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ON COMPLETENESS OF MUSIELAK -ORLICZ SPACES

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Abstract

The paper deals with completeness theorems on Musielak–Orlicz spaces of vector valued functions. Generally speaking, those spaces are not assumed to be locally convex. There are given sufficient conditions for the Musielak–Orlicz space to be complete in terms of the function Φ.

§0. Introduction

It is necessary to know whether the investigated space is complete in many athematical problems concerning topological vector spaces. Non–complete pological vector spaces are very often replaced by their completions. However, e structure of the space can change and that can be inconvenient in many uations. So, the theorems on completeness of topological vector spaces are very iportant.

This paper deals with completeness theorems of a large class of function spaces–called Musielak–Orlicz spaces–which contains the spaces like $L^p(0<p\leqslant\infty)$. These aces (locally convex or not) are very useful tool in investigating various oblems of functional analysis (cf. [2, 10]). Moreover, they are very interesting emselves and their properties are intensively studied by many mathematicians. lthough the case of Musielak–Orlicz space being a normed space is well known and e completeness theorems concerning them can be found in Krasnosielskii–Rutickii] or Kozek [3], the authors have not determined which property of the function yielding the Musielak–Orlicz space is sufficient for it to be complete. And that is ie main purpose of this paper.

§1. Preliminaries

Throughout the paper, by (T, Σ, μ) we shall denote a measure space with a nonnegative, complete, atomless and σ–finite measure μ.

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(X, τ) will be a real topological vector space (t. v. s.), $\mathscr{B}_X$ — the σ-algebra of all Borel subsets of X and $\mathscr{C}$ — an arbitrary base of closed neighbourhoods of 0. Further, $\mathfrak{M}(T, X)$ will be the set of all Borel measurale functions $f: T \to X$, i. e. functions satisfying $f^{-1}(U) \in \Sigma$ for every set $U \in \mathscr{B}_X$. By $M(T, X)$ (or shortly M) we shall denote an arbitrary linear subset of $\mathfrak{M}(T, X)$. As usual, $f = g$ means that $f(t) = g(t)$ for almost every (a. e.) $t \in T$.

Definition 1.1. *A function $\Phi: X \times T \to [0, +\infty]$ is said to be a Φ-function if there is a set T_0 of measure zero such that*

a) *Φ is $\mathscr{B}_X \times \Sigma$-measurable,*

b) *Φ is lower semicontinuous on X for $t \notin T_0$*, i. e. *for every $t \notin T_0$, $a \geqslant 0$, x such that $a < \Phi(x, t)$ there is a neighbourhood y of x such that $a < \Phi(y, t)$ for every V,*

c) *$\Phi(0, t) = 0$, $\Phi(x, t) = \Phi(-x, t)$ for every $x \in X$, $t \notin T_0$,*

d) *$\Phi(ux + vy, t) \leqslant \Phi(x, t) + \Phi(y, t)$ for every $u, v \geqslant 0$, $u + v = 1$, $x, y \in X$ and T_0,*

e) *$\lim_{u \to 0} \Phi(ux, t) = 0$ for every $x \in X$ and $t \notin T_0$.*

If, moreover, the functions $\Phi(\cdot, t)$ are continuous on $\{x \in X : \Phi(x, t) < +\infty\}$ $t \notin T_0$ then Φ is called a continuous Φ-function.

Remarks. 1. 2. Condition a) of the above definition is necessary for composition $\Phi(f(t), t)$, $f \in \mathfrak{M}(T, X)$, being measurable. However (cf. [6]), i is the Euclidean space $\mathbb{R}$, then instead of a) and b) we can assume the Carathéo. conditions:

a$_1$) $\Phi(\cdot, t)$ is left-continuous on $[0, +\infty]$ for $t \notin T_0$;

a$_2$) $\Phi(x, \cdot)$ is Σ-measurable for all $x \in X$.

Note that Φ fulfils d) iff $\Phi(0, t)$ is nondecreasing on $[0, +\infty)$ for $t \notin T_0$.

In the following, the symbol Φ will be used only for functions which ar functions at least.

By I we shall denote the functional from $\mathfrak{M}(T, X)$ into $[0, +\infty]$ define $I(f) = \int_T \Phi(f(t), t) d\mu$. Then $I|_M$ is a pseudomodular on $M = M(T, X)$ in the s of [7] and [9].

Definition 1. 3. *By Musielak–Orlicz space $L^\Phi(M)$ we mean the set of functions $f \in M$ such that $I(af) < +\infty$ for some $a > 0$ equipped with the top determined by the F-seminorm*

$$\|f\|_\Phi = \inf\{a > 0 : I(a^{-1}f) \leqslant a\}.$$

Remarks 1. 4. If Φ is a convex function (i. e. the functions $\Phi(\cdot, t)$ convex for a. e. $t \in T$), then we can define a seminorm on the space $L^\Phi(M)$

$$\|f\|_\Phi^0 = \inf\{a > 0 : I(a^{-}f) \leqslant 1\}.$$

Note that $\|f_n\|_\Phi \to 0$ iff $I(af_n) \to 0$ for all $a>0$. If Φ is convex then also $\|f_n\|_\Phi^0 \to 0$ iff $I(af_n) \to 0$ for all $a>0$. Thus the topologies determined by $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^0$ re equivalent provided Φ is a convex function.

Definition 1.5. *A sequence (f_n) of elements of M is said to be*

a) *I-convergent to $f \in M$, if $I(a(f_n - f)) \xrightarrow[n\to+\infty]{} 0$ for some $a>0$,*

b) *N-convergent to $f \in M$, if $I(a(f_n - f)) \xrightarrow[n\to+\infty]{} 0$ for all $a>0$* (i. e. $\|f_n - \|_\Phi \xrightarrow[n\to+\infty]{} 0$),

c) *μ-convergent to $f \in M$ if*

$$\forall_{G\in\mathscr{G}}\ \forall_{\varepsilon>0}\ \exists_{n_0\in\mathbb{N}}\ \forall_{n\geq n_0}\ \mu(\{t\in T: f_n(t) - f(t) \notin G\}) < \varepsilon,$$

d) *a. e.-convergent to $f \in M$ if there is a set T_0 of measure zero such that*

$$\forall_{t\notin T_0}\ \forall_{G\in\mathscr{G}}\ \exists_{n_0\in\mathbb{N}}\ \forall_{n\geq n_0}\ f_n(t) - f(t) \in G.$$

§2. The Property (C) of the Set $M(J, X)$

We shall say that the set $M(T, X)$ has the property (C) if from every μ-auchy sequence (f_n) of elements of $M(T, X)$ we can choose a subsequence which a. e.-convergent to a function $g \in M(T, X)$.

Lemma 2. 1. *Assume that $\|\cdot\|$ is a semimetric on X such that the space $(X, \|\ \|)$ is complete. Further, let the space $M(T, X)$ be closed with respect to the a. e. onvergence. Then $M(T, X)$ has the property (C).*

Proof Let (f_n) be a μ-Cauchy sequence. In virtue of the Riesz theorem, we n choose a subsequence (f_{n_k}) such that $(f_{n_k}(t))$ is a Cauchy sequence with respect $\|\cdot\|$ for all $t \notin A$ where $A \in \Sigma$ and $\mu(A) = 0$. Now define

$$g(\ ) = \begin{cases} \lim\limits_{k\to\infty} f_{n_k}(t) & \text{for } t \notin A, \\ 0 & \text{otherwise.} \end{cases}$$

hen, by the completeness of X, g is well defined and, by assumption, $g \in M(T, X)$.

Exaples 2. 2. The following spaces have property (C):

1) $\mathfrak{M}(T, \mathbb{R})$,

2) $M_c(T, X)$——the space of all constant functions provided X is a Frechet omplete metric) space,

3) $\mathscr{M}(T, X)$——the space of all strongly measurable functions (i. e. functions hich are limits of sequences of simple functions with respect to the a. e. convergence) provided X is a Frechet space,

4) $\mathfrak{M}(T, X)$——if X is a separable Banach space (then $\mathfrak{M}(T, X) = \mathscr{M}(T, X)$ by the Pettis theorem).

§ 3. The Main Result

The Musielak–Orlicz space is said to be I–complete (resp. N–complete) if it is complete with respect to the I–convergence (resp. N–convergence).

In the next theorem, the following conditions will be a key of importance:

$(I\mu)$ for every set $G\in\mathscr{C}$ there are a set T_0 of measure zero and a measurable function $\eta_G:T\to(0, +\infty]$ such that

$$\inf_{x\notin G}\Phi(x, t)\geqslant\theta_G(t)\quad\text{for every } t\notin T_0,$$

$(N\mu)$ for every set $G\in\mathscr{C}$ there are a set T_0 of measune zero and a sequence (measurable functions

$$\eta_{G,n}:T\to[0, +\infty]\quad\text{such that}$$

$$\inf_{x\notin G}\Phi(nx, t)\geqslant\eta_{G,n}(t)\quad\text{and}\quad\sup_{n\in\mathbf{N}}\eta_{G,n}(t)>0\quad\text{for}\quad t\notin T_0.$$

Remarks 3.1. We shall use the following consequence of condition ($I\mu$ (resp. $N\mu$)): If $\mu(T)<\infty$ then every sequence I–convergent (resp. N–convergent is μ–convergent at the same timo (cf. [11, Theorems 2.3 and 2.7]).

If $(X, \|\cdot\|)$ is a metric space then instead of condition $(I\mu)$ one can conside the following one:

$$\inf_{\|x\|>\delta}\Phi(x, t)>0\quad\text{for all } \delta>0 \text{ and a. e. } t\in T$$

although the function $t\mapsto\inf_{\|x\|>\delta}\Phi(x, t)$ may be non-measurable. However, th consequence mentioned above remains true (cf. [1, Proposition 2.3]).

Theorem 3.2. *If $M(T, X)$ has the property (C) and Φ satisfies conditi $(I\mu)$ (resp. $(N\mu)$) then the Musielak-Orlicz space $L^\Phi(M)$ is I-complete (resp. N complete).*

Proof Since μ is a σ–finite measure, there is a function $\xi:T\to(0, +\infty)$ su that $\int_T\xi(t)d\mu<+\infty$. Let us denote $\overline{\Phi}=\Phi\cdot\xi^{-1}$ and $d\overline{\mu}=\xi\cdot d\mu$. Then $\overline{\Phi}$ is a 9 function, $\overline{\mu}$ is a finite measure and

$$\int_T\overline{\Phi}(f(t))t,\ d\overline{\mu}=\int_T\Phi(f(t), t)d\mu.$$

Hence, it suffices to provo the theorem in the case of finite measure μ only.

Let (f_n) be an I–Cauchy sequence with a constant $a>0$. We claim that (f_n) a μ–Cauchy sequence as well. On the contrary, if (f_n) were not a μ–Cauc sequence then we can find a set $G\in\mathscr{C}$, a number $\delta>0$ and sequences $n_k>m_k\geqslant k$ su that

$$\mu(\{t\in T:\ a(f_{n_k}(t)-f_{m_k}(t))\notin a\cdot G\})\geqslant\delta\text{ for } k\in\mathbb{N}.$$

Let $h_k=a(f_{n_k}-f_{m_k})\in M(T, X)$. Then, obviously, (h_k) is not μ–convergent to O. On the other hand, for every $\varepsilon>0$,

$$I(h_k)=I(a(f_{n_k}-f_{m_k}))<\varepsilon$$

for sufficiently large $k\in\mathbb{N}$ i. e. (h_k) is I-convergent to O. Thus, by $(I\mu)$, (h_k) has to be μ-convergent to O——a contradiction.

Now, by property (C), there are a subsequence (f_{n_k}) and a function $g\in M(T, X)$ such that $f_{n_k}(t)\to g(t)$ for a. e. $t\in T$. Therefore, in virtue of the Fatou Lemma,

$$\int_T \Phi(a(f_n(t)-g(t)),\ t)d\mu=\int_T \liminf_{k\to\infty}\Phi(a(f_n(t)-f_{n_k}(t))t)d\mu$$

$$\leqslant\liminf_{k\to\infty}\int_T \Phi(a(f_n(t)-f_{n_k}(t),\ t)d\mu<\varepsilon$$

for all sufficiently large $n\in\mathbb{N}$. Thus (f_n) is I-convergent to g and $g\in L^\Phi(M)$.

In an analogous way (using condition $(N\mu)$) one can prove the N-completeness of the space $L^\Phi(M)$.

Corollaries 3.3. The consequences of Theorem 3.2 are collected in the following table. Let us look at the assumptions concerning X and Φ. we have 8$\Rightarrow$ $(N\mu)$ (in that case we take $G=\{x\in X:\|x\|\leqslant b\}$ and deduce that $\eta_{G,n}(t)=\inf_{\|x\|>b}\Phi(nx, t)$ are measurable functions). Next, using the Pettis theorem and [11, Proposition 2.9] we get $5\Rightarrow6\Rightarrow8$. Analogously, $7\Rightarrow(I\mu)$ [1, Proposition 2.3]. Further, $4\Rightarrow7$ and $3\Rightarrow7$ (cf. [11, Remark 2.4]). Finally, $1\Rightarrow(I\mu)$ and $2\Rightarrow(N\mu)$ even when Φ is not continuous since the functions $t\mapsto\inf_{|x|<b}\Phi(nx, t)$ are measurable (Φ is left-continuous and nondecreasing on $(0, +\infty)$).

Proposition 3.4. *If* $(X, \|-\|)$ *is a linear metric space and* Φ *satisfies condition* $(N\mu)$ *then the F-seminorm* $\|\cdot\|_\Phi$ *becomes an F-norm i. e.* $\|f\|_\Phi=0$ *iff* $f=0$.

Proof Suppose that $\|f\|_\Phi=0$ and $\mu(t\{\in T:f(t)\neq0\})=\mu(\{t\in T:\|f(t)\|>0$. Then $\mu(\{t\in T:\|f(t)\|>r^{-1}\})=c>0$ for some $r\in\mathbb{N}$. Denote $C_r=\{t\in T:\ \|f(t)\|>r^{-1}\}$. Without loss of generality we can assume that $\mu(C_r)=c<+\infty$. Further, let $G\in\mathscr{C}$ be such that $G\subset\{x\in X:\|x\|\leqslant r^{-1}\}$. In virtue of $(N\mu)$

$$\mu(C_r)=\mu(\{t\in C_r:\sup_{n\in\mathbb{N}}\eta_{G,n}(t)>0\})=\mu\Big(\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\{t\in C_r:\eta_{G,n}(t)\geqslant m^{-1}\}\Big).$$

Thus, we can find numbers $m, n\in\mathbb{N}$ such that the set $A_{m,n}=\{t\in C_r:\eta_{G,n}(t)\geqslant m^{-1}\}$ is of positive measure.

On the other hand, there is a sequence $a_l\geqslant0$, $a_l\to0$ such that $I(a_l^{-1}f)\leqslant a_l$ for all $l\in\mathbb{N}$. Let $l_0\geqslant n$ be such a number that $a_l<\min\{n^{-1}, m^{-1}\cdot\mu(A_{m,n})\}$ for all $l\geqslant l_0$. Then

$$\int_T\Phi(a_l^{-1}f(t),\ t)d\mu\geqslant\int_{A_{m,n}}\Phi(nf(t),t)d\mu$$

$$\geqslant\int_{A_{m,n}}\eta_{G,n}(t)d\mu\geqslant m^{-1}\cdot\mu(A_{m,n})>a_l$$

for all $l\geqslant l_0$——a contradiction.

Corollaries 3.3.

	X	Φ	$L^{\Phi}(\mathfrak{m}(T, X))$	$L^{\Phi}(\mathcal{M}(T, X))$
1	$(R, \lvert\cdot\rvert)$	$\Phi(\cdot, t)$ vanishes only at O for a. e. $t \in T$	I_{Φ}-complete	
2		$\Phi(\cdot, t)$ is not identically equal to O for a. e. $t \in T$	N_{Φ}-complete	
3	$(X, \lVert\cdot\rVert)$-a Montel space	Φ is continuous and $\Phi(\cdot, t)$ vanishes only at O for a. e. $t \in T$	I_{Φ}-complete	
4	$(X, \lVert\cdot\rVert)$-a Banach sp.	$\inf_{\lVert x\rVert>b} \Phi(x, t)>0$ for every $b>0$ and a. e. $t \in T$	I_{Φ}-complete	
5	$(X, \lVert\cdot\rVert)$-a separable Banach sp.	Φ is continuous and $\liminf_{\lVert x\rVert\to\infty} \Phi(x, t)>0$ for a. e. $t \in T$	N_{Φ}-complete	
6	$(X, \lVert\cdot\rVert)$-aeparable p-Banach space with a p-homogeneous norm $0<p\leqslant 1$	Φ is continuous and $\liminf_{\lVert x\rVert\to\infty} \Phi(x, t)>0$ for a. e. $t \in T$		N_{Φ}-complete
7	$(X, \lVert\cdot\rVert)$-a Frechet space	$\inf_{\lVert x\rVert>b} \Phi(x, t)>0$ for every $b>0$ and a. e. $t \in T$		I_{Φ}-complete
8	$(X, \lVert\cdot\rVert)$-a separable Frechet space	Φ is continuous and $\sup_{n\in N} \inf_{\lVert x\rVert>b} \Phi(nx, t)>0$ for every $b>0$ and a. e. $t \in T$		N_{Φ}-complete

§ 4. Applications

Our first application deals with a special case of modular spaces (in the se of [9]). Those spaces can be used in the nonlinear integral equations of t Urysohn type

$$f(t)=a\int_T k(|f(t)|, t, s)d\mu(s)+f_0(t)$$

(cf. [6, p. 153] or [5]), where $k:[0, +\infty)\times T\times T\to[0, +\infty)$ is a $\Sigma_L\times\Sigma$ -measurable function (Σ_L denotes the σ-algebra of all Lebesgue measurable sub of $[0, +\infty)$), $k(0, t, s)=0$, $k(\cdot, t, s)$ is strictly increasing convex and continu for all $(t, s)\in T\times T$.

Define a modular $\rho:\mathfrak{M}(T, R)\to[0, +\infty]$ by

$$\rho(f)=\int_T\int_T k(|f(t)|, t, s)d_{\mu}(s)d_{\mu}(t).$$

The following theorem has been presented in [6, Theorm 18. 18].

Theorem 4. 1. *If*

$$\int_T k(u, t, s)d_\mu(t)>0 \quad \textit{for all } u>0 \textit{ and a. e. } s\in T \tag{2}$$

hen the modular space

$$X_\rho=\{f\in\mathfrak{M}(T, \mathbb{R}): \lim_{u\to 0}\rho(uf)=0\}$$

quipped with the F-norm

$$|f|_\rho=\inf\{u>0: \rho(u^{-1}f)\leqslant u\}$$

s complete.

We are going to prove that Theorem 4.1 remains true under some weaker ssumption.

Proposition 4. 2. *Let $A=T\times T$ (he product of two measure spaces) and $=\mu\times\mu$. The modular space $(X_\rho, |\cdot|_\rho)$ is comple e provided*

the functions $k(\cdot, t, s)$ are no identically equal to zero for ν-a. e. $(t, s)\in A$. (3)

Proof Let

$$M(A, \mathbb{R})=\}g:T\times T\to\mathbb{R}: \text{there is } f\in\mathfrak{M}(T, \mathbb{R}) \text{ such that } g(t, s)=f(t) \text{ for } \nu\text{-a. e. } (t, s)\in A\}.$$

f course, there is one-to-one and "onto" correspondence between the spaces $M(A,$) and $\mathfrak{M}(T, \mathbb{R})$, so we can identify them. Further, let

$$\Phi: \mathbb{R}\times A\to[0, +\infty), \ \Phi(x, t, s)=k(|x|, t, s).$$

is easy to verify that Φ is a Φ-function. Moreover, by the Tonelli and Lebe- ue dominated convergence theo ems we infer that

$$L^\Phi(M)=L^\Phi(\mathfrak{M}(T, \mathbb{R}))=\left\{f\in\mathfrak{M}(T, \mathbb{R}):\int_{T\times T}\Phi(af(t), t, s)d\nu<+\infty \text{ for some } a>0\right\}$$

$$=\left\{f\in\mathfrak{M}(T, \mathbb{R}):\int_T\int_T k(a|f(t)|, t, s)d\mu(t)d\mu(s)<+\infty \text{ for some } a>0\right\}$$

$$=X_\rho.$$

Of course, $|f_n|_\rho\to 0$ iff $\|f_n\|_\Phi\to 0$. Hence, in virtue of Corllary 3. 3. 2 and 3 the ace X_ρ is $|\cdot|$-complete.

Corollary 4. 3. *The space X_ρ is ρ-complete provided $k(\cdot, t, s)$ vanishes only at ro for ν-a. e. $(t, s)\in A$.*

The next application deals with families of modules depending on rameter [6, p. 129].

Let (T, Σ, μ), (S, Ξ, ν) be two spaces with nonnnegative, complete, atomless, finite measures and let

$$\varphi:[0, +\infty)\times S\to[0, +\infty)$$

be a function such that $\varphi(u, \cdot)$ is measurable for all $u\geqslant 0$. Moreover, let $\varphi(\cdot, s)$ be a continuous, nondecreasing function for ν-a. .e $s\in S$. A modular ρ_s is defined by

$$\rho_s:\mathfrak{M}(T,\ \mathbb{R})\to[0,\ +\infty]$$
$$f\mapsto\int_S\Big[\int_T\varphi(|f(t)|,\ s)d\mu(t)\Big]d\nu(s).$$

In an analogous way as in Theorem 4.1 we define the modular space X_{ρ_s} and the F-seminorm $|\cdot|_{\rho_s}$.

Proposition[8, p.74] **4. 4.** *If*

$$\varphi(u,\ s)=0 \text{ for } \nu\text{-a. e. } s\in S\Leftrightarrow u=0,$$

then the modular space $(X_{\rho_s},\ |\cdot|_{\rho_s})$ *is complete.*

We shall prove that the assumption of the above proposition can be replace by a weaker one:

Proposition 4. 5. *If* $\varphi(\cdot,\ s)$ *is not identically equal to zero for* ν*–a. e.* $s\in$ *then the space* $(X_{\rho_s},\ |\cdot|_{\rho_s})$ *is complete.*

Proof Let $A=T\times S$ a space with the measure $\mu\times\nu$. Moreover, let

$M(A,\ R)=\{g:T\times S\to\mathbb{R}$: there is $f\in\mathfrak{M}(T,\ \mathbb{R})$ such that

$g(t,\ s)=f(t)$ for $\mu\times\nu$–a. e. $(t,\ s)\in A\}$

and $\Phi:\mathbb{R}\times T\times S\to[0,\ +\infty),\ \Phi(x, t,\ s)=\varphi(|x|,\ s).$

Then $M(A,\ \mathbb{R})$ is a linear subset of $\mathfrak{M}(A,\ \mathbb{R})$ and can be identify with $\mathfrak{M}(T,\ \mathbb{R})$ We claim that Φ is a Φ–function. We shall prove only the $\Sigma\times\Xi$–measurability the function $\Phi(x,\ \cdot,\ \cdot)$ for every fixed $x\in\mathbb{R}$ (cf. Remark 1.2). Let $a>0$. Then

$$\{(t,\ s)\in A:\Phi(x,\ t,\ s)<a\}=\{(t,\ s)\in T\times S:\varphi(|x|,\ s)<a\}$$
$$=T\times\{s\in S:\varphi(|x|,\ s)<a\}\in\Sigma\times\Xi.$$

Further,

$$L^{\Phi}(M(A,\ \mathbb{R}))=\Big\{f\in\mathfrak{M}(T,\ \mathbb{R}):\ \underset{a>0}{\exists}\int_{T\times S}\Phi(af(t),\ t,\ s)d\mu(t)d\nu(s)<+\infty\Big\}$$
$$=\Big\{f\in\mathfrak{M}(T,\ \mathbb{R}):\ \underset{a>0}{\exists}\int_T\Big[\int_S\varphi(a|f(t)|,\ s)d\mu(t)\Big]d\nu(s)<+\infty\Big\}$$
$$=X_{\rho_s}$$

Hence, in virtue of Corollary 3.3.2 the space $(X_{\rho_s},\ |\cdot|_{\rho_s})$ is complete.

Corollary 4. 6. *If* $\varphi(u,\ s)=0\Leftrightarrow u=0$ *for* ν*–a. e.* $s\in S$*, then the space* X_{ρ_s} *ρ_s–complete.*

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