Chin. Ann. of Math.
10B (2) 1989

EXISTENCE OF LIMIT CYCLES FOR QUADRATIC SYSTEMS WITH ONE INFINITE SINGULAR POINT

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Abstract

This paper discusses the existence of limit cycles for quadratic systems with one infinite singular point. In particular, the author gives necessary and sufficient conditions for the existence of a unique limit cycle of a class of quadratic systems with an integral line under certain conditions. As a special consequence, all the results of [6] and [7] can be implied by means of affine transformations. The method used here is much simpler than those of [6] and [7].

We know that any quadratic systems with a focus can be changed into t following form

$$\dot{x}=-y+ax^2+bxy,\quad \dot{y}=x+\delta y+lx^2+mxy+ny^2 \qquad ($$

by means of linear transformations. And the following result is proved in pap [1].

Theorem A[1]. *All solutions of the system* (1) *are bounded for* $t\geqslant 0$ *iff one of t following conditions holds*:

(i) $n=0,\ a+m=0,\ a^2+bl=0,\ b^2+a(a-b\delta)=0,\ ab<0$;

(ii) $n=0,\ b(b\delta+m)<0,\ (a-m)^2+4bl<0$;

(iii) $n=b\delta+m=0,\ b+l+a\delta=0,\ ab\leqslant 0,\ (a-m)^2+4bl<0$.

Under the conditions of the above theorem we have $b\neq 0$. Thus without loss generality, we can suppose $b=-1$, and consider

$$\dot{x}=-y+ax^2-xy,\ \dot{y}=x+\delta y+lx^2+mxy+ny^2. \qquad ($$

System (2) can be transformed into the equation on the region $x>-1$ ([1]):

$$\dot{x}=y-F(x),\ \dot{y}=-g(x), \qquad ($$

where

$$F(x)=\int_0^x f(x),\ f(x)=\psi(x)(1+x)^{n-2},\ g(x)=x\varphi(x)(1+x)^{2n-3},$$

$$\psi(x)=-(a+m+2na)x^2-(\delta+2a+m)x-\delta,$$

$$\varphi(x)=(a^2n+l+am)x^3+(am+1+a\delta+2l)x^2+(l+2+a\delta)x+1.$$

Manuscritp received December 9, 1986. Revised April 9, 1987.

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Put

$$\varphi_0(x) = (am+l)x^2 + (l+1+a\delta)x + 1.$$

It is easy to see that

$$\varphi(x) = a^2nx^3 + (1+x)\varphi_0(x).$$

Theorem 1. *Suppose in* (2) *that* $n=0$, $0<\delta/m<1$, $(a-m)^2<4l$. *Let*

$$\Delta = (l+1+a\delta)^2 - 4(am+l).$$

(i) *If* $\Delta<0$, *system* (2) *has a unique limit cycle.*

(ii) *Assume, in addition, that* $ma \geqslant 0$, $a(2m-\delta)+1 \leqslant 1$. *Then* (2) *has a unique (resp., at least one) limit cycle if* $\Delta=0$ (*resp.*, $\Delta>0$).

Proof We can suppose $m>0$ (otherwise, change the signs of y and t in (2)). If $\Delta<0$, (2) has a unique singular point $O(0, 0)$, and it is unstable. We also note that all solutions of (2) are bounded for $t \geqslant 0$ by Theorem A. Thus (2) has at least one limit cycle. The uniqueness of limit cycles follows from [2]. If, in addition, $a \geqslant 0$, $a(2m-\delta)+l \leqslant 1$, then (2) has only one singular point on the region $x>-1$, and $\dot{x}|_{x=-1} \geqslant 0$. Thus (2) has at least one limit cycle. For $\Delta=0$ the uniqueness follows from Theorem C[3] (which says that any bounded quadratic systems with at most two singular points possess at most one limit cycle). The proof is complete.

Now we use the following special case of [4] to discuss the existence of limit cycles for the system (2).

Theorem B[4]. *Consider the system*

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{4}$$

where $F, g \in C^1(x_2, x_1)$, $x_2<0<x_1$. *Suppose that the following conditions are satisfied:*

(1) $O(0, 0)$ *is an unstable focus or node point of* (4);

(2) $xg(x)>0$ *for* $x \in (x_2, x_1)$ *and* $x \neq 0$;

(3) $G(x_2+0) = \infty = G(x_1-0)$, *where*

$$G(x) = \int_0^x g(x)\,dx.$$

Let $F_i(Z) = F(x_i(Z))$, $i=1, 2$, *where* $x_1(Z) \geqslant 0 \geqslant x_2(Z)$ *are inverse functions of* $Z = G(x)$.

(I) *If for some* $Z_0>0$,

$$\int_0^{Z_0} (F_1(Z) - F_2(Z))\,dZ > 0,$$

and $F_1(Z) \geqslant F_2(Z)$, $F_1(Z) > -a\sqrt{Z}$, $F_2(Z) < a\sqrt{Z}$ *for* $Z \geqslant Z_0$, *where* $0<a<\sqrt{8}$, *then* (4) *has an odd number of limit cycles.*

(II) *If for some* $Z_0>0$,

$$\int_0^{Z_0} (F_1(Z) - F_2(Z))\,dZ < 0,$$

and $F_1(Z) \leqslant F_2(Z)$, $F_1(Z) < a\sqrt{Z}$, $F_2(Z) > -a\sqrt{Z}$ *for* $Z \geqslant Z_0$, *then* (4) *has an even number of limit cycles.*

We shall consider the system (2) as according that $a=0$, $n=0$ and $an\neq 0$. First for the case $a=0$ we have the following theorem.

Theorem 2. *Suppose* $a=0$, $0<l<1$, $-1<n\leqslant 0$, and $m^2<4l(n+1)$. *Then*

(i) *the system* (2) *has at most one limit cycle, and has at least two singular points* $O(o, o)$ *and* $B(-1/l, 0)$;

(ii) *if, in addition,* $l+n>0$, *then* (2) *has a unique limit cycle around* 0 *iff* $m\neq 0$, *and*

$$0<\delta/\mathrm{m}<1-\sqrt{\frac{n(l-1)}{l(n+1)}};$$

(iii) *if in addition,* $l<n+1$ *then* (2) *has a unique limit cycle around* B *iff* $m\neq 0$, *and*

$$1+\sqrt{\frac{n(l-1)}{l(n+1)}}<\delta/m<1/l.$$

Proof Since $a=0$, we have

$$f(x)=-(mx+\delta)(1+x)^{n-1},\quad g(x)=x(lx+1)(1+x)^{2n-1}.$$

It follows from $0<l<1$ that $xg(x)>0$ for $x>-1$, $x\neq 0$. By putting

$$Z=G(x)=\int_0^x g(x)\,dx$$

we get two functions of Z, $F_i(Z)$, $i=1, 2$. Clearly, conclusion (i) is true. For (ii) and (iii) we suppose $m\geqslant 0$ as in Theorem 1. Now we compute the limits

$$\lim_{Z\to\infty}\frac{F_1(Z)}{\sqrt{Z}}=\lim_{x\to\infty}\frac{F(x)}{\sqrt{G(x)}}\quad\text{and}\quad\lim_{Z\to\infty}\frac{F_2(Z)}{\sqrt{Z}}=\lim_{x\to-1}\frac{F(x)}{\sqrt{G(x)}}.$$

It is straightforward that

$$F(x)=-\frac{m}{n+1}x^{n+1}+o(x^{n+1}),\quad G(x)=\frac{1}{2n+2}x^{2n+2}+o(x^{2n+2})$$

when $x\to\infty$. Thus

$$\lim_{x\to\infty}\frac{F(x)}{\sqrt{G(x)}}=-\frac{m\sqrt{2}}{\sqrt{l(n+1)}}=\lambda$$

and

$$-\sqrt{8}<\lambda\leqslant 0 \tag{5}$$

from $m\geqslant 0$, and $m^2<4l(n+1)$.

We also have

$$F(x)=\begin{cases}\dfrac{m-\delta}{n}(1+x)^n+o((1+x)^n), & n<0,\ m\neq\delta;\\ (m-\delta)\ln(1+x)+o(\ln(1+x)), & n=0,\ m\neq\delta;\\ -\dfrac{\delta(1+x)^{n+1}}{n+1}+o((1+x)^{n+1}), & m=\delta,\end{cases}$$

$$G(x)=\begin{cases}\dfrac{l-1}{2n}(1+x)^{2n}+o((1+x)^{2n}), & n<0;\\ (l-1)\ln(1+x)+o(\ln(1+x)), & n=0,\end{cases}$$

when $x \to -1+o$. Hence for $-1<n\leqslant 0$

$$\lim_{x\to -1+o}\frac{F(x)}{\sqrt{G((x)}}=\begin{cases}-\dfrac{\sqrt{2}\,(m-\delta)}{\sqrt{n(l-1)}}=\mu, & n<0;\\ -\infty, & n=0,\ m>\delta;\\ 0, & n=0,\ m=\delta.\end{cases}$$

Clearly,

$$\mu\leqslant 0<\sqrt{8}\ \text{for}\ n<0,\ m\geqslant\delta. \tag{6}$$

Denote

$$D=1-\sqrt{\frac{n(l-1)}{l(n+1)}}.$$

Then it is easy to show that

$$\mu<\lambda\ \text{iff}\ \delta<mD\ \text{for}\ n<0\ \text{and}\ \delta<m,$$
$$\mu\geqslant 0>\lambda\ \text{for}\ n<0\ \text{and}\ \delta\geqslant m>0.$$

And thus we obtain easily

$$\lim_{Z\to\infty}(F_1(Z)-F_2(Z))=\begin{cases}+\infty, & \delta<mD,\\ -\infty, & \delta>mD.\end{cases} \tag{7}$$

Note that 0 is unstable for $\delta>0$. It follows from formulae (5) to (7) and Theorem B that (2) possesses a unique limit cycle around 0 for $m\neq 0$, $0<\delta<mD$, and no limit cycles around 0 for $mD<\delta\leqslant m$, or $n=0$, $\delta=m$.

We note that

$$\begin{vmatrix} P & Q \\ \dfrac{\partial P}{\partial\delta} & \dfrac{\partial Q}{\partial\delta}\end{vmatrix}=-g(x)\frac{\partial F}{\partial\delta}=\left.\begin{cases}-\dfrac{g(x)}{n}(1-(1+x)^n), & n<0\\ g(x)\ln(1+x), & n=0\end{cases}\right\}>0(x\neq 0),$$

where $\qquad P(x,y)=y-F(x),\quad Q(x,y)=-g(x).$

Then by the theory of rotation vector fields, the system (2) has no limit cycles for $\delta\leqslant 0$ or $\delta\geqslant mD$. Thus conclusion (ii) follows.

Now making the change $x:=-x-1/l$, $y:=y$ we get from (2)

$$\dot{x}=-\frac{1-l}{l}y-xy,\ \dot{y}=x+\frac{\delta l-m}{l}y+lx^2-mxy+ny^2.$$

Then putting

$$x:=\frac{l}{1-l}x,\ y:=\sqrt{\frac{l}{1-l}}\,y,\ t:=\sqrt{\frac{1-l}{l}}\,t$$

we have form the above system

$$\dot{x}=-y-xy,\ \dot{y}=x+\frac{\delta l-m}{\sqrt{l(1-l)}}y+(1-l)x^2-m\sqrt{\frac{1-l}{l}}\,xy+ny^2$$

which has the same form as the system (2). Then by a simple computation conclusion (iii) follows from (ii). The proof is complete.

Similarly, we can prove the following theorem.

Theorem 3. *Suppose that* $a=0$, $0<l<1$, $-1<n<0$, $(\delta-m)^2<4n(l-1)$.

(1) *If* $l+n<0$, *then the system* (2) *has a unique limit cycle around* 0 *iff* $m\neq 0$, $1-\theta<\delta/m<0$, *where*

$$\theta=\sqrt{\frac{n(l-1)}{l(n+1)}}.$$

(2) *If* $l>1+n$, *then* (2) *has a unique limit cycle around B iff* $m\neq 0$, $1/l<\delta/m<1+\theta$.

In fact, if we suppose $m\leqslant 0$, then

$$\lambda\geqslant 0,\ 0\leqslant\mu<\sqrt{8}$$

for $m\leqslant\delta$, and (7) remains true. Thus the theorem follows in the same way.

It is easy to prove that $l+n=0$ iff $D=0$, and $(l+n)D>0$ as long as $l+n\neq$ Then from (7), the system (2) has no limit cycle if $l+n=0$. Hence by Theorems and 3 we have immediately the following corallary.

Corollary 1. *Suppose* $a=0$, $0<l<1$, $-1<n<0$, $m^2<4l(n+1)$, *and* $(\delta-m)^2<$ $4n(l-1)$. *Then*

(i) (2) *has a unique limit cycle around O iff* $m\neq 0$ *and* δ/m *lies strictly betwee* 0 *and* $1-\theta$;

(ii) (2) *has a unique limit cycle around B iff* $m\neq 0$ *and* δ/m *lies strictly betwee* $1/l$ *and* $1+\theta$.

Under the conditions of Corollary 1, (2) has two infinite separatrix cycles denoted by L_1 and L_2, which consist of the line $x=-1$ and the right or the left hal of the equator respectively. From Theorems 2 and 3, Corollary 1 and formula (7 we have easily the following corollary.

Corollary 2. *Suppose* $a=0$, $0<l<1$, $-1<n<0$, $m^2<4l(n+1)$, *and* $(\delta-m)^2<$ $4n(l-1)$. *Denote*

$$\theta=\sqrt{\frac{n(l-1)}{l(n+1)}}$$

as before.

(I) *The separatrix cycle* L_1 *is negatively (resp., positively) asympotctically stabl iff* $\delta<m(1-\theta)$ *or* $\delta=m(1-\theta)<0$ *(resp.,* $\delta>m(1-\theta)$ *or* $\delta=m(1-\theta)>0$*). All th orbits contained in the interior of* L_1 *are closed iff* $\delta=0$ *and* $m(l+n)=0$.

(II) *The separatrix cycle* L_2 *is negatively (resp., positively) asympototically stable iff* $\delta<m(1+\theta)$ *or* $\delta=m(1+\theta)$ *and* $\delta l<m$ *(resp.,* $\delta>m(1+\theta)$ *or* $\delta=m(1+\theta)$ *and* $\delta l>m$*) All the orbits contained in the interior of* L_2 *are closed iff*

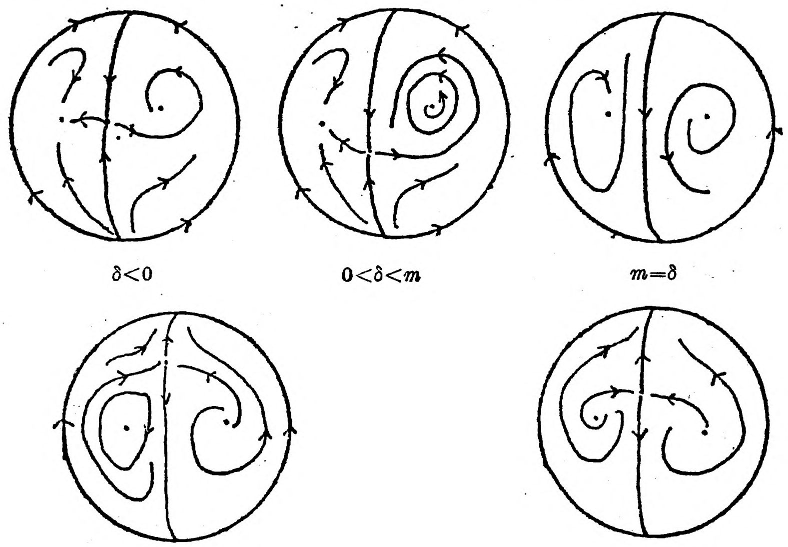
$$\delta l=m,\ m(1+n-l)=0.$$

In [6] K. R. Zhou corrected a mistake in Cerkas's work [5]. Some results of [6] were reobtained by C. A. Holmes[7] recently by a simple method, and the problem left over of [6] was solved by [7] completely. Obviously, our Corollary 1 is eqivalent to Theorems 2 and 2′ of [7], and Corollary 2 contains Theorem 1 of [7] by means of affine transformations. And all the results of [6] can also be implied from these two corollaries.

Under the conditions of Theorem 2 we can analyze the global orbit structure of (2) as δ varies. For the case $n=0$, (2) has only one singular point

$$A\left(-1, \frac{l-1}{m-\delta}\right)$$

on the line $x=-1$, and it is easy to get the following Figure 1 for $m>0$.



$\delta<0$ $0<\delta<m$ $m=\delta$

$m<\delta<m/l$ $\delta\geqslant m/l$

Fig. 1 $m>0$ $n=0$

Suppose $-1<n<0$, $-n<l<1+n$, and $m>0$ as before. We also have two cases to consider according as $m^2\leqslant 4n(l-1)$ or not. For example, we take the case that $m^2>4n(l-1)$. If $0<\delta<m-2\sqrt{n(l-1)}$, (2) has only two singular points $A_i(-1, y_i)$ on the line $x=-1$, where

$$y_i=(m-\delta\pm\sqrt{(m-\delta)^2-4n(l-1)})/(2n),\ y_1<y_2<0.$$

It is easy to see that A_1 is a node, and A_2 a saddle. When

$$\delta=m-2\sqrt{n(l-1)},\quad A_1=A_2$$

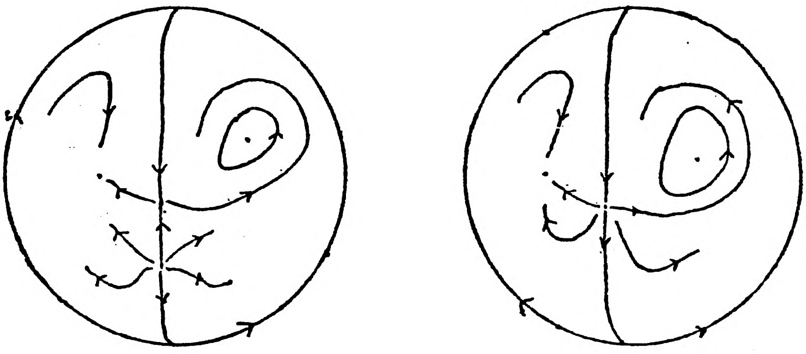
is a saddle-node point. Then we have Figure 2. If

$$\delta_0=m+2\sqrt{n(l-1)}\geqslant m/l,$$

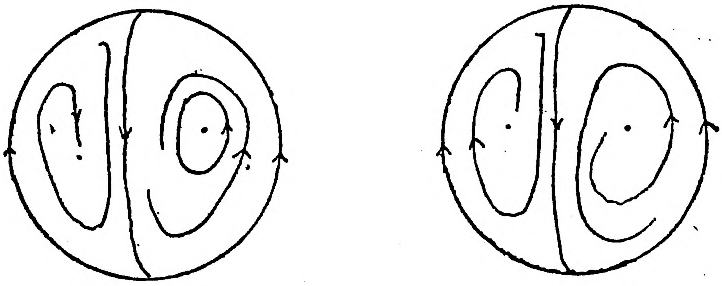
then continuing Figure 2, we have Figure 3. If $\delta_0<m/l$, then continuing Figure 2 we have Figure 3′.

The following theorem treat the case $l=1$. We first prove Lemma 1.

Lemma 1. *Consider the system* (4). *Suppose that* F, $g\in C^1(x_2, x_1)$, $x_2<0<x_1$, *and the following are satisfied*:

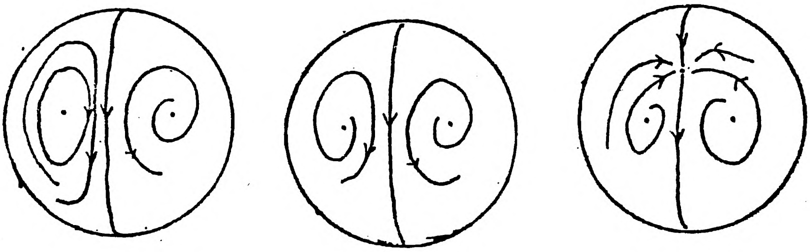


$0<\delta<m-2\sqrt{n(l-1)}$ $\delta=m-2\sqrt{n(l-1)}$



$m-2\sqrt{n(l-1)}<\delta<m(1-\theta)$ $m(1-\theta)\leqslant\delta\leqslant m(1+\theta)$

Fig. 2



$m(1-\theta)<\delta<m/l$ $\delta=m/l<\delta_0$ $\delta=m/l=\delta_0$

Fig. 3 ($\delta_0\geqslant m/l$)

(1) *$O(0, 0)$ is an unsatble focus or node of (4).*

(2) $xg(x)>0$, $x\in(x_2, x_1)$, $x\neq0$.

(3) $G(x_1-0)=\infty$, $G(x_2+0)<\infty$, *where*

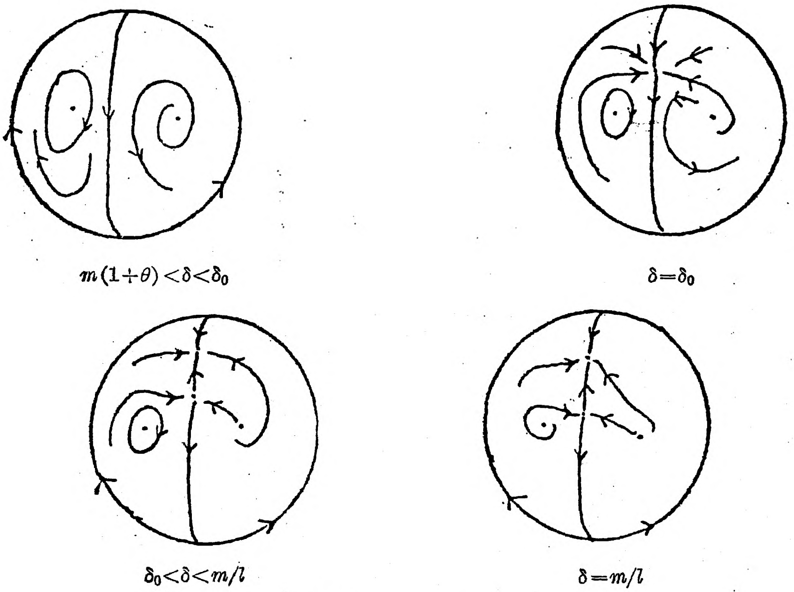
$$G(x)=\int_0^x g(x)\,dx.$$

(4) $\lim\limits_{x\to x_2+0} F(x)=-\infty$, $\lim\limits_{x\to x_1-0} F(x)/\sqrt{G(x)}=\lambda$, *and* $\lambda>-\sqrt{8}$.

Then the system (4) has at least one limit cycle.

Proof Since there is $x_0\in(x_2, 0)$ such that

$$F(x)<-\sqrt{8G(x)}$$



$m(1+\theta)<\delta<\delta_0$ $\delta=\delta_0$

$\delta_0<\delta<m/l$ $\delta=m/l$

Fig. 3′ ($\delta_0<m/l$)

or $x\in(x_2, x_0)$, we can prove in a similar manner to the proof of Lemma 2.3 of [1] hat there is a monotone negative semi-orbit L^- which lies above the curve $y=F(x)$ n the region $x_2<x<0$. From the condition (4), L^+ must cross the negative y-axis, nd then return to the positive y-axis, and so on. Thus the positive limit set of L is closed orbit of (4), and the proof is complete.

Theorem 4. *Suppose $a=0$, $l=1$, $-1<n\leqslant 0$, $m^2<4(n+1)$. Then the system* (2) *ossesses a unique limit cycle (around O) iff $m\neq 0$, and $0<\delta/m<1$.*

Proof Still suppose $m\geqslant 0$. Since $l=1$, we have $g(x)=x(1+x)^{2n}$. And as before,

$$\lim_{x\to\infty}\frac{F(x)}{\sqrt{G(x)}}=-\frac{m\sqrt{\ }}{\sqrt{n+1}}=\lambda,\quad -\sqrt{8}<\lambda<0.$$

f $2n+1\leqslant 0$, then

$$G(x)=\begin{cases}-\dfrac{(1+x)^{2n+1}}{2n+1}+o((1+x)^{2n+1}), & 2u+1<0,\\ -\ln(1+x)+o(\ln(1+x)), & 2n+1=0,\end{cases}$$

nd $F(x)$ is the same as in the proof of Theorem 2 when $x\to-1+0$. Thus we have

$$\lim_{x\to-1+0}\frac{F(x)}{\sqrt{G(x)}}=\begin{cases}-\infty, & \delta<m,\ 2n+1\leqslant 0,\\ 0, & \delta=m,\ 2n+1\leqslant 0,\end{cases}$$

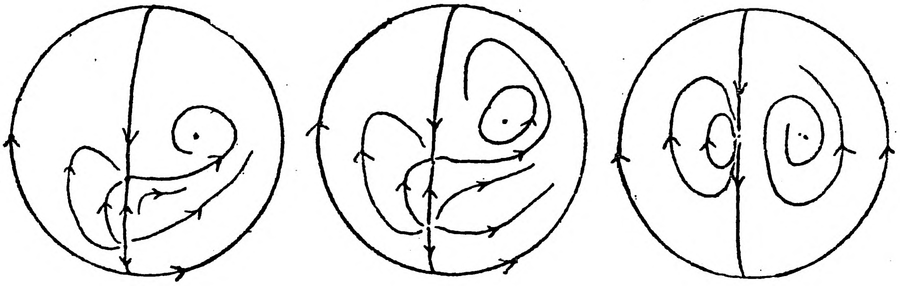
nd similarly, with Theorem 2, Theorem 4 follows from Theorem B. If $2n+1>0$, ıen $G(-1+0)<\infty$, and

$$\lim_{x\to-1+0}F(x)=-\infty$$

for $\delta<m$. Thus from Lemma 1, the system (2) has a unique limit cycle for $0<\delta<m$. If $\delta=m$, then $f(x)=-m(1+x)^n<0$ for $x>-1$, in this case (3) has no periodic orbit.

Hence from the theory of rotation vector fields, the system (2) has no limit cycle for $\delta \leqslant 0$ or $\delta \geqslant m$. This finishes the proof.

Under the conditions of Theorem 4, we have the Figure 4 below. (For the behavour of the singular point $A(-1, 0)$ when $\delta = m$, use Theorem 7.2, Chapter 2 of [8].)



$\delta \leqslant 0$ $\qquad$ $0<\delta<m$ $\qquad$ $\delta = m$

Fig. 4 $(-1<n<0)$

The global orbit structure of (2) for $n=0$ is easy to give.

We now treat the case that $a \neq 0$ but $n=0$.

Theorem 5. *Suppose* $n=0$, $ma>0$, $l+am<1$ *and* $(m-a)^2<4l$. *Then the syste* *(2) has at least one limit cycle if*

$$\max\left\{0,\ 2-\frac{1-l}{am}\right\}<\delta/m\leqslant 1.$$

Proof Suppose $m>0$. We have

$$f(x)=\psi(x)(1+x)^{-2}, \quad g(x)=x\varphi_0(x)(1+x)^{-2},$$

where

$$\psi(x)=-(a+m)x^2-(\delta+2a+m)x-\delta,$$

$$\varphi_0(x)=(am+l)x^2+(l+1+a\delta)x+1.$$

A simple computation gives

$$\lim_{x\to\infty}\frac{F(x)}{\sqrt{G(x)}}=-\frac{\sqrt{2}\,(a+m)}{\sqrt{l+am}}=\lambda,$$

$$\lim_{x\to-1+0}\frac{F(x)}{\sqrt{G(x)}}=-\infty \text{ for } m\geqslant\delta.$$

By $(m-a)^2<4l$ we have $-\sqrt{8}<\lambda<0$. Note that (3) has only one singulrr point on the region $x>-1$ since $\varphi_0(-1)\geqslant 0.\varphi_0'(-1)>0$ from the assumption. The theore follows from Theorem B. Thus the proof is complete.

Corollary 3. *If* $n=0$, $ma>0$, $l+2am\leqslant 1$, $(a-m)^2<4l$ *and* $0<\delta/m\leqslant 1$, *the the system* (2) *admits at least one limit cycle.*

We finally consider the case where $an\neq 0$.

Theorem 6. *Suppose*

(1) $-1<n<0$, $a\delta>0$, $(m-a)^2<4l(n+1)$;

(2) $\varphi(x)>0$ *for* $x>-1$, *where* φ *is given in the system* (3). *Then the system* (2) *has at least one limit cycle.*

Proof As before we can suppose $a>0$. Let

$$A=l+am+a^2n,\ A_1=am+1+a\delta+2l,$$
$$B=-(a+m+2an),\ B_1=-(\delta+2a+m).$$

Then $A>0$ from $(m-a)^2<4l(n+1)$. And when $x\to\infty$

$$F(x)=\begin{cases}\dfrac{B}{n+1}x^{n+1}+o(x^{n+1}),\ B\neq0,\ n+1\neq0;\\ B\ln x+o(\ln x),\ B\neq0,\ n+1=0;\\ \dfrac{B_1}{n}x^n+o(x^n),\ B=0,\ n<0;\end{cases}$$

$$G(x)=\begin{cases}\dfrac{A}{2n+2}x^{2n+2}+o(x^{2n+2}),\ n+1\neq0;\\ A\ln x+o(\ln x),\ n+1=0.\end{cases}$$

Thus

$$\lim_{x\to\infty}\frac{F(x)}{\sqrt{G(x)}}=\begin{cases}\dfrac{\sqrt{2}B}{\sqrt{A(n+1)}}=\lambda,\ n+1>0;\\ +\infty,\ n+1=0,\ B>0;\\ 0,\ n+1=0,\ B=0.\end{cases}$$

It is easy to show that $|\lambda|<\sqrt{8}$ iff $B^2<4A(n+1)$ iff $(m-a)^2<4l(n+1)$.

While $x\to-1+0$, we have

$$f(x)=a(1-2n)(1+x)^{n-2}+o((1+x)^{n-2}),$$
$$g(x)=a^2n(1+x)^{2n-3}+o((1+x)^{2n-3}).$$

Hence

$$\lim_{x\to-1+0}\frac{F(x)}{\sqrt{G(x)}}=-\frac{\sqrt{2}(1-2n)}{\sqrt{n(n-1)}}=\mu<0.$$

We next prove that $\mu<\lambda$. It suffices to show that

$$A>\frac{B^2}{4(n+1)}\text{ implies }\mu<\lambda. \tag{8}$$

In fact, $\mu<\lambda$ iff $A>\dfrac{n(n-1)}{(n+1)(1-2n)^2}B^2$.

Set $r(n)=n(n-1)/(1-2n)^2$. Then $r'(n)=-1/(1-2n)^3<0$, and hence

$$\max\{r(n),\ -1<n<0\}=r(-1)=2/9<1/4.$$

Thus (8) follows.

Noting that the system (3) has only one singular point O on the region $x>-1$ and it is completely unstable, the theorem follows from Theorem B.

Corollary 4. *Let* $-1<n<0$, $a\delta>0$, $(m-a)^2<4l(n+1)$ *and*

$$0\leqslant3a^2n+a(m-\delta)\leqslant1-l-am.$$

Then there exists at least one limit cycle of (2) *around* O.

Proof In fact, by the last condition of the corollary

$$\varphi(-1)>0,\ \varphi'(-1)\geqslant 0,\ \varphi''(-1)\geqslant 0.$$

Thus $\varphi(x)>0$ for $x>-1$ since it is cubic. Then the conclusion follows from Theorem 6.

As an example, if $m=a>0$, $-1/3<n<0$, $l>0$, $l+a^2(3n+2)<1$ and

$$0<\delta<a(3n+1),$$

then (2) has a limit cycle by Corollary 4.

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