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SENTINELS FOR PERIODIC DISTRIBUTED SYSTEMS

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Abstract

For periodic distributed systems the author reduces the "sentinels" problem to a problem of controllability type and uses suitable adaptations of HUM (Hilbert Uniqueness Method) to gives olutions to the original "sentinels" problem.

§1. Introduction

Let Ω be an open set of $\mathbf{R}^n$, bounded or not, with a smooth boundary Γ.

Let $T>0$ be given.

We consider the periodic (in time) system given as follows: the state equati is given by

$$y'-\Delta y+f(y)=0 \quad \text{in} \quad \Omega\times(0, T), \qquad (1.$$

where $y'=\dfrac{\partial y}{\partial t}$, $\Delta y=\dfrac{\partial^2 y}{\partial x_1^2}+\cdots+\dfrac{\partial^2 y}{\partial x_n^2}$ and where $f(\lambda)$ is a non-necessarily line function, smooth and such that the conditions below are satisfied.

We are interested intime periodic solutions of (1.1), i.e. such that

$$y(0)=y(T), \qquad (1.$$

where $y(s)$ stands for the function $x\to y(x, s)$.

We add now the boundary conditions. We are interested in systems n completely known, where some of the conditions are not entirely available. In t present situation we assume that

$$y=\bar{y}+\tau\hat{y} \quad \text{on} \quad \Sigma=\Gamma\times(0, T), \qquad (1.$$

where $\bar{y}$ is known (sufficiently smooth) and where $\tau\hat{y}$ denotes the "perturbation" (the unknown part of the data). The function $\hat{y}$ is arbitrary and $\tau\in\mathbf{R}$ is sma enough.

We assume that (1.1) (1.2) (1.3) admits a unique solution in a suitable spac Let

$$y=y(\tau, \hat{y})=y(\tau) \qquad (1.4$$

be this solution.

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We denote by $\bar{y}$ the solution for $\tau=0$. We assume that $\bar{y}$ can be computed approximately).

We now introduce, following J. L. Lions[3], the notion of "sentinel".

Let ω be an open set of Ω. Let h_0 be a given function on $\omega\times(0, T)$, such that

$$h_0\geqslant 0, \quad \iint_{\omega\times(0,T)} h_0\,dx\,dt=1. \tag{1.5}$$

We introduce the functional

$$\mathscr{S}(\tau)=\iint_{\omega\times(0,T)}(h_0+w)y(\tau)\,dx\,dt, \tag{1.6}$$

where $w\in L^2(\omega\times(0, T))$ is to be determined.

We shall say that (1.6) defines a sentinel (for the system (1.1)(1.2)(1.3)) if the two following conditions are satisfied:

$$\frac{d}{d\tau}\mathscr{S}(\tau)\,|_{\tau=0}=0 \quad \forall \hat{y} \tag{1.7}$$

and

$\|w\|_{L^2(\omega\times(0,T))}=$ minimum, among all w's such that that (1.7) holds true. (1.8)

Remark 1.1. Of course it will be necessary to verify that $h_0+w\not\equiv 0$ on $\omega\times(0, T)$.

Remark 1.2. The notion of "sentinel" as introduced above, following the author [3], is completely general. We refer to [3], [4], [9], [10] for other situations.

Remark 1.3. Very many situations of the type (1.1)(1.2)(1.3) arise in physical situations, in particular in natural sciences.

Remark 1.4. In what follows we are going to construct w satisfying (1.8), associated to any open set ω and to any function h_0 satisfying (1.5). There are an infinite number of sentinels for a given system.

§2. Sentinels and Controllability

Let us introduce

$$\dot{y}=\frac{d}{d\tau}y(\tau)\,|_{\tau=0}. \tag{2.1}$$

The function $\dot{y}$ is given by

$$\begin{cases}\dot{y}-\Delta\dot{y}+f'(\bar{y})\dot{y}=0,\\ \dot{y}(0)=\dot{y}(T),\ \dot{y}=\hat{g}\ \text{on}\ \Sigma.\end{cases} \tag{2.2}$$

We suppose that (2.2) admits a unique solution. It is very simple to give sufficient conditions for this hypothesis to be satisfied.

Condition (1.7) is equivalent to

$$\iint_{\omega\times(0,T)} (h_0+w)\dot{y}\,dx\,dt=0\ \forall\hat{y}. \tag{2.3}$$

We introduce q given by the solution of

$$\begin{cases} -q'-\Delta q+f'(\bar{y})q=(h_0+w)\chi_\omega \text{ in } \Omega\times(0,T), \\ q(0)=q(T), \\ q=0 \text{ sur } \Sigma, \end{cases} \tag{2.4}$$

where χ_ω = characteristic function of ω.

Multiplying the first equation (2.4) by $\dot{y}$ and integrating by parts gives

$$\iint_{\omega\times(0,T)} (h_0+w)\chi_\omega\dot{y}\,dx\,dt=-\int_\Sigma \frac{\partial q}{\partial\nu}\dot{y}\,d\Gamma\,dt=-\int_\Sigma \frac{\partial q}{\partial\nu}\hat{y}\,d\Gamma\,dt,$$

where $\frac{\partial}{\partial\nu}$ denotes the normal derivative to Γ, directed towards the exterior of

Then (2.3) is equivalent to

$$\frac{\partial q}{\partial\nu}=0 \text{ on } \Sigma. \tag{2}$$

This is now a problem of controllability type.

Indeed, let us introduce q_0 and z by

$$\begin{cases} -q_0'-\Delta q_0+f'(\bar{y})q_0=h_0\chi_\omega, \\ q_0(0)=q_0(T),\ q_0=0 \text{ on } \Sigma, \end{cases} \tag{2}$$

$$\begin{cases} -z'-\Delta z+f'(\bar{y})z=w\chi_\omega, \\ z(0)=z(T),\ z=0 \text{ on } \Sigma. \end{cases} \tag{2.}$$

Then $q=q_0+z$ and (2.5) is equivalent to

$$\frac{\partial z}{\partial\nu}=-\frac{\partial q_0}{\partial\nu} \text{ on } \Sigma. \tag{2.}$$

We can now state the problem in the framework of controllability. We want find a "control" w such that the "state" $z=z(w)$ (solution of (2.7)) satisfies (2. and (1.8) among all w's such that (2.8) holds true.

Remark 2.1 Let us take an arbitrary function q, smooth enough, such th $q=\frac{\partial q}{\partial\nu}=0$ on Σ, $q(0)=q(T)$ and such that q has its support in $\bar{\omega}\times(0,T)$. We th compute $-q'-\Delta q+f'(\bar{y})q=F=F\chi_\omega$ and we define w by

$$F=h_0+w.$$

Then (2.5) is true (by construction). Therefore there exists functions w su that (2.5) is satisfied, in other words: there is controllability. The question is th to find the "best" possible choice for w, namely such that (1.8) holds true.

Remark 2.2. Condition (1.8) means that the "sentinel" is "as close possible" of a mean value on $\omega\times(0,T)$. We could introduce other norms than $\|w\|_{L^2(\omega\times(0,T))}$ but we do not pursue this matter here.

§ 3. Solution of the Problem of Exact Controllability

We have introduced, for other purposes, in [5], and we have developed in [6], [7], a general method HUM (Hilbert Uniqueness Method) for the solution of problems of Exact Controllability of the classical type, i. e. the question of steering a given system from a given state to another given state in a given finite time.

We show that suitable adaptations of HUM give solutions for the sentinels problems.

This "program" has been initiated in [3]. Another situation has been considered in [4]. We now indicate how one can solve (2.8) (1.8) using techniques somewhat associated with HUM.

Let σ be an arbitrary smooth function given on Σ. Let ρ be the solution of

$$\rho' - \Delta\rho + f'(\bar{y})\rho = 0 \text{ in } \Omega\times(0, T), \quad \rho(0) = \rho(T),\ \rho|_{\Sigma} = \sigma. \tag{3.1}$$

We then define ζ as the solution of

$$-\zeta' - \Delta\zeta + f'(\bar{y})\zeta = \rho\chi_{\omega} \text{ in } \Omega\times(0, T), \quad \zeta(0) = \zeta(T),\ \zeta = 0 \text{ on } \Sigma. \tag{3.2}$$

We then introduce a linear operator M by

$$M\sigma = -\frac{\partial\zeta}{\partial\nu} \text{ on } \Sigma. \tag{3.3}$$

If we multiply (3.2) by ρ and if we integrate by parts, we obtain

$$\langle M\sigma, \sigma\rangle = \int_{\Sigma} (M\sigma)\sigma\, d\Gamma\, dt = \iint_{\omega\times(0,T)} \rho^2\, dx\, dt. \tag{3.4}$$

This leads to the following. We set

$$\|\sigma\|_F = \left(\iint_{\omega\times(0,T)} \rho^2\, dx\, dt\right)^{1/2}. \tag{3.5}$$

We define in this way a prehilbertian semi norm on the space of smooth functions σ on Σ.

But in fact, provided $f'(\bar{y})$ is smooth enough, we have a norm: if $\rho = 0$ on $\omega\times(0, T)$ and if ρ satisfies (3.1) then $\rho \equiv 0$ so that $\sigma = 0$.

We then denote by F the Hilbert space obtained by completion of smooth functions for the norm (3.5).

Remark 3.1. Because of the very strong regularization properties in solving (3.1), F will consist of very general ultra distributions (assuming Γ very smooth) on Σ. A complete characterization of F is an open question.

Remark 3.2. The space F will in general depend on f and on $\bar{y}$, i. e. in fact of $f'(\bar{y}) = b$. It would be interesting to find classes of functions b on $\Omega\times(0, T)$

giving the same spaces F.

Remark 3.3. New spaces necessary for the solution of optimal control problems in distributed systems are not unusual. New spaces where introduced for pointwise control in J. L. Lions[8], a question of the type of Remark 3.2 being solved by Li Tatsien [2]. Other spaces (in very large number) have been introduced in J. L. Lions [6], [7].

We now observe that for any smooth functions σ and $\hat{\sigma}$, we have

$$\langle M\sigma, \hat{\sigma}\rangle = \langle \sigma, M\hat{\sigma}\rangle = \iint_{\omega\times(0,T)} \rho\hat{\rho}\,dx\,dt \tag{3.7}$$

(where $\hat{\rho}$ is given by (3.1) with $\hat{\sigma}$ instead of σ). It follows from (3.6) and (3.7) (which reads $\langle M\sigma, \sigma\rangle = \|\sigma\|_F^2$) that

$$M \text{ is an isomorphism from } F \text{ onto } F' \tag{3.8}$$

and

$$M^* = M. \tag{3.8}$$

Let us now verify that

$$\partial q_0/\partial\nu \in F'. \tag{3.9}$$

Indeed if we multiply (2.6) by ρ we obtain

$$\int_\Sigma \frac{\partial q_0}{\partial\nu}\sigma\,d\Gamma\,dt = -\iint_{\omega\times(0,T)} h_0\rho\,dx\,dt, \tag{3.10}$$

hence (3.9) follows.

Therefore the equation

$$M\sigma = -\partial q_0/\partial\nu,\ \sigma\in F \tag{3.11}$$

admits a unique solution.

We then define

$$w=\rho \text{ on } \omega\times(0,T), \text{ where } \rho \text{ is the solution of (3.1)}, \tag{3.12}$$

$$\text{corresponding to } \sigma \text{ given by (3.11)}.$$

For this choice of w we have $z=\zeta$ and (2.8) is equivalent to (3.11). We n verify that the "control" w given by (3.12) minimizes $\|w\|_{L^2(\omega\times(0,T))}$. Indeed le be any function in $L^2(\omega\times(0,T))$ satisfying (2.8), and let $\hat{z}$ be the correspond solution of (2.7). We have to show that

$$\|w\|_{L^2(\omega\times(0,T))} \leqslant \|\hat{w}\|_{L^2(\omega\times(0,T))}. \tag{3.13}$$

We introduce

$$\psi = \zeta - \hat{z} \tag{3.14}$$

and we observe that

$$\begin{gathered} -\psi' - \Delta\psi + f'(\bar{y})\psi = (w-\hat{w})\chi_\omega, \\ \psi(0) = \psi(T), \\ \psi = \frac{\partial\psi}{\partial\nu} = 0 \text{ on } \Sigma. \end{gathered} \tag{3.15}$$

If we multiply (3.15) by ρ (solution of (3.1) which corresponds to (3.11)), we
btain

$$\iint_{\omega\times(0,T)} (w-\hat{w})\, w\, dx\, dt=0,$$

nce (3.13) follows.

It remains to see if $h_0+w\not\equiv 0$ in $\omega\times(0, T)$. Since $w=\rho$ and since ρ satisfies
3.1), it suffices to take h_0 such that

$$h_0'-\Delta h_0+f'(\bar{y})h_0\neq 0 \text{ in } \omega\times(0, T), \tag{3.16}$$

r having $h_0+w\not\equiv 0$ in $\omega\times(0, T)$.

Let us take for instance

$$h_0=T^{-1} \quad (\text{volume } \omega)^{-1}. \tag{3.17}$$

Then (3.16) is satisfied, provided $f'(\bar{y})\not\equiv 0$ on $\omega\times(0, T)$.

Summing up *We assume f smooth (at least C^1) and that* (1.1)(1.2)(1.3) *lmits a unique solution.*

Let ω be an arbitrary open set of Ω and let h_0 be given with (1.5) *and* (3.16).

Then there exists a sentinel

$$\iint_{\omega\times(0,T)} (h_0+w)\, y\, dx\, dt, \tag{3.18}$$

here w is given by (3.12) *and* (3.11).

§4. Various Remarks

Remark 4.1. Let us assume that the state is given by

$$y'-\Delta y+f(y)=v\chi_{\mathcal{O}} \tag{4.1}$$

with (1.2)(1.3) (unchanged) where $\mathcal{O}\subset\Omega$, $\chi_{\mathcal{O}}$ = characteristic function of $\mathcal{O}$ and here v is a control variable. For instance the term $v\chi_{\mathcal{O}}$ can represent some sort of ollution, arising in the region $\mathcal{O}$. let us compute

$$\iint_{\omega\times(0,T)} (h_0+w)(y-\bar{y})\, dx\, dt.$$

Using q as defined by (2.4) and satisfying (2.5) we have

$$\iint_{\omega\times(0,T)} (h_0+w)(y-\bar{y})\, dx\, dt$$

$$=\iint_{\omega\times(0,T)} qv\, dx\, dt-\iint_{\omega\times(0,T)} q[f(y)-f(\bar{y})-f'(\bar{y})(y-\bar{y})]\, dx\, dt. \tag{4.2}$$

Therefore, if we want to let y stay "not too far" from $\bar{y}$, we have to maintain

$$\left|\iint_{\omega\times(0,T)} qv\, dx\, dt\right| \text{ as small as possible.} \tag{4.3}$$

We can even try to do that with several sentinels.

Remark 4.2. Everything which we have introduced here can be generalized or adapted in many directions: systems of equations, periodic problems for coupled systems, for hyperbolic systems, systems with sources partially known. A systematic account will be presented in [9].

Remark 4.3. The method presented here can be used from a constructive numerical view point. But a large number of investigations are still needed in order to validate the above methods for practical applications. The method H--- has been tested in the "classical" situation of exact controllability in Glowinski, C. Li and the Author [1]-where one has to introduce a number regular ization procedures.

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