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THE NUMBER OF LIMIT CYCLESOF THE PLANE AUTONOMOUS SYSTEMS

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Abstract

In this paper the author discusses the vector fields satisfying the conditions (C1) and (C2) and proves some theorems for the number of limit cycles of the plane autonomous systems. Some applications of the theorems and lemmas in § 1 are given in § 2.

§1. Basic Theorems

Let
$$X(x)=X_1(x)\frac{\partial}{\partial x_1}+X_2(x)\frac{\partial}{\partial x_2},$$
$$x=(x_1,\ x_2)\in R^2$$
be a C^1 vector field on the x_1x_2-plane, its trajectories are the trajectories of the pla autonomous system
$$\frac{dx_1}{dt}=X_1(x),\quad \frac{dx_2}{dt}=X_2(x).$$

We now give the theorems, which can be used to obtain the results in [2, 3, and some new conclusions.

If X, Y are C^1 vector fields, we define
$$\Delta=X_1[X,\ Y]_2-X_2[X,\ Y]_1,$$
$$B=X_1Y_2-X_2Y_1,$$
where $[X,\ Y]$ is the Lie bracket of X and Y,
$$[X,\ Y]=[X,\ Y]_1\frac{\partial}{\partial x_1}+[X,\ Y]_2\frac{\partial}{\partial x_2},$$
$$[X,\ Y]_1=X_1\frac{\partial Y_1}{\partial x_1}+X_2\frac{\partial Y_1}{\partial x_2}-Y_1\frac{\partial X_1}{\partial x_1}-Y_2\frac{\partial X_1}{\partial x_2},$$
$$[X,\ Y]_2=X_1\frac{\partial Y_2}{\partial x_1}+X_2\frac{\partial Y_2}{\partial x_2}-Y_1\frac{\partial X_2}{\partial x_1}-Y_2\frac{\partial X_2}{\partial x_2}.$$

Lemma 1. *If X, Y are C^1 vector fields, then*
$$\frac{\partial B}{\partial x_1}X_1+\frac{\partial B}{\partial x_2}X_2=\Delta+B\operatorname{div}X.$$

Proof We take $\omega=X_2\,dx_1-X_1\,dx_2$ in the following formula[1]
$$\langle X\wedge Y,\ d\omega\rangle=X\langle Y,\ \omega\rangle-Y\langle X,\ \omega\rangle-\langle[X,\ Y],\ \omega\rangle$$

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o obtain Lemma 1.

For C^1 vecter field X we suppose that there is a C^1 vector field Y such that

(C1) $\Delta \geqslant 0$, on whole plane.

(C2) The sets

$$W=\{x \mid \Delta=0,\ x\in R^2\}$$

nd

$$H=\{x \mid B=0,\ x\in R^2\}$$

ave notwo dimensional subsets. $W\cap H$ has no accumulation points on plane.

Lemma 2. *Let X be a C^1 vector field with properties* (C1), (C2), *and $I\subset H$, be segment without multiple points. Then function B has different signs on the two sides f I (in a neighborhood of I).*

Proof If B has the same sign on the two sides of I, then B takes maximal alues or minimal values at all points of I. So

$$\frac{\partial B}{\partial x_1}=\frac{\partial B}{\partial x_2}=0$$

n I and Lemma 1 means $\Delta=0$ on I, i. e., $I\subset W\cap H$. It contradicts (C2) and the rool is completed.

Lemma 3. *Let X be a C^1 vector field with properties* (C1), (C2). *Then the set of he multiple points of the curves in H has no accumulation points on plane and the nultiple points are the singular point of X.*

Proof At the multiple points of the curves in H we have $B=0$,

$$\frac{\partial B}{\partial x_1}=\frac{\partial B}{\partial x_2}=0$$

nd so $\Delta=0$ (Lemma 1). Hence the set of the multiple points is contained in $W\cap H$ nd (C2) means the set has no accumulation points.

Let A be a multiple point of the curves in H. Then there is a neighborhood U f point A such that there are no other multiple points in U and the curves in H assing A divide U into several parts, in each part function B does not change its gn. Lemma 2 means B has different signs in two adjacent parts. On H we have Lemma 1)

$$\left.\frac{dB}{dt}\right|_X=\frac{\partial B}{\partial x_1}X_1+\frac{\partial B}{\partial x_1}X_1+\frac{\partial B}{\partial x_2}X_2=\Delta\geqslant 0.$$

he fact that $W\cap H$ has no accumulation points and the lemma*) on page 60 in [2] ow that in U the trajectories meeting $H\cap U-\{A\}$ of X go from the parts in hich B is negative into the parts in which B is positive. It is easy to show A is a ngular point of X.

Lemma 4. *Let X be a C^1 vector field with properties* (C1), (C2). *Then the set of the isolated points of H has no accumulation points on plane.*

*) The lemma holds if we take segment I, the sides (in a neighborhood) of I instead of the simple closed curve, the interior and exterior in the lemma respectively.

Proof At the isolated points of H we have $B=0$,

$$\frac{\partial B}{\partial x_1}=\frac{\partial B}{\partial x_2}=0$$

and so $\Delta=0$. Hence the set of the isolated points of H is contained in $W\cap H$ and (C2) means the set has no accumulation points.

Lemma 5. *Let X be a C^1 vector field with properties (C1), (C2). Then the closed trajectories cannot meet the curves (except the isolated points) in H.*

Proof On H we have

$$\left.\frac{dB}{dt_X}\right|_X=\frac{\partial B}{\partial x_1}X_1+\frac{\partial B}{\partial x_2}X_2=\Delta\geqslant 0.$$

The fact that $W\cap H$ has no accumulation points and the lemma on page 60 in [
show that the trajectories meeting a curve I in H must cross I from the side
which B is negative to the side in which B is positive. Noticing Lemma 2 a
Lemma 3 we assert that the conclusion of Lemma 5 is true.

Theorem 1. *Let X be a C^1 vector field with properties (C1), (C2). Then the f
cycles of X are in W and the hyperbolic stable (unstable) cycles of X are in the set*

$$H_+=\{x\,|\,B\geqslant 0,\ x\in R^2\}$$
$$(H_-=\{x\,|\,B\leqslant 0,\ x\in R^2\}).$$

Proof Let Γ: $q(t)$ be a closed trajectory with period T. Γ meets H only at t
isolated points of H (Lemma 5) and the number of the isolated points on Γ is fin
(Lemma 5 and the compactness of Γ). We suppose that Γ contains only one isolat
point A for simplicity and $q(t_1)=A$, $0<t_0<t_1<t_2<T$. Then we have (Lemma 1)

$$\begin{aligned}\int_\Gamma(\operatorname{div}X)\,dt&=\int_0^T(\operatorname{div}X)(q(t))\,dt\\&=\left(\int_0^{t_0}+\int_{t_2}^T\right)(\operatorname{div}X)(q(t))\,dt+\int_{t_0}^{t_2}(\operatorname{div}X(q(t))\,dt\\&=\left(\int_0^{t_0}+\int_{t_2}^T\right)\left(\left(\frac{\partial B}{\partial x_1}X_1+\frac{\partial B}{\partial x_2}X_2\right)\Big/B-\Delta/B\right)(q(t))\,dt+\\&\quad\int_{t_0}^{t_2}(\operatorname{div}X)(q(t))\,dt\\&=\ln\left|\frac{B(q(t_0))}{B(q(t_2))}\right|+\int_{t_0}^{t_2}(\operatorname{div}X)(q(t))\,dt\\&\quad-\left(\int_0^{t_0}+\int_{t_2}^T\right)(\Delta/B)(q(t))\,dt,\end{aligned}$$

$$\int_\Gamma(\operatorname{div}X)\,dt=\lim_{t_2-t_0\to 0}\left(-\int_0^{t_0}-\int_{t_2}^T\right)(\Delta/B)(q(t))\,dt,$$

where we take t_2-t_0 sufficiently small and

$$B(q(t_0))=B(q(t_2)).$$

Hence the sign of

$$\int_\Gamma(\operatorname{div}X)\,dt$$

depends on that of Δ/B. From Lemma 5 we know B does not change its sign on Γ i. e., $B\geqslant 0$ on Γ or $B\leqslant 0$ on Γ and $B=0$ at A only. So we complete the proof of Theorem 1 by Theorem 2.3 in [2].

Theorem 2. *Suppose that X is a C^1 vector field with properties* (C1), (C2), *H contains k isolated simple closed curves $L_i(i=1, 2, \cdots, k)$, surrounding only a singular point A of X $\left(\text{no other singular points of } X \text{ in the set } \bigcup_{i=1}^{k}\overline{\text{int } L_i}\right)$ and X has no closed trajectories in W. Then X has r limit cycles which only surround the singular point A, where $k-1\leqslant r\leqslant k+1$ for $k\geqslant 1$ and $0\leqslant r\leqslant 1$ for $k=0$.*

Proof From Theorem 1 we know X has no the fine cycles.

If Γ_1, Γ_2 are two adjacent limit cycles of X, which only surround the singular point A, D is the open annular region bounded by Γ_1 and Γ_2, then H has curves (which are not the isolated points of H) in D (Because Γ_1 and Γ_2 have different stability, B has different signs on Γ_1, Γ_2.) and the curves have no multiple points and can not meet Γ_i, $i=1, 2$. (X has no singular points in D and Lemma 5). So H only has the isolated simple closed curves in D and they are not null-homotopic in D (Note that a null-homotopic simple closed curve of H surrounds a singular point of X at least). The following proof shows that H has a unique isolated simple closed curve in D, which is not null-homotopic in D

Let L_1 and L_2 be two adjacent isolated simple closed curves as that in Theorem 2 and E be the open annular region bounded by L_1 and L_2. Hence X has no singular points in $\overline{E}$ and H has no curves (except the isolated points of H) in E. Function B does not change its sign in E, say $B\geqslant 0$ in E and $B=0$ only at the isolated points of H. On L_1 and L_2 we have

$$\left.\frac{dB}{dt}\right|_X=\Delta\geqslant 0.$$

So from Lemma 2 and (C2) we know the trajectories meeting L_1 or L_2 must enter E and X has a closed trajectory in E at least. Theorem 2 shows the closed trajectory in E is unique and stable.

The above statement gives the proof of Theorem 2.

Theorem 3. *If X is a C^1 vector field with properties* (C1), (C2), *then every trajectory on Y meets a closed trajectory of X at most one point.*

Proof From Lemma 5 we know that B does not change its sign on a closed trajectory Γ of X, i.e., $B\geqslant 0$ on Γ or $B\leqslant 0$ on Γ and $B=0$ only at the isolated points of H. Using the Lemma 4 and the lemma on page 60 in [2] we complete the proof.

Theorem 4. *If X is a C^1 vector field with properties* (C1), (C2), *and X has no closed trajectories in W, then Y (resp. $[X, Y]$) has the singular points or the closed trajectories in the closure of the annular region bounded by two adjacent cycles of X with* *the same (resp. opposite) orientation.*

Proof If Γ_1, Γ_2 are two adjacent cycles of X with the same (resp. opposite) orientation and D is the open annular region bounded by Γ_1 and Γ_2, noting that X has no the fine cycles we know that function B has different signs on Γ_1 and Γ_2 (resp. $\Delta \neq 0$ on Γ_1 and Γ_2). From the lemma on page 60 in [2] we obtain the proof.

Theorem 5. *If X is a C^1 vector field with properties* (C1), (C2), *the Y in* (C1) *and* (C2) *is a Hamiltonian vector field, then X has no closed trajectories.*

Proof Suppose that

$$Y = -\frac{\partial F(x)}{\partial x_2}\frac{\partial}{\partial x_1} + \frac{\partial F(x)}{\partial x_1}\frac{\partial}{\partial x_2},$$

then

$$B = \frac{\partial F}{\partial x_1} X_1 + \frac{\partial F}{\partial x_2} X_2.$$

The Theorem 1.9 in [2] shows that B must change its sign on any closed trajectoı of X. So from Lemma 5 we assert that X has no closed trajectories.

Remark. (1) If for C^1 vector field X there is a C^1 function B satisfying (C: and (C2), where we define

$$\Delta = \frac{\partial B}{\partial x_1} X_1 + \frac{\partial B}{\partial x_2} X_2 - B \operatorname{div} X,$$

then the above Lemmas 1—5 and Theorems 1—2 hold.

(2) If we take $\Delta \leqslant 0$ instead of $\Delta \geqslant 0$ in (C1), then the above Lemmas 2—5 ar Theorems 2—5 hold. In the case H_+ and H_- in Theorem 1 exchange places.

§2. Applications

In this section we give some applications of the above lemmas and theorems.

Example 1 Consider

$$\begin{aligned} \frac{dx_1}{dt} &= ax_1 + x_2 - a(x_1+x_2)f, \\ \frac{dx_2}{dt} &= -x_1 + ax_2 - a(-x_1+x_2)f, \end{aligned} \tag{1}$$

where $f = x_1^2 + x_2^2 + bx_1x_2$, the parameters a and b satisfy $0<a<1$, $|b|<2$.

We take

$$Y = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2},$$

then

$$[X, Y] = 2a(x_1+x_2)f\frac{\partial}{\partial x_1} + 2a(-x_1+x_2)f\frac{\partial}{\partial x_2},$$

$$\Delta = 2a(x_1^2+x_2^2)(1-a)f,$$

$$B = (1-af)(x_1^2+x_2^2).$$

Noting that $f \geqslant 0$ and $f=0$ only at (0, 0), we know that the vector field X given by (1) satisfies (C1) and (C2). $H = \{(0, 0)\} \cup I$, where I: $1-af=0$, an ellipse surrounding (0, 0). X has no closed trajectories in $W(W=\{(0, 0)\})$. So (1) has at most two cycles surrounding (0, 0) (Theorem 2). However, we note that Y and $[X, Y]$ only have singular point (0, 0). Theorem4 means (1) has at most one cycle surrounding (0, 0).

We note that (0, 0) is an unstable focus of (1) and $B\geqslant 0$ in the inside of the ellipse I ($B=0$ only at (0, 0)) and $B<0$ in the outside of I. So all the trajectories meeting I must enter the inside of I and from the above statement of uniqueness we assert (1) has unique limit cycle ((0, 0) is the unique singular point of (1)). which is star shaped about (0, 0) (The trajectories of Y are rays and Thearem 3).

Example 2 Consider Liénard system

$$\frac{dx_1}{dt}=-x_2-F(x_1),\quad \frac{dx_2}{dt}=x_1, \tag{2}$$

where F is a C^2 function and $F(0)=0$.

We take

$$Y=x_1\frac{\partial}{\partial x_1}+(x_2+F(x_1)-x_1F'(x_1))\frac{\partial}{\partial x_2},$$

where "′" denotes the derivative. Then

$$\Delta=-x_1F''(x_1)(x_2+F(x_1))^2,$$

$$B=-(x_2+F(x_1)-x_1F'(x_1))(x_2+F(x_1))-x_1^2.$$

H has no closed curves surrounding (0, 0) (the intersection of H and the curve $x_2+F(x_1)=0$ is (0, 0)). So from Remark (2) and Theorem 2 we have the following result[3].

If $x_1F''(x_1)\geqslant 0$ (resp. $\leqslant 0$) and the set $\{x_1|F''(x_1)=0\}$ has no accumulation points, then (2) has at most one limit cycle.

Example 3 Consider

$$\begin{aligned}\frac{dx_1}{dt}&=-\varphi(x_2)-F(x_1),\\ \frac{dx_2}{dt}&=g(x_1),\end{aligned} \tag{3}$$

where g, F and φ are C^1 functions satisfying $F(0)=0$, $x_1 g(x_1)>0$ for $x_1\neq 0$, $x_2\varphi(x_2)>0$ for $x_2\neq 0$.

Let

$$G(x_1)=\int_0^{x_1} g(x_1)\,dx_1,$$

$$\Phi(x_2)=\int_0^{x_2}\varphi(y)\,dy.$$

Suppose that limits

$$\lim_{x_1\to 0}\frac{G}{g}$$

and

$$\lim_{x_2\to 0}\frac{\Phi}{\varphi}$$

exist. We define functions Y_1 and Y_2 as follows

$$Y_1(x_1)=\begin{cases}\dfrac{G}{g} & \text{for } x_1\neq 0,\\ \lim\limits_{x_1\to 0}\dfrac{G}{g} & \text{at } x_1=0;\end{cases} \tag{4}$$

$$Y(x_2)=\begin{cases}\dfrac{\Phi}{\varphi} & \text{for } x_2\neq 0,\\ \lim\limits_{x_2\to 0}\dfrac{\Phi}{\varphi} & \text{at } x_2=0.\end{cases} \tag{5}$$

Theorem 6. *Under the above conditions, let the functions Y_1, Y_2 defined in* (4 (5) *be C^1 functions and the following conditions hold.*

1. $Fg\Phi\varphi'-Gf\varphi^2\geqslant 0$ (*resp.* $\leqslant 0$),

where the set in which the equality holds has not any two dimensional regionand X h no closed trajectories in the set.

2. *The set*

$$\{x\,|\,Fg\Phi\varphi'-Gf\varphi^2=0\}\cap\left\{x\,|\,(-\varphi-F)\frac{\Phi}{\varphi}-G=0\right\}$$

has no accumulation points.

Then system (3) *has at most one limit cycle.*

Proof We take

$$Y=Y_1\frac{\partial}{\partial x_1}+Y_2\frac{\partial}{\partial x_2},$$

where Y_1, Y_2 are defined in (4), (5).

$$A=(Fg\Phi\varphi'-Gf\varphi^2)/\varphi^2,$$
$$B=(-\varphi-F)\Phi/\varphi-G.$$

So the proof is completed by Theorem 2 and Remark (2) immediately.

If $$\varphi(x_2)=x_2,\ g(x_1)=x_1,$$

then we have the following result[4] from Theorem 6.

If $$F'(x_1)-\frac{F(x_1)}{x_1}\geqslant 0,\quad (\text{resp. }\leqslant 0)$$

for $x_1\neq 0$, and the equality cannot hold in any interval, then system (2) has at m(one limit cycle.

Example 4 Consider

$$\frac{dx_1}{dt}=-(x_2+x_2^3)-\left(\frac{x_1^3}{3}-x_1\right),$$
$$\frac{dx_2}{dt}=x_1. \tag{}$$

Taking

$$\varphi(x_2)=x_2+x_2^3,\ g(x_1)=x_1,$$
$$F(x_1)=\frac{x_1^3}{3}-x_1$$

in (3) we get (6). It is easy to chock that

$$Fg\Phi\varphi'-Gf\varphi^2\leqslant 0.$$

Hence from Theorem 6, system (6) has at most one limit cycle.

Example 5 Consider

$$\frac{dx_1}{dt}=P(x_1, x_2), \quad \frac{dx_2}{dt}=Q(x_1, x_2), \tag{7}$$

where P, Q are C^1 functions

We take $Y=Q\frac{\partial}{\partial x_2}$, then

$$\Delta=P^2\frac{\partial Q}{\partial x_1}+Q^2\frac{\partial P}{\partial x_2},$$

$$B=PQ.$$

Noting that tho closed trajectories of (7) must meet $PQ=0$ and Lemma 5, we see that if P, Q are C^1 functions and

$$\frac{\partial Q}{\partial x_1}>0 \quad (\text{resp. } <0),$$

$$\frac{\partial P}{\partial x_2}>0 \quad (\text{resp. } <0),.$$

then system (7) has no closed trajectories.

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