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A NOTE ON FULLER'S THEOREM

ZHU SHENGLIN (朱胜林)*

Abstract

In 1972, Fuller proved that a complete additive subcategory ${}_RC$ of R-Mod isequivalent to amodule category Δ-Mod if and only if ${}_RC=\mathrm{Gen}({}_RU)$ for some quasiprogenerator ${}_RU$ and $\Delta\cong\mathrm{End}\ {}_RU$ canonically. In this note the author gives a characterization of ${}_RC$ which makes ${}_RU$ a projective R-module in the case when R is a right perfect ring with identity, and shows that R-Mod is the unique complete additive subcategory of R-Mod which is equivalent to R-Mod for a left Artinian ring R.

In 1972, Fuller[1] proved that a complete additive subcategory ${}_RC$ of R-Mod is equivalent to a module category Δ-Mod if and only if ${}_RC=\mathrm{Gen}\ ({}_RU)$ for some quasiprogenerator ${}_RU$ and $\Delta\cong\mathrm{End}_R U$ canonically. In this note we give a characterization of ${}_RC$ which makes ${}_RU$a projective module in the case when R is a right perfect ring with identity, and show that R-Mod is the unique complete additive subcategory which is equivalent to R-Mod provided that R is left Artinian.

Throughout this note all rings are associative with identity, and all modules are unital. Each endomorphism will be written on the other side of the module. We write R-Mod for the category of all left R-modules. A full subcategory of R-Mod is called complete additive[1] if it is closed under taking submodules, epimorphic images and direct sums. A left R-module ${}_RU$ is a quasiprogenerator if ${}_RU$ is finitely generated quasiprojective and ${}_RU$ generates each of its submodules. A ring R is semiperfect if every finitely generated left R-module has a projective cover, and R is right perfect if and only if R is semiperfect and left semiartinian (Theorem 27.6 in [2]; p. 151 in [3]).

Proposition 1. *Let R and Δ be rings. Let ${}_RC$ be a complete additive subcategory of R-Mod and assume that T: Δ-Mod $\to {}_RC$ is a (additive) category equivalence. Let ${}_RU$ be the canonical left R-module $U=T(\Delta)$. Then the following are equivalent*

(a) ${}_RC=\mathrm{Gen}({}_RP)$ *for some projective module* ${}_RP$;

(b) ${}_RC=\mathrm{Gen}({}_RP)$ *for some finitely generated projective R-module* ${}_RP$;

(c) U *is a projective left R-module;*

(d) *There exists a ring Δ' and a Morita equivalence T': Δ'-Mod$\to\Delta$-Mod such that*

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* Institute of Mathematics, Fudan University, Shanghai, China.

$P=T\cdot T'(\Delta')$ is projective.

Proof Obviously.

Given a ring R, let Ω denote the set of isomorphism classes of simple left R-modules. If Ω_1 is a subset of Ω, then a hereditary torsion radical $\bar{S}_{\Omega_1}$ can be constructed as follows: for a module $_RM$, the Ω_1-socle $\mathrm{soc}_{\Omega_1}(M)$ of M is the sum of all simple submodules of M with isomorphism classes in Ω_1; let

$$s^0_{\Omega_1}(M)=0,$$

$$s^\alpha_{\Omega_1}(M)/s^{\alpha-1}_{\Omega_1}(M)=\mathrm{soc}_{\Omega_1}(M/s^{\alpha-1}_{\Omega_1}(M)),$$

$$s^\alpha_{\Omega_1}(M)=\sum_{\beta<\alpha}s^\beta_{\Omega_1}(M)$$

when α is a limit ordinal,

$$\bar{S}_{\Omega_1}(M)=s^\alpha_{\Omega_1}(M)$$

where α is the first ordinal for which

$$s^\alpha_{\Omega_1}(M)=s^{\alpha+1}_{\Omega_1}(M).$$

Then $S_{\Omega_1}(M)=M$ if and only if $\mathrm{soc}_{\Omega_1}(M/L)\neq 0$ for every proper submodule L of (Prop. 3.1, VIII in [4]).

Theorem 2. *Let R, Δ, $_RC$ be as in Proposition 1. If $_RU$ is projective, then satisfies the following*

(1) $_RC$ is closed under extension;

(2) Let S, S' be simple left R-modules with $S\in {}_RC$. If $\mathrm{Ext}^1_R(S, S')\neq 0$, t $S'\in {}_RC$.

Moreover, if R is right perfect with each simple modules in $_RC$ being finitely pr ented (e.g. $_RJ(R)$ is finitely generated), then the converse holds (i. e., $_RU$ is projecti

Proof If $_RU$ is projective, (1) is obvious by $_RC=\mathrm{Gen}(_RU)$. To see (2), let S be simple R-modules with $S\in {}_RC$ and $\mathrm{Ext}^1_R(S, S')\neq 0$. Then there exist nonsplitting exact sequence

$$0\to {}_RS'\to {}_RN\to {}_RS\to 0;$$

and $_RN$ is uniform with minimal submodule $_RS'$. Since $S\in\mathrm{Gen}(_RU)$, and $_RU$ projective, we have $\mathrm{Trace}_N\, U\neq 0$, hence $S'\in\mathrm{Gen}(_RU)={}_RC$.

To see the converse, let R be a right perfect ring with each simple module in finitely presented. Let $S_1, S_2, \cdots, S_n$ be a representive set of all simple R-module $_RC$, andset $\Omega_1=\{[S_1],[S_2],\cdots,[S_n]\}$; $\Omega_2=\Omega-\Omega_1$. Denote the projective cover of S_i

$P_i\xrightarrow{\pi_i} S_i\to 0$, then P_i is a local module with largest submodule $K_i=\ker\pi_i$.

Set $P^1_i=S_{\Omega_1}(P_i)$, we will show that $P^1_i=P_i$. If it it not the case, then $P^1_i\subseteq$ Since $P_i/P^1_i/K_i/P^1_i\cong S_i$ and $[S_i]\in\Omega_1$, we have $S_{\Omega_1}(P_i/P^1_i)\neq P_i/P^1_i$. Let

$$P^2_i/P^1_i=S_{\Omega_2}(P_i/P^1_i),$$

then $\mathrm{soc}_{\Omega_2}(P_i/P^2_i)=0$. Since P_i is semiartinian, we have a submodule P^3_i of P_i such that $P^3_i\supsetneqq P^2_i$ and $P^3_i/P^2_i\cong S$ with $[S]\in\Omega_1$. But S is finitely presented, the functor

$\mathrm{Ext}_R^1(S, -)$ commutes with direct limits. By $\mathrm{Ext}_R^1(S, S')=0$ for any simple R-module S' with $[S'] \in \Omega_2$, we have $\mathrm{Ext}_R^1(S, P_i^2/P_i^1)=0$. Thus the exact sequence

$$0 \to P_i^2/P_i^1 \to P_i^3/P_i^1 \to P_i^3/P_i^2 \cong S \to 0$$

splits. Hence there exists a submodule P_i^4 of P_i^3 such that

$$P_i^4/P_i^1 \cong P_i^3/P_i^2 \cong S.$$

This contradicts

$$P_i^1 = S_{\Omega_1}(P_i).$$

Hence

$$P_i = S_{\Omega_1}(P_i),$$

and condition (1) gives that $P_i \in {}_RC$.

Let $P = \bigoplus_{i=1}^{n} P_i$, then P is a projective R-module in ${}_RC$ which generates each simple objects (modules) of ${}_RC$. Hence ${}_RC = \mathrm{Gen}({}_RP)$. By Prop. 1, we have that ${}_RU$ is projective.

Example 3. Let

$$R=\begin{pmatrix} S & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix},$$

where S is the localization of $\mathbb{Z}$ at $2\mathbb{Z}$, then R is semiperfect but R is not right perfect. Let

$$I=\left\{\begin{pmatrix} 0 & 0 \\ q & q' \end{pmatrix} \middle| \; q, q' \in \mathbb{Q}\right\},$$

then I is an ideal of R. ${}_RU = R/I$ is a quasiprogenerator such that ${}_RC = \mathrm{Gen}({}_RU)$ satisfies (1), (2) of Theorem 2. Moreover, $J(R)$ is a finitely generated left ideal. But ${}_RU$ is not a projective R-module.

Example 4. Let S be the ring of all the $\aleph_0$-square upper triangular matrices over a field F that are constant on the diagonal and have only finitely many nonzero entries off the diagonal. Then S is a right perfect ring (Ex. 28.2 in [2]). Let M be all the $\aleph_0$-square matrices with finitely many nonzero entries, then M can be seen as an S-S-bimodule canonically, and $MJ(S)=M$. Let

$$R=\begin{pmatrix} S & 0 \\ M & S \end{pmatrix},$$

then R is a right perfect ring. Let

$$I=\begin{pmatrix} 0 & 0 \\ M & S \end{pmatrix},$$

then I is an ideal of R. ${}_RU = R/I$ is a quasiprogenerator and ${}_RC = \mathrm{Gen}({}_RU)$ satisfies (1) and (2) of Theorem 2. But ${}_RU$ is not projective. In this example the only simple R-module in ${}_RC$ is not finitely presented.

Theorem 5. *Let R be a left Artinian ring, and let ${}_RC$ be a complete additive subcategory of R-Mod. If ${}_RC$ is category equivalent to R-Mod, then ${}_RC = R$*-Mod.

Proof Let T: R-Mod$\to {}_R\mathcal{C}$ denote the category equivalence. Let $S_1, S_2, \cdots, S_n$ be a set of representitives of all nonisomorphic simple R-modules. As ${}_R\mathcal{C}$ is complete additive, we have ${}_RT(S_i)=S_{\sigma(i)}$ for a permutation σ on $\{1, 2, \cdots, n\}$.

Let $P_i \xrightarrow{\pi_i} S_i \to 0$ be a projective cover in R-Mod, then P_i is finitely generated (hencs has finite length). Since T is a category equivalence, we see that $T(P_i) \xrightarrow{T(\pi_1)} T(S_i)\to 0$ is a projective cover of $T(S_i)$ in ${}_R\mathcal{C}$. As in the category R-Mod, we have a commutative diagram

$$\begin{array}{ccccccc} 0\to \ker T(\pi_i) \to & T(P_i) & \xrightarrow{T(\pi_i)} & T(S_i) \to 0 \\ & \uparrow f_i & & \wr\Vert \\ & P_{\sigma(i)} & \xrightarrow{\pi_{\sigma(i)}} & S_{\sigma(i)} \to 0. \end{array}$$

Since $\ker\ T(\pi_i)$ is small in $T(P_i)$ (also in the category R-Mod), f_i is epimorphism. If we denote the length of a module ${}_RM$ by $l\text{th}(M)$, then

$$l\text{th}(T(P_i))\leqslant l\text{th}(P_{\sigma(i)}).$$

On the other hand, there is a lattice isomorphism T:

$$\text{Lat}({}_RP_i)\to \text{Lat}({}_RT(P_i)),$$

thus

$$l\text{th}(T(P_i))=l\text{th}(P_i).$$

While $\sum_{i=1}^{n} l\text{th}(T(P_i))\leqslant \sum_{i=1}^{n} l\text{th}(P_{\sigma(i)})=\sum_{i=1}^{n} l\text{th}(P_i)=\sum_{i=1}^{n} l\text{th}(T(P_i))$,

we get $l\text{th}(T(P_i))=l\text{th}(P_{\sigma(i)})$, hence f_i are isomorphisms, $i=1, 2, \cdots, n$; and $P_i\in$

Since $\bigoplus_{i=1}^{u} P_i$ is a generator of R-Mod, we have ${}_R\mathcal{C}=R$-Mod.

In Theorem 5 the Artinian condition cannot be replaced by right perfect n

Example 6. Let R be the ring S in Example 4. Let $I=\{x\in R\,|\,$all the ent of x except the first row equal to zero$\}$, then I is an ideal of R and ${}_RU=R/I$ quasiprogenerator which is not a generator of R-Mod. Moreover, End ${}_RU\cong R$. He the complete additive subcategory Gen$({}_RU)$ is a proper subcategory of R-Mod wl is equivalent to R-Mod.

Proposition 7. *Let R, Δ be rings and ${}_R\mathcal{C}$ b$_2$ a complete additive subcategor R-Mod such that ${}_R\mathcal{C}\cong\Delta$-Mod. Then Δ is (semi)-perfect whenever R is. When local, Δ always has the form $M_n(R/I)$, for some ideal I of R and $n\in\mathbb{Z}^+$.*

Proof Let T: Δ-Mod$\to {}_R\mathcal{C}$ be the category equivalence and ${}_RU=T(\Delta)$, t ${}_RU$ is finitely generated quasiprojective and $\Delta\cong\text{End}_RU$.

Let $P \xrightarrow{\pi} {}_RU\to 0$ be the projective cover of ${}_RU$, then P is a finitely gener projective R-module and ker π is fully invariant in P. Hence there exis canonical ring surjection $\text{End}_RP\to\text{End}_RU$, setting $f\in\text{End}_RP$ to

$$\bar f\in\text{End}_RU\colon (x+\ker\pi)\bar f=(x)f+\ker\pi.$$

When R is (semiperfect) left perfect, End ${}_RP$ is (semiperfect) left perlect, and so s End ${}_RU \cong \Delta$.

Let R be a local ring with largest left ideal J. Then all simple R-modules are somorphic (to $S=R/J$). Thus all simple objects of ${}_RC$ are isomorphic, and so is -Mod. Let ${}_\Delta Q \xrightarrow{\pi} {}_\Delta S' \to 0$ be a projective cover of a simple Δ-module ${}_\Delta S'$, then ${}_\Delta Q$ a progenerator of Δ-Mod and $T({}_\Delta Q) \to T({}_\Delta S') \to 0$ is a projective cover of $T({}_\Delta S')$ in C, ${}_RU'=T({}_\Delta Q)$ is a progenerator of ${}_RC$. As in the category R-Mod, we have the llowing commutative diagram

$$\begin{array}{ccccccc} 0 \to \ker T(\pi) & \to & {}_RU' & \to & {}_RT(\Delta^{S'}) & \to & 0 \\ & & \uparrow f & & \wr\Vert & & \\ & & R & \longrightarrow & R/J & \longrightarrow & 0 \end{array}$$

is an epimorphism, hence ${}_RU' \cong R/I$, for some left ideal I of R. From the uasiprojectivity we see that I is an ideal of R. Δ and $R/I \cong \mathrm{End}_R(R/I)$ are Morita quivalent, and since R/I is local, we have $\Delta \cong M_n(R/I)$ for some $n \in \mathbb{Z}^+$.

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