}{3m+1}$$

nd $\Delta_0(x)=\Delta(x)$.

§2. Preliminaries

To prove our theorem we need several almost well-known lemmas. We state hem below.

Lemma 1[4] *If $\sigma>1/2$, then*

$$\sum_{n=1}^{\infty}\frac{l(n)}{n^s}=\frac{\zeta(2s)\zeta(3s)}{\zeta(6s)}. \tag{1}$$

Lemma 2[5]. *On the assumption of RH, we have as $|t|\to\infty$,*

$$\zeta(\sigma+it)\ll|t|^{\varepsilon} \qquad \text{if} \quad \sigma\geqslant 1/2,$$

$$\zeta^{-1}(\sigma+it)\ll|t|^{\varepsilon} \qquad \text{if} \quad \sigma\geqslant 1/2,$$

$$\zeta(\sigma+it)\ll|t|^{1/2-\sigma+\varepsilon} \qquad \text{if} \quad \sigma\leqslant 1/2.$$

Lemma 3[5]. *If we assume RH, then*

$$\int_0^T |\zeta(\sigma+it)|^{2k}\,dt\sim T\sum_{n=1}^{\infty}\frac{d_n^2(n)}{n^{2\sigma}}, \qquad \text{as } T\to\infty,$$

olds for $\sigma>1/2$.

Using the method of Titchmarsh in [5], § 7.9, we could easily show the following lemma.

Lemma 4. *On the assumption of RH, we have*

$$\int_0^T |\zeta(\sigma+it)^{-4}dt \sim c(\sigma)T$$

as $T\to\infty$, *for* $\sigma>\frac{1}{2}$, *where* $c(\sigma)>0$ *is constant.*

Lemma 5[6]. *If* $f(t)\in L^2(-\infty,\infty)$, *then the Fourier transform* F *of* f *defined by*

$$F(x)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)e^{ixt}\,dt$$

exists and belongs to $L^2(-\infty,\infty)$. *Also we have*

$$\int_{-\infty}^{\infty}|F(x)|^2dx=\int_{-\infty}^{\infty}|f(t)|^2dt.$$

Lemma 6[5]. *Let*

$$f(s)=\sum_{n=1}^{\infty}a_n n^{-s}\quad (s=a+it)$$

be absolutely convergens for $\sigma>1$. *Then*

$$\sum_{n=1}^{\infty}\frac{a_n}{n^s}e^{-\delta n}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(w-s)f(w)\delta^{s-w}dw$$

for $\delta>0$, $c>\max(1, a)$.

Lemma 7[7]. *If* $\{a_n\}$ *is a sequence of complex numbers and*

$$\sum_{n=1}^{\infty}n|a_n|^2<+\infty,$$

then

$$\left|\sum_{m\neq n}\frac{a_m\bar{a}_n}{\log m/n}\right|=O\left(\sum_{n=1}^{\infty}n|a_n|^2\right).$$

Lemma 8[8]. *If we have*

$$\int_0^{\infty}f(x)e^{-xy}\,dx\sim\frac{c}{y}\,a.$$

as $y\to0$, *and* $f(x)\geqslant0$ *then*

$$\int_0^T f(x)dx\sim cT$$

as $T\to\infty$.

§3. Main Results

We first prove a mean value theorem for $\zeta(s)$.

Theorem 1. *If we assume the RH, then*

$$\int_0^T\left|\frac{\zeta(\sigma_1+2it)\zeta(\sigma_2+3it)}{\zeta(\sigma_3+6it)}\right|^2dt\sim c(\sigma_1\sigma_2,\ \sigma_3)T$$

for $\sigma_1,\sigma_2,\sigma_3>1/2$, *as* $T\to\infty$, *where* $c(\sigma_1,\sigma_2,\sigma_3)$ *is a positive continuous function* $(\sigma_1,\sigma_2,\sigma_3)$ *for* $\sigma_i>\frac{1}{2}$, $1\leqslant i\leqslant3$.

Proof Let

$$f(t)=\zeta(\sigma_1+2it)\zeta(\sigma_2+3it)/\zeta(\sigma_3+6it),$$

$$F(s)=\zeta(\sigma_1+2s)\zeta(\sigma_2+3s)/\zeta(\sigma_3+6s).$$

en

$$F(it)=f(t).$$

Since

$$F(s)=\sum_{n=1}^{\infty} a_n n^{-s}$$

ıbsolutely convergent for Re $s>1$, applying Lemma 6, we have

$$\sum_{n=1}^{\infty} a_n n^{-s} e^{-\delta n}=\frac{1}{2\pi i}\int_{2-\infty}^{2+i\infty}\Gamma(w-s)F(w)\delta^{s-w}\,dw. \tag{5}$$

king $s=it$ in (5), we obtain

$$\sum_{n=1}^{\infty} a_n n^{-it} e^{-\delta n}=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(w-it)F(w)\delta^{it-w}dw. \tag{6}$$

ving the contour to Re$w=-\alpha$,

$$\alpha=\frac{1}{2}\min\left(\frac{\sigma_1-1/2}{2},\ \frac{\sigma_2-1/2}{3},\ \frac{\sigma_3-1/2}{6}\right)>0.$$

pass the pole of $\Gamma(w-it)$ at $w=it$ with residue $F(it)=f(t)$, and the poles of $w)$ at $w=(1-\sigma_1)/2$ and $w=(1-\sigma_2)/3$, with residues $O(e^{-A|t|})$.

Hence

$$f(t)=\sum_{n=1}^{\infty} a_n n^{-it}e^{-\delta n}-\frac{1}{2\pi i}\int_{-\alpha-i\infty}^{-\alpha+i\infty}\Gamma(w-it)F(w)\delta^{it-w}dw+O(e^{-A|t|}). \tag{7}$$

; us call the first two terms on the right Z_1 and Z_2. Then

$$\int_{T/2}^{T}|z_1|^2dt=\frac{T}{2}\sum_{n=1}^{\infty}|a_n|^2e^{-2\delta n}+O\left(\left|\left(\sum_{m\neq n}\frac{a_m a_n e^{-(m+n)\delta}}{\log m/n}e^{it\log m/n}\right)\Big|_{T/2}^{T}\right|\right). \tag{8}$$

plying Lemma 7 to the O-term in (8) we have

$$\int_{T/2}^{T}|Z_1|^2dt=\frac{T}{2}\sum_{n=1}^{\infty}|a_n|^2e^{-2\delta n}+O(\sum_n n|a_n|^2e^{-2\delta n}).$$

ıce

$$a_n=\sum_{n_1^2n_2^3n_3^6=n} n_1^{-\sigma_1}n_2^{-\sigma_2}n_3^{-\sigma_3}\mu(n_3), \tag{9}$$

trivially have

$$a_n=O(1).$$

nce

$$\int_{T/2}^{T}|Z_1|^2dt=\frac{T}{2}\sum_{n=1}^{\infty}|a_n|^2e^{-2\delta n}+O(\delta^{-2}).$$

us we obtain

$$\int_{0}^{T}|Z_1|^2dt=T\Sigma(\delta)+O(\delta^{-2}\log T), \tag{10}$$

ere

$$\Sigma(\delta)=\sum_{n=1}^{\infty}|a_n|^2e^{-2\delta n}. \tag{11}$$

For Z_2, using Hölder's inequality, we obtain

$$Z_2\ll\delta^{\alpha}\left(\int_{-\infty}^{\infty}|\Gamma(w-it)|dv\right)^{1/2}\left(\int_{-\infty}^{\infty}|\Gamma(w-it)||F(w)|^2dv\right)^{1/2}.$$

The first integral is $O(1)$, while for $|t|\leqslant T$

$$\left(\int_{-\infty}^{-2T}+\int_{2T}^{\infty}\right)|\Gamma(w-it)|\,|F(w)|^2\,dv$$
$$=\left(\int_{-\infty}^{-2T}+\int_{2T}^{\infty}\right)|v|^{c}e^{-A|v\pm t|}dv=O(e^{-AT}),\quad w=-\alpha+iv.$$

Then for
$$w=-\alpha+iv,$$
$$\int_{T/2}^{T}|z_2|^2dt\ll\delta^{2\alpha}\int_{-2T}^{2T}|Fw^2\,dv\int_{T/2}^{T}(\Gamma(w-it)|\,dt+\delta^{2\alpha}$$
$$\ll\delta^{2\alpha}\int_{-2T}^{2T}|F(-\alpha+iv)|^2dv+\delta^{2\alpha}.$$

From (*) we know that
$$\sigma_1-2\alpha>\frac{1}{2},\quad \sigma_2-3\alpha>1/2,\quad \sigma_3-6\alpha>1/2.$$

Then by Lemma 3 and Lemma 4 we have
$$\int_{-2T}^{2T}|F(-\alpha+iv)|^2\,dv$$
$$\leqslant\left(\int_{-2T}^{2T}|\zeta(\sigma_1-2\alpha+2iv|^8\,dv\right)^{1/4}\cdot\left(\int_{-2T}^{2T}|\zeta(\sigma_2-3\alpha+3iv)|^8dv\right)^{1/4}$$
$$\cdot\left(\int_{-2T}^{2T}|\zeta(\sigma_3-6\alpha+6iv)^{-4}dw\right)^{1/2}\ll T.$$

Hence
$$\int_{T/2}^{T}|Z_2|^2dt\ll T\delta^{2\alpha},$$
$$\int_{0}^{T}|Z_2|^2dt\ll T\delta^{2\alpha}.$$

We show that
$$\sum_{n=1}^{\infty}|a_n|^2<+\infty.$$

Notice that (12) is also true for $\alpha=0$, we have
$$\int_0^T|f(t)|^2dt\ll T.$$

So we have
$$\int_0^T|Z_1|^2dt=T\Sigma(\delta)+O(\delta^{-2}\log T)$$
$$\ll\int_0^T f(t)|^2dt+\int_0^T|z_2|^2\,dt+1$$
$$\ll T+T\delta^{2\alpha}\ll T.$$

Then we get
$$\Sigma(\delta)=\Sigma(T^{-1/3})\ll 1,$$
$$\sum_{n\leqslant T^{1/3}}|a_n|^2\ll\sum_{n\leqslant T^{1/3}}|a_n|^2e^{-2\delta n}\ll\Sigma(T^{-1/3})\ll 1.$$

Letting $T\to\infty$ we obtain (14).

Let
$$c(\sigma_1,\ \sigma_2,\ \sigma_3)=\sum_{n=1}^{\infty}|a_n|^2,\tag{16}$$

$$\delta=T^{-1/3}.$$

rom (7)、(10)、(13), we have

$$\int_0^T |f(t)|^2 dt=c(\sigma_1,\ \sigma_2,\ \sigma_3)T+O(\delta^{-2}\log T)$$
$$+O(T\delta^{2\alpha})+O(T\delta^{\alpha})+O(T^{1/2}\delta^{-1+\alpha}\log T)$$
$$=cT+o(T)\quad \text{as } T\to\infty.$$

ıus (4) follows.

From the proof of (14) it is easy to see that

$$\sum_{n=1}^{\infty}|a_n|^2,\quad a_n=a_n(\sigma_1,\ \sigma_2,\ \sigma_3)$$

convergent uniformly for

$$\sigma_1>1/2+\varepsilon,\quad \sigma_2>1/2+\varepsilon,\quad \sigma_3>1/2+\varepsilon$$

r every $\varepsilon>0$. So $c(\sigma_1,\ \sigma_2,\ \sigma_3)$ is continuous for $(\sigma_1,\ \sigma_2,\ \sigma_3)$ when

$$\sigma_i>1/2,\ i=1,\ 2,\ 3.$$

Now we prove our mean value theorem for $\Delta(x)$.

Theorem 2. *On the assumption of RH we have*

$$\int_1^T \frac{\Delta^2(t)}{t^{6/5}}\,dt\sim c_1\log T \tag{17}$$

$T\to\infty$, *where $c_1>0$ is a constant.*

Proof Using Perron's sum formula, we deduce from Lemma 1

$$L(x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{\zeta(2w)\zeta(3w)}{\zeta(6w)}\,\frac{x^w}{w}\,dw, \tag{18}$$

›0, nonintegral.

Since the integer values of x form a denumerable set, we can ignore the havior of the sum-function at these points.

Moving the contour to Re $w=a$, we pass the pole at $w=1/2$ and $w=1/3$, with ɔ residues

$$\frac{\zeta(3/2)}{\zeta(3)}x^{1/2}\quad\text{and}\quad\frac{\zeta(2/3)}{\zeta(2)}x^{1/3}$$

pectively. Thus

$$\Delta(x)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)(a+it)}\,x^{a+it}\,dt. \tag{19}$$

Now we make the substitution $x=e^u$, so that the right side is a Fourier nsform. Then

$$e^{-au}\Delta(e^u)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)(a+it)}\,e^{iut}dt. \tag{20}$$

ɔm Lemma 2 we know that

$$\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)}\ll|t|^{1-5a+\varepsilon},\quad 1/10<a<1/6,$$

then

$$\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)(a+it)}\in L^2(-\infty,\infty),\text{ if } 1/10<a<1/6.$$

By Lemma 5, we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-2au}\Delta^2(e^u)\,du\\&=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left|\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)(a+it)}\right|^2 dt\\&=\frac{1}{\pi}\int_0^{\infty}\left|\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)(a+it)}\right|^2 dt.\end{aligned}$$

From the functional equation of $\zeta(s)$,

$$\zeta(s)=2^s\pi^{s-1}\sin\frac{\pi s}{2}\,\Gamma(1-s)\zeta(1-s),$$

we have

$$|\zeta(a+it)|\sim c_2(a)|t|^{1/2-a}|\zeta(1-a-it)|$$

as $|t|\to\infty$, for $0<a<1$.

Now we show

$$\int_0^{\infty}\left|\frac{\zeta(2a+2it)\zeta(3a+3it)}{\zeta(6a+6it)}\right|^2\frac{dt}{a^2+t^2}\sim\frac{c_3}{a-1/10}$$

as $a\to\frac{1}{10}+$.

From (22) we know that it is sufficient to prove

$$\int_1^{\infty}\frac{1}{t^{10a}}\left|\frac{\zeta(1-2a+2it)\zeta(1-3a+3it)}{\zeta(6a+6it)}\right|^2 dt\sim\frac{c_3}{a-1/10}.$$

Let $I(a)$ denote the above integral, and

$$\psi(t)=\int_1^t\left|\frac{\zeta(1-2a+2iu)\zeta(1-3a+3iu)}{\zeta(6a+6iu)}\right|^2 du.$$

For $1/10<a<4/30$, by Theorem 1 we have

$$\begin{aligned}I(a)&=\int_1^{\infty}\frac{1}{t^{10a}}\,d\psi(t)\\&=\frac{1}{t^{10a}}\psi(t)\Big|_1^{\infty}+10a\int_1^{\infty}\frac{\psi(t)}{t^{10a+1}}dt\\&=10a\int_1^{\infty}\frac{c(1-2a,\,1-3a,\,6a)}{t^{10a}}dt+10a\int_1^{\infty}\frac{E(t)}{t^{10a+1}}dt,\end{aligned}$$

where

$$E(t)=\psi(t)-c(1-2a,\,1-3a,\,6a)t=o(t),\ t\to\infty,$$

$$c(1-2a,\,1-3a,\,6a)\to c_4 \text{ as } a\to1/10.$$

Hence

$$I(a)=\frac{c_3}{a-1/10}+o\left(\frac{1}{a-1/10}\right),\ as\ a\to\frac{1}{10}+,\tag{26}$$

(24) follows immediately from (26).

o we have

$$\int_{-\infty}^{\infty} e^{-2au}\Delta^2(e^u)\,du \sim \frac{c_5}{a-1/10}, \text{ as } a\to 1/10+. \tag{27}$$

; is easy to see that

$$\int_{-\infty}^{0} e^{-2au}\Delta^2(e^u)\,du = 0.$$

'hen

$$\int_{0}^{\infty} e^{-2(a-1/10)u}e^{-u/5}\Delta^2(e^u)\,du \sim \frac{c_5}{a-1/10},\ a\to 1/10+. \tag{28}$$

y Lemma 8, we obtain

$$\int_0^T e^{-u/5}\Delta^2(e^u)\,du \sim c_1 T.$$

'hence a change of variable $t=e^u$ yields (12).

From Theorem 2 we immediately obtain two corollaries.

Corollary 1. *On the assumption of RH, we have for* $0<q\leqslant 2$,

$$\frac{1}{T}\int_1^T |\Delta(t)|^q\,dt = O(T^{q/10}(\log T)^{q/2}). \tag{29}$$

ı particular, taking $q=1$, *we have*

$$\frac{1}{T}\int_1^T |\Delta(t)|\,dt = O(T^{1/10}(\log T)^{1/2}), \tag{30}$$

hich is better than the result of [2].

Proof Let

$$u=\frac{2}{q},\quad v=\frac{2}{2-q}\quad (0<q<2),$$

$$\frac{1}{u}+\frac{1}{v}=1.$$

sing Hölder's inequality we have

$$\int_1^T |\Delta(t)|^q\,dt$$

$$\leqslant\left(\int_1^T\left(\frac{|\Delta(t)|^q}{t^{3q/5}}\right)^{2/q}dt\right)^{q/2}\cdot\left(\int_1^T t^{3q/5\cdot 2/(2-q)}\,dt\right)^{(2-q)/2}$$

$$\ll T^{1+q/10}(\log T)^{q/2}.$$

or $q=2$, (29) follows trivially from

$$\int_{T/2}^{T} |\Delta(t)|^2\,dt \ll \int_{T/2}^{T}\frac{|\Delta(t)|^2}{t^{6/5}}\,dt\cdot T^{6/5} \ll T^{6/5}\log T.$$

Corollary 2. *On the assumption of RH we have*

$$\Delta(x)=\Omega(x^{1/10}).$$

I would like to thank Professor Pan Chengdong for his helpful guidance and ıcouragement.

References

[1] Bateman, P. T. & Grosswald, E. On a theorem of Erdös and Szekeres, *Illinois J. Math.*, 2(1958), 88—

98.

[2] Suryanarayana, D. & Sita, R. Rama Chandra Rao, The distribution of square-full integers, *Ark. Mat.* *II* (1973), 195—201.

[3] Suryanarayana, D., On the order of the error function of the squale-full integers, II, *Period. Math. Hungar*, **14** (1983), 69—75.

[4] Golomb, S.,Powerful numbers, *Amer. Math. Monthly*, **77** (1970), 848—855.

[5] Titchmarsh, E. C., The theory of the Riemann Zeta function, Oxford University Press, 1951.

[6] Titchmarsh, E. C.,Introduction to the theory of Fourier integrals(second edition), Oxford University Press,1948.

[7] Vaughan, R. C., Hilbert's inequality, *J. London Math. Soc.*, **82** (1974), 73—81.

[8] Bellman, R., Analytic number theory, An introductioa, The Benjrmin/Cummings Publishing Comp Inc., 1980.