Chin. Ann. of Math.
10B (2) 1989

ON TFE THIRD CONJECTURE OF K.OGIUE*

TONG TAIHENG (董太亨)** SHUI NAXIANG (水乃翔)*

Abstract

In this paper, the authors prove following result:

Let M^n be a complete Bechner-Kaehler submanifold of complex dimension ($n\geqslant 4$) in a complex projective space $CP^{n+p}(1)$ of complex dimension $n+p$, endowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. If the sectional curvature K of M^n satisfies $K<1$, then codimension p of M^n is not less then $n(n+1)/2$.

§ 1. Introduction

Let M^n be a compact Kaehler submanifold of complex dimension n, immersed the complex projective space CP^{n+p} (1) of complex dimension $n+p$, endowed ith the Fubini-Study metric of constant holomorphic sectional curvature 1. The ird conjecture of K. Ogiue is as follows[1]:

If the sectional curvature K of M^n is greater than 0 and $p<n(n+1)/2$, then M^n a totally geodesic submanifold of $CP^{n+p}(1)$, i.e., there is a gap in the codimension M^n in $CP^{n+p}(1)$.

It is well known[2] that $M^n(n\geqslant 2)$ is a totally geodesic submanifold of $CP^{n+p}(1)$, and only if the sectional curvature of M^n satisfies

$$1/4\leqslant K\leqslant 1.$$

Therefore, we need only to consider the conjecture in the case

$$0<K\leqslant 1.$$

In this paper, we will prove the following theorem.

Theore. *Let M^n be a complete Bochner-Kaehler submanifold of complex mension $n(\geqslant 4)$ in a complex projective space $CP^{n+p}(1)$ of complex dimension $v+p$, lowed with the Fubini-Study metric of constant holomorphic sectional curvature 1. the sectional curvature K of M^n satisfies*

$$K<1, \tag{1}$$

m codimension p of M^n is not less than $\frac{n(n+1)}{2}$.

Manuscript received September 10, 1986.

* Projects Supported by the Science Fund of the Chinese Academy of Sciences.

** Department of Mathematncs, Hangzhou University, Hangzhou, Zhejiang, China.

Namely, in the upper special case, the conjecture of K. Ogiue is correct.

Using this theorem, we can obtain the Theorem 4.2 of [2] at once:

If $CP^n(c)$ is a Kaehler submanifold immersed in $CP^{n+p}(1)$ and if $p<n(n+1)/2$, then $CP^n(c)$ is totally geodesic in $CP^{n+p}(1)$.

§ 2. Preliminaries

Let M^n be a Kaehler submanifold of $CP^{n+p}(1)$. We choose a local fields orthonormal frames

$$e_1, \cdots, e_n; e_{n+1}, \cdots, e_{n+p}; e_1^*=Je_1, \cdots, e_n^*=Je_n, \cdots, e_{n+p}^*=Je_{n+p}$$

in $CP^{n+p}(1)$ in such a way that restricted to M^n, $e_1, \cdots, e_n$ are thangent to M^n.

The Fubini-Study metric of constant holomophic sectional curvature 1 $CP^{n+p}(1)$ and the induced metric on M^n will both be denoted by $\langle \quad \rangle$. The comp structure of $CP^{n+p}(1)$ and the induced complex structure on M^n will both be deno by J.

In th later, the range of indices is the following:

$$a, b, c, d, \cdots=1, \cdots, n: A, B, C, D, \cdots=1, \cdots, n, n+1, \cdots, n+p;$$
$$I, J, \cdots=1, \cdots, n; 1^*, \cdots, n^*.$$

The curvature tensor $\bar{K}$ of $CP^{n+p}(1)$ satisfies:

$$\bar{K}(\bar{X}, \bar{Y})\bar{Z}=\frac{1}{4}(\langle\bar{Y}, \bar{Z}\rangle\bar{X}-\langle\bar{X}, \bar{Z}\rangle\bar{Y}+\langle J\bar{Y}, \bar{Z}\rangle J\bar{X}-\langle J\bar{X}, \bar{Z}\rangle J\bar{Y}$$
$$+2\langle\bar{X}, J\bar{Y}\rangle J\bar{Z}),$$

where vector fields $\bar{X}$, $\bar{Y}$, $\bar{Z}$ are tangent to $CP^{n+p}(1)$.

M^n is supposed to be a Bochner-Kaehler manifold of $CP^{n+p}(1)$ if the curvatu tensor K of M^n satisfies

$$K(X, Y)Z=L(Y, Z)X-L(X, Z)Y+\langle Y, Z\rangle NX-\langle X, Z\rangle NY$$
$$+M(Y, Z)JX-M(X, Z)JY+\langle JY, Z\rangle PX$$
$$-\langle JX, Z\rangle PY-2(M(X, Y)JZ+\langle JX, Y\rangle PZ),$$

where

$$L(X, Y)=\frac{1}{2(n+2)}\left\{-\sum_I\langle K(X, e_I)Y, e_I\rangle+\frac{1}{4(n+1)}\langle X, Y\rangle e\right\},$$
$$M(X, Y)=+\sum_I L(X, e_I)\langle JY, e_I\rangle,$$

vector fields X, Y, $Z\cdots$ are tangent to M^n, NX, PX are defined by

$$\langle NX, Y\rangle=L(X, Y), \langle PX, Y\rangle=M(X, Y),$$

ρ is the scalar curvature of M^n

$$\rho=-\sum_{I,H}\langle K(e_H, e_I)e_H, e_I\rangle.$$

The Ricci tensor $\bar{R}$ for a Kaehler manifold satisfies

$$\bar{R}(J\bar{X}, J\bar{Y})=\bar{R}(\bar{X}, \bar{Y}). \tag{8}$$

By (4), (5), (8), we have

$$L(e_a, e_b)=L(e_{a^*}, e_{b^*}),\ L(e_a, e_{b^*})=-L(e_{a^*}, e_b), \tag{9}$$

$$M(e_a, e_b)=L(e_a, e_{b^*})=-L(e_{a^*}, e_b)=-L(e_b, e_{a^*})=-M(e_b, e_a), \tag{10}$$

$$M(e_a, e_b)=M(e_{a^*}, e_{b^*}), \tag{11}$$

$$M(e_a, e_{b^*})=-L(e_a, e_b)=-M(e_{a^*}, e_b)=-L(e_b, e_a)=-M(e_{b^*}, e_a). \tag{12}$$

Therefore

$$M(e_I, e_J)=-M(e_J, e_I),\quad L(e_I, e_J)=L(e_J, e_I). \tag{13}$$

The corresponding Gauss equation of M^n are

$$\langle K(X, Y)Z, W\rangle=\langle \overline{K}(X, Y)Z, W\rangle-\langle\sigma(X, Z), \sigma(Y, W)\rangle$$
$$+\langle\sigma(Y, Z), \sigma(X, W)\rangle. \tag{14}$$

The second fundamental form σ of the immersion satisfies relations

$$\sigma(JX, Y)=\sigma(X, JY)=J\sigma(X, Y). \tag{15}$$

In the later, we call

$$\sigma(e_I, e_J)\ (I, J=1, \cdots, n;\ 1^*, \cdots, n^*)$$

vectors of the normal curvature on M^n.

§ 3. Proof of the Theorem

We first prove the following lemma.

Lemma. *The $n(n+1)$ local fields of vectors of normal curvature $\sigma(e_a, e_b)$, $\sigma(e_a, e_b)^1$ $(1\leqslant a\leqslant b\leqslant n)$ are orthogonal.*

Proof From Gauss equation (14),

when $X=Z=e_a$, $Y=W=e_b$ $(a\neq b)$, we have

$$\langle\sigma(e_a, e_a), \sigma(e_b, e_b)\rangle-\|\sigma(e_a, e_b)\|^2=-1/4+L(e_a, e_a)+L(e_b, e_b); \tag{16}$$

when $X=Z=e_a$, $Y=W=e_{b^*}$,

$$\langle\sigma(e_a, e_a), \sigma(e_b, e_b)\rangle+\|\sigma(e_a, e_b)\|^2=\frac{1}{4}-L(e_a, e_a)-L(e_b, e_b). \tag{17}$$

(16)+(17), using (14), gives

$$\langle\sigma(e_a, e_a), \sigma(e_b, e_b)\rangle=0, \tag{18}$$

$$\|\sigma(e_a, e_b)\|^2=1/4-L(e_a, e_a)-L(e_b, e_b).$$

When $X=Z=e_a$, $Y=e_b$, $W=e_d$ $(a, b, d\neq)$, we have

$$-L(e_b, e_d)=+\langle\sigma(e_a, e_b), \sigma(e_a, e_d)\rangle-(\sigma(e_a, e_a), \sigma(e_b, e_d)\rangle; \tag{19}$$

when $X=Z=e_a$, $Y=e_{b^*}$, $W=e_{d^*}$.

Then

$$-L(e_b, e_d)=+\langle\sigma(e_a, e_b), \sigma(e_a, e_d)\rangle+\langle\sigma(e_a, e_a), \sigma(e_b, e_d)\rangle. \tag{20}$$

(19)+(20) gives

$$\langle\sigma(e_a, e_a), \sigma(e_a, e_d)\rangle=0, \tag{21}$$

$$\langle\sigma(e_a, e_b), \sigma(e_a, e_d)\rangle=-L(e_b, e_d). \tag{22}$$

When
$$X=e_a,\ Z=e_{a^*},\ Y=e_b,\ W=e_{d^*}(a,\ b,\ d\neq),$$
$$L(e_b,\ e_d)=-M(e_b,\ e_{d^*})=-\langle\sigma(e_a,\ e_a),\ \sigma(e_b,\ e_d)\rangle+\rangle\sigma(e_b,\ e_a),\ \sigma(e_a,\ e_d)\rangle; \qquad (23)$$
when
$$X=e_a,\ Z=e_{a^*},\ Y=e_{b^*},\ W=e_d,$$
$$L(e_b,\ e_d)=+M(e_{b^*},\ e_b)=\langle\sigma(e_a,\ e_a),\ \sigma(e_b,\ e_d)\rangle+\rangle\sigma(e_d,\ e_a),\ \sigma(e_a,\ e_d)\rangle. \qquad (24)$$
(23) + (24) gives
$$\langle\sigma(e_a,\ e_a),\ \sigma(e_b,\ e_d)\rangle=0, \qquad (25)$$
$$\langle\sigma(e_a,\ e_b),\ \sigma(e_a,\ e_d)\rangle=L(e_b,\ e_d). \qquad (2$$
From (22), (26), We obtain
$$L(e_b,\ e_d)=0. \qquad (2$$
Similarly changing basic vectors of the orthogonal ennuple, we have
$$L(e_{b^*},\ e_d)=0. \qquad (2$$
From (14), when $X=Z=e_a,\ Y=e_{a^*},\ W=e_{d^*}(a\neq d)$, we obtain
$$\langle\sigma(e_a,\ e_a),\ \sigma(e_a,\ e_d)\rangle=L(e_a,\ e_d)=0; \qquad (2$$
when $X=Z=e_a,\ Y=e_{a^*},\ W=e_d\ (a\neq d)$,
$$\langle\sigma(e_a,\ e_a),\ \sigma(Je_a,\ e_d)\rangle=0.$$
In (14), setting $X=Z=e_a,\ Y=e_b,\ W=e_{d^*}$, *and* $X=Z=e_a,\ Y=e_{b^*},\ W=e_d$, separate we obtain
$$\langle\sigma(e_a,\ e_a),\ \sigma(Je_b,\ e_d)\rangle=0,$$
$$\langle\sigma(e_a,\ e_b),\ \sigma(Je_a,\ e_d)\rangle=0. \qquad ($$
Finally, spreading out
$$\left\langle\sigma\left(\frac{Je_a+e_c}{\sqrt{2}},\ \frac{Je_a+e_c}{\sqrt{2}}\right),\ \sigma\left(\frac{e_b+e_d}{\sqrt{2}},\ \frac{e_b+e_d}{\sqrt{2}}\right)\right\rangle=0$$
we have
$$\langle\sigma(Je_a,\ e_b),\ \sigma(e_c,\ e_d)\rangle=0 \quad (a,\ b,\ c,\ d\neq). \qquad ($$
Similarly from
$$\left\langle\sigma\left(\frac{e_a+e_c}{\sqrt{2}},\ \frac{e_a+e_c}{\sqrt{2}}\right),\ \sigma\left(\frac{e_b+e_d}{\sqrt{2}},\ \frac{e_b+e_d}{\sqrt{2}}\right)\right\rangle=0,$$
we get
$$\langle\sigma(e_a,\ e_b),\ \sigma(e_c,\ e_d)\rangle=0\ (a,\ b,\ c,\ d\neq), \qquad ($$
by (18). (21), (26), (28)⋯(32) we come to the conclusion: All vectors of nor curvature $\sigma(e_a,\ e_b)$, $J\sigma(e_a,\ e_b)$ are orthogonal.

We begin to prove the theorem now.

From (14) and (2), the sectional curvature $K(e_I,\ e_J)$ and the holomor sectional curvature $H(e_I)$ of M^n have the relations respectively
$$K(e_I,\ e_J)=\frac{1}{4}(1+3\langle e_I,\ Je_J\rangle)+\langle\sigma(e_Ie_I),\ \sigma(e_J,\ e_J)\rangle-\|\sigma(e_I,\ e_J)\|^2 \qquad ($$
and
$$H(e_I)=1-2\|\sigma(e_I,\ e_I)\|^2.$$
By (1)
$$K(e_{a^*},\ e_a)=1-2\|\sigma(e_I,\ e_I)\|^2<1, \qquad (35)$$

namely

$$0<\|\sigma(e_a, e_a)\|^2=\|\sigma(Je_a, Je_a)\|^2=\|\sigma(e_a, Je_a)\|^2. \tag{36}$$

Since

$$\left\langle\sigma\left(\frac{e_a+e_b}{\sqrt{2}}, \frac{e_a+e_b}{\sqrt{2}}\right), \sigma\left(\frac{e_a-e_b}{\sqrt{2}}, \frac{e_a-e_b}{\sqrt{2}}\right)\right\rangle=0 \quad (a\neq b),$$

we have

$$\|\sigma(e_a, e_b)\|^2=\frac{1}{2}(\|\sigma(e_a, e_a)\|^2+\|\sigma(e_b, e_b)\|^2)>0. \tag{37}$$

From (15), (36), (37), we reach the conclusion: Vectors of normal curvature $\sigma(e_a, e_b)$, $J\sigma(e_a, e_b)$ are non-null. Using the lemma, we know that the $n(n+1)$ vectors of $\sigma(e_a, e_b)$, $J\sigma(e_a, e_b)$ are independent, i. e., the theorem are proved.

References

[1] Ros, A., A characterization of seven compact Kaehler submanifold by holomorphic pinching, *Ann. of Math.*, **121** (1985), 377—382.

[2] Ogiue, K., Differential geometry of Kaehler submanifolds, *Adv, Math.*, **13** (1974), 73—144.

[3] Houh, Chornghsi, Totally real submanifolds in a Bochner-Käehler manifold, *Tensor. N. S.*, **32** (1978), 293—296.