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THE DIRICHLET PROBLEM FOR DIFFUSION EQUATION

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Abstract

Let D be a bounded domain in the $d+1$-dimensional Euclidean space R^{d+1}. This paper aims at giving a probabilistic treatment of the Dirichlet problem for the following diffusion equation on D

$$(1/2\Delta+q)u(x,t)=\frac{\partial}{\partial t}u(x,t),\quad (x,t)\in D,$$

where q is a function to be specified later and Δ is the Laplace operator $\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$. The existence and uniqueness theorems are given, and furthermore, the probabilistic representation and martingale characterization of the solutions for diffusion equations are obtained.

§0. Introduction

Let D be a bounded domain in R^d. In 1954, J. L. Doob gave an intensive stuc of the Laplace,s equation on D from a probabilistic point of view (see [1]). Rece tly, the probabilistic treatment of the following Schrödinger equation on D

$$(\Delta/2+q)u(x)=0,\ x\in D, \tag{0.:}$$

becomes an active topic, where $\Delta=\sum_{i=1}^{d}\frac{\partial^2}{\partial x_i^2}$, q is a suitable function on D. A seri of results have been obtained (see [4, 6]). The tool employed there is Brownia motion.

Now let D be a bounded domain in R^{d+1}. A point in R^{d+1} is denotedby $X=(x,$ with $x\in R^d$ and $t\in R^1$. In [2], J. L. Doob studied the following heat equatic (0.2) by probability methods

$$\frac{1}{2}\Delta u(x,t)=\frac{\partial}{\partial t}u(x,t),\quad (x,t)\in D. \tag{0.:}$$

We have observed that the key in his probabili ty approach is the use of stochast process——space-time Brownian motion to be defined below.

Keeping the previous reviews in mind, a detailed observation yields a questio Can we approach the following most typical diffusion equation (0.3) on

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probabilistically?

$$\left(\frac{1}{2}\Delta+q\right)u(x, t)=\frac{\partial}{\partial t}u(x, t), \quad (x, t)\in D, \tag{0.3}$$

where q is a suitable function on D. It seems that this work has not been explored up to now. This paper is contributed to the answer of such a question to someextent.

§1. Preliminaries

The set $I=(a_1, b_1)\times\cdots\times(a_d, b_d)\times(s_1, s_2)$ in R^{d+1} is called an interval. Particularly, for any $r>0$, $X=((x_1, \cdots, x_d), s)\in R^{d+1}$, we set $R(X, r)=(x_1-r, x_1+r)\times\cdots\times(x_d-r, x_d+r)\times(s-r, s+r)$. Throughout this paper, $l_d(dx)$ denotes the Lebesgue measure in R^d. Let $D\subset R^{d+1}$ be an open set, we use $C^{2,1}(D)$ to denote the class of functions on D which are twice continuously differentiable with respect to x and continuously differentiable with respect to t.

A function $u(X)$ on an open set $D\subset R^{d+1}$ is called parabolic iff

$$\frac{1}{2}\Delta u(x, t)=\frac{\partial}{\partial t}u(x, t), \quad (x, t)\in D. \tag{1.1}$$

Let $X_0=(x_0, s_0)\in R^{d+1}$ and $\{x(t), \mathscr{F}_t, t\geqslant 0\}$ be a d-dimensional standard Brownian motion in R^d starting from x_0. The process

$$\{X(t), \mathscr{F}_t, t\geqslant 0\}, =\{(x(t), s_0-t), \mathscr{F}_t, t\geqslant 0\}$$

with state space R^{d+1} is called a standard space-time Brownian motion (denoted simply by SSTBM) from X_0 (see [3], p. 575).

Define

$$p(t, x)\triangleq\begin{cases}(2\pi)^{-d/2}\exp\left\{\dfrac{-|x|^2}{2t}\right\}, & \text{if } t>0,\\ 0, & \text{if } t\leqslant 0,\end{cases} \quad (x, t)\in R^{d+1}, \tag{1.2}$$

and set for $x, y\in R^d$, $t\geqslant 0$,

$$p(t, x, y)\triangleq p(t, x-y).$$

For any $X=(x, s)$, $Y=(y, t)\in R^{d+1}$, define

$$p(X, Y)\triangleq p(s-t, x-y). \tag{1.3}$$

The SSTBM defined above is a continuous strong Markov process ([3], p. 579) and has the following transition density with respect to $l_d(dy)$ (see [3], p. 576).

$$p(t, (x, s), (y, s-t))$$
$$\triangleq p(d(x, s), (y, s-t)), \quad t>0, \quad X=(x, s), \ Y=(y, s-t)\in R^{d+1}.$$

We use P^X and E^X to denote the probability and expectation associated with the SSTBM from $X\in R^{d+1}$.

Let B be a Borel subset of R^{d+1}, $r>0$, define

$$\tau_B=\inf\{t>0, X(t)\notin B\}, \tag{1.5}$$
$$\tau_r=\inf\{t>0, X(t)\notin R(X(0), r)\}, \quad (\inf\phi=\infty). \tag{1.6}$$

Then it can be proved that τ_B and τ_r are optional relative to $\{\mathscr{F}_t\}_{t>0}$.

For an open set $D\subset R^{d+1}$, ∂D denotes its boundary, a point $Z\in\partial D$ is said to bearegular boundary point iff $P^Z(\tau_D=0)=1$. By the zero-one law for SSTBM (see [3], p. 583), $Z\in\partial D$ is irregular iff $P^Z(\tau_D=0)=0$.

For $X=(x, s)$, $Y=(y, t)\in D$, define

$$p_D(X, Y)\triangleq p(s-t, x-y)-E^{(x,s)}\{p(s-t-\tau_D, x(\tau_D)-y)\}. \quad (1.7)$$

Then $p_D(X, Y)$ is continuous in $D\times D\backslash\{(X, X), X\in D\}$ (see [3], p. 593, p. 661 Theorem 17(a) and p. 298–299).

The SSTBM in an open set D is a SSTBM from a point in D, killed at th irst exit time of D. It is a Markov process with submarkovian transition functio given by

$$p_D(t, (x, s), B)\triangleq P^{(x,s)}\{X(t)\in B, t<\tau_D\}, t>0, B\subset R^{d+1}.$$

The transition density with respectto $l_d(dy)$ is given by

$$p_D(t, (x, s), (y, s-t))\triangleq p_D((x, s), (y, s-t))$$
$$=p((x, s), (y, s-t))-r_D((x, s), (y, s-t)), \quad (1.8)$$

where

$$r_D((x, s), (y, s-t))=E^{(x,s)}\{p((x(\tau_D), s-\tau_D), (y, s-t))\}. \quad (1.9)$$

As in the situation of Brownian motion, we have the following proposition. Fo the proof see Proposition 2.2.1[7].

Proposition 1. 1. *Let D be an open subset of R^{d+1}. Then for any $t>0$, $P^{\cdot}\{\tau_D<t\}$ is lower semicontinuous in R^{d+1}.*

An immediate consequence of this proposition is the following.

Corollary 1.1. *Let D be an open subset of R^{d+1}, and $z\in\partial D$ be a regular boundary point. For any $r>0$, τ_r is defined by* (1.6). *Then we have*

$$\lim_{D\ni X\to Z} P^X\{\tau_D<\tau_r\}=1.$$

§2. Basic Definitions and Results

From now on, D is always assumed to be a bounded domain in R^{d+1} unless otherwise stated.

Definition 2.1. *Let q be a measurable function defined on D. We say that $q\in K_d$, iff the following condition is satisfied*

$$\lim_{t\to 0}\sup_{X\in D} E^X\left[\int_0^t |q|(X_s)ds\right]=0, \quad (2.1)$$

here and hereafter, $q(X)$ is understood to be zero outside D.

Keeping the convention on q in mind, we can easily prove an equivalent condition of (2.1).

Proposition 2.1. *Let q be a measurable function on D. Then $q\in K_d$ iff*

$$\lim_{t\to 0}\sup_{X\in R^{d+1}} E^X\left[\int_0^t |q|(X_s)\,ds\right]=0. \tag{2.2}$$

Proof It is enough to show that for any $X_0\in R^{d+1}\setminus\overline{D}$, $t>0$

$$E^{X_0}\left[\int_0^t |q|(X_s)\,ds\right]\leqslant\sup_{X\in\overline{D}} E^X\left[\int_0^t |q|(X_s)\,ds\right].$$

In fact, by strong Markov property

$$E^{X_0}\left[\int_0^t |q|(X_s)\,ds\right]=E^{X_0}\left[\int_{\tau_{D^c}}^t |q|(X_s)\,ds,\ \tau_{D^c}<t\right]$$

$$\leqslant E^{X_0}\left[\int_{\tau_{D^c}}^{t+\tau_{D^c}} |q|(X_s)\,ds,\ \tau_{D^c}<t\right]=E^{X_0}\left[E^{X_{\tau_{D^c}}}\left[\int_0^t |q|(X_s)\,ds\right],\ \tau_{D^c}<t\right]$$

$$\leqslant\sup_{X\in\overline{D}} E^X\left[\int_0^t |q|(X_s)\,ds\right]$$

as desired.

Let $q\in K_d$, define

$$e_q(t)=\exp\left[\int_0^t q(X_s)\,ds\right] \tag{2.3}$$

where $\{X_t,\ t\geqslant 0\}$ is the SSTBM in R^{d+1}.

Proposition 2.2. *Suppose that $q\in K_d$, then*

$$\sup_{X\in R^{d+1}} E^X\{e_{|q|}(\tau_D)\}<\infty. \tag{2.4}$$

Proof The boundedness of D guarantees that τ_D is bounded, say by T_0, i. e. for $X\in R^{d+1}$, $P^X\{\tau_D<T_0\}=1$. Since $q\in K_d$, there exists a $t_0>0$ such that

$$\sup_{X\in R^{d+1}} E^X\left[\int_0^{t_0} |q|(X_s)\,ds\right]<1/2.$$

Then by Khas' minskii's lemma[6], we obtain

$$\sup_{X\in R^{d+1}} E^X[e_{|q|}(t_0)]<2.$$

Choose an integer n so that $n-1\leqslant T_0/t_0<n$, then using the strong Markov property, we have

$$E^X\{e_{|q|}(\tau_D)\}\leqslant E^X\left\{\exp\left\{\int_0^{T^0} |q|(X_t)\,dt\right\}\right\}$$

$$=E^X\left\{\exp\left\{\int_{t_0}^{t_0}+\int_{t_0}^{2t_0}+\cdots+\int_{(n-1)t_0}^{T_0} |q|(X_t)\,dt\right\}\right\}$$

$$\leqslant(\sup_{X\in R^{d+1}} E^T\{e_{|q|}(t_0)\})^n<2^n<\infty.$$

Therefore

$$\sup_{X\in R^{d+1}} E^X\{e_{|q|}(\tau_D)\}<\infty.$$

§ 3. Dirichlet Problem

The formulation of Dirichlet problem for diffusion equation (0.3) is as follows: Given a bounded measurable function f on ∂D, a function u on D is to be found such that

$$\begin{cases}\left(\frac{1}{2}\Delta+q-\frac{\partial}{\partial t}\right)u(x,\ t)=0,\ X=(x,\ t)\in D, & (3.1)\\ \lim\limits_{\overline{D}\in X\to Z} u(X)=f(Z), & (3.2)\end{cases}$$

where Z is any regular boundary point at which f is continuous.

Suppose that f is a bounded measurable function on ∂D, $q\in K_d$, define

$$u(X)\triangleq E^X\{e_q(\tau_D)f(X_{\tau_D})\},\ X\in\overline{D}. \tag{3.3}$$

It follows from Proposition 2.2 that $\|u\|_{\overline{D}}=\sup\limits_{X\in\overline{D}}|u|(X)<\infty$.

The following theorem characterizes the boundary behavior of $u(X)$ defined by (3.3).

Theorem 3.1. *Suppose that $q\in K_d$ and f is a bounded measurable function on ∂D. If $Z\in\partial D$ is a regular boundary point at which f is continuous, then we have*

$$\lim_{\overline{D}\ni X\to Z} u(X)=f(Z). \tag{3.4}$$

Proof It can be easily deduced that

$$u(X)=P_Df(X)+E^X\left\{f(X_{\tau_D})\int_0^{\tau_D}q(X_s)\exp\left[\int_s^{\tau_D}q(X_t)\,dt\right]ds\right\} \tag{3.5}$$

where

$$P_Df(X)\triangleq E^X\{f(X_{\tau_D})\},\ X\in\overline{D}. \tag{3.6}$$

Using Fubini theorem, the second term on the right side of (3.5) is equal to

$$E^X\left\{\int_0^\infty f(X_{\tau_D})q(X_s)I_{(s>\tau_D)}\exp\left[\int_s^{\tau_D}q(X_t)\,dt\right]ds\right\}$$
$$=\int_0^\infty E^X\left\{f(\bar{x}_{\tau_D})q(X_s)I_{(s<\tau_D)}\exp\left[\int_s^{\sigma_D}q(X_t)\,dt\right]\right\}ds. \tag{3.7}$$

By the strong Markov property, the integrand on the right side above relative to ds reduces to

$$E^X\left[E^X\left[f(X_{\tau_D})q(X_s)I_{(s<\tau_D)}\exp\left[\int_s^{\tau_D}q(X_t)dt\right]\Big|\mathscr{F}_s\right]\right]$$
$$=E^X[q(X_s)I_{(s<\tau_D)}E^{X_s}\{e_q(\tau_D)f(X_{\tau_D})\}]$$
$$=E^X[q(X_s)u(X_s)I_{(s<\tau_D)}]. \tag{3.8}$$

Combining (3.5) with (3.7) and (3.8), and using Fubini theorem we get

$$u(X)=P_Df(X)+E^X\left\{\int_0^{\tau_D}q(X_s)u(X_s)ds\right\}=P_Df(X)+G_D(qu)(X), \tag{3.9}$$

where $G_D(qu)(X)\triangleq E^X\left\{\int_0^{\tau_D}q(X_s)u(X_s)ds\right\}$.

It is known (see [2], Theorem 3.1) that

$$\lim_{\overline{D}\ni X\to Z}P_Df(X)=f(Z). \tag{3.10}$$

Hence, to prove (3.4), it is sufficient to show that

$$\lim_{\overline{D}\in X\to Z}G_D(qu)(X)=0.$$

In fact, for any $\delta>0$, using strong Markov property, $\exists C_1, C_2>0$, s. t.,

$$G_D(|qu|)(X)=E^X\left\{\int_0^{\tau_D}|qu|(X_s)ds,\ \tau_D\leqslant\delta\right\}+E^X\left\{\int_0^{\tau_D}|qu|(X_s)ds,\ \tau_D>\delta\right\}$$

$$\leqslant C_1\sup_{X\in\bar{D}}E^X\left[\int_0^{\delta}|q|(X_s)ds\right]+C_2P^X[\tau_D>\delta]\cdot\sup_{X\in\bar{D}}E^X\left[\int_0^{\tau_D}|q|(X_s)ds\right]$$

Hence $\lim\limits_{D\in X\to Z}G_D(qu)(X)=0.$

Before giving the main result in this paper, let us first state several classical results to be needed later.

Let G be a bounded domain in R^d, $T_1, T_2\in R^d$ with $T_1<T_2$. In the following three lemmas, $D=G\times(T_1, T_2)$.

Lemma 3.1. *Suppose that $g(x, t)$ is a bounded measurable function on $\bar{D}=\bar{G}\times[T_1, T_1]$. Then the function $V_Dg(x, s)$ defined by*

$$V_Dg(x, s)\triangleq\int_0^T\int_G g(y, s-t)p(t, (x, s), (y, s-t))l_d(dy)dt,\ (x, s)\in\bar{D}\qquad(3.12)$$

is continuous in $\bar{D}$, where $T=T_2-T_1$, $p(t, (x, s), (y, s-t))$ is defined by (1.4) and $g(x, t)$ is identified with zero outside $\bar{D}$.

Lemma 3.2. *Suppose that $g(x, t)$ is continuous in $\bar{D}$, $V_Dg(x, s)$ is defined by (3.12). Then $\frac{\partial}{\partial x_i}V_Dg(x, s)$ exists $(i=1, \cdots, d)$ for $(x, s)\in D$ and is continuous in $(x, s)\in D$.*

Before proceeding further, letus giveadefinition.

Definition 3.1. *A function $g(x, t)$ on $\bar{D}=\bar{G}\times[T_1, T_2]$ is said to be Hölder continuous in $x\in G$ uniformly for t, iff $\exists$ constants $c, \alpha>0$ s. t.*

$$|g(x, t)-g(y, t)|\leqslant C|x-y|^{\alpha},\ \forall x, y\in G,\ \forall t\in[T_1, T_2].$$

Lemma 3. 3. *Suppose that $g(x, t)$ is continuous in $\bar{D}$ and Hölder continuous in $x\in G$ uniformly for t. Then $V_Dg(x, s)$ defined by (3.12) is twice continuously differentiable with respect to $x\in G$, continuously differentiable relative to $s\in(T_1, T_2)$. Furthermore, we have*

$$\frac{1}{2}\Delta V_Dg(x, s)+g(x, s)=\frac{\partial}{\partial s}V_Dg(x, s),\quad(x, s)\in D.$$

These lemmas are the versions of the corresponding results in [10] and the proofs are referred to [10], Chapter 1.

Theorem 3.2. *Let D be a bounded domain in R^{d+1}. Assume that $q\in K_d$ is continuous in D and for any $X_0\in D$, any $r>0$ such that $\overline{B(X_0, r)}\subset D$, $q(x, t)$ is Hölder continuous in x uniformly for t in $B(X_0, r)$. Suppose that f is bounded measurable on ∂D. Then the function*

$$u(X)=E^X\{e_q(\tau_D)f(X_{\tau_D})\},\quad X\in\bar{D}$$

gives a solution for the Dirichlet problem (3.1) and (3.2).

Proof We shall prove this result in four steps.

Step 1. As in the proof of Theorem 3.1, we have

$$u(X)=P_D f(X)+G_D(qu)(X),\quad X\in D. \tag{3.13}$$

Step 2. In this step, we shall assume that $D=G\times(T_1, T_2)$ and $q(X)$ is identified with zero outside D, where G is a bounded domain in R^d, $T_1, T_2\in R^1$ and $T_1<T_2$. We shall decompose $G_D(qu)(X)$ above.

For $X=(x, s)\in D$, using Fubiui theorem, we obtain

$$\begin{aligned} G_D(qu)(X) &= E^X\left\{\int_0^{\tau_D} q(X_t)u(X_t)dt\right\} \\ &\xlongequal{T=T_2-T_1} E^X\left\{\int_0^T q(X_t)u(X_t)I_{(t<\tau_D)}dt\right\}=\int_0^T E^X\{q(X_t)u(X_t)I_{(t<\tau_D)}\}dt \\ &=\int_0^T\int_G q(y, s-t)u(y, s-t)p_D(t, (x, s), (y, s-t))l_d(dy)dt \\ &=\int_0^T\int_G q(y, s-t)u(y, s-t)p(t, (x, s), (y, s-t))l_d(dy)dt \\ &\quad -\int_0^T\int_G q(y, s-t)u(y, s-t)r_D((x, s), (y, s-t))l_d(dy)dt \\ &\triangleq V_D(qu)(X)-Q_D(qu)(X), \text{ say,} \end{aligned}$$

where in the fourth equality, we have used the convention on q and

$$Q_D(qu)(X)=\int_0^T\int_G q(y, s-t)u(y, s-t)r_D((x, s), (y, s-t))l_d(dy)dt.$$

We note that the intgral interval $[0, T]$ above can be replaced by $[0, +\infty$ Then by the definition of $r_D((x, s), (y, s-t))$ and Fubini theorem, we have

$$\begin{aligned} Q_D(qu)(X) &= \int_0^\infty\int_G q(y, s-t)u(y, s-t)E^{(x,s)}\{p(t-\tau_D, x(\tau_D)-y)\}l_d(dy)dt \\ &= E^{(x,s)}\left\{\int_0^\infty\int_G q(y, s-t)u(y, s-t)p(t-\tau_D, x_{\tau_D}-y)l_d(dy)dt\right. \\ &= E^{(x,s)}\left\{\int_0^\infty\int_G q(y, s-\tau_D-t)u(y, s-\tau_D-t)p(t, x_{\tau_D}-y)l_d(dy)dt\right\}. \end{aligned}$$

Set

$$\varphi(x, s)\triangleq\int_0^\infty\int_G q(y, s-t)u(y, s-t)p(t, x, y)l_d(dy)dt,\quad (x, s)\in\partial D. \tag{3.1}$$

Since u is bounded in $\bar{D}$, by Khas' minskii's lemma[6], we can easily conclude th φ is bounded measurable on ∂D. Hence

$$Q_D(qu)(X)=E^{(x,s)}\varphi(x_{\tau_D}, s-\tau_D)=P_D\varphi(X).$$

Finally we obtain

$$G_D(qu)(X)=V_D(qu)(X)-P_D\varphi(X). \tag{3.1}$$

Step 3. For any $X_0\in D$, choose an $r>0$ such that $R(X_0, r)\subset R(X_0, 2r)$ $R(X_0, 3r)\subset\overline{R(X_0, 3r)}\subset D$. By the strong Markov property, for any $X\in R($ $3r)$, we have

$$u(X)=E^X\{e_q(\tau_{R(X_0,3r)})u(X\tau_{R(X_0,3r)})\}.$$

We note from this equality that we can regard q as zero outside $\overline{R(X_0, 3r)}$ in the following without affecting ihe value of $u(X)$ on $R(X_0, 3r)$. Repeating the procedure in step 1 and step 2 with f and D there being replaced by u and $R(X_0, 3r)$, we obtain decompositions:

$$u(X)=P_{R(X_0,3r)}u(X)+G_{R(X_0,3r)}(qu)(X), \tag{3.15}$$

$$G_{R(X_0,3r)}(qu)(X)=V_{R(X_0,3r)}(qu)(X)-P_{R(X_0,3r)}\varphi(X), \tag{3.15$'$}$$

where $\varphi(X)$ is defined by (3.14) with D there being replaced by $R(X_0, 3r)$.

Since f is bounded, $u(X)$ is bounded on $\overline{D}$ by Proposition 2.2. Hence qu is bounded on $\overline{R(X_0, 3r)}$. Using Lemma 3.1, we know that $V_{R(X_0, 3r)}(qu)(X)$ is continuous in $\overline{R(X_0, 3r)}$. It is known that $P_{R(X_0,3r)}\varphi(X)$ is parabolic on $R(X_0, 3r)$ (see [2] or [3]). Hence $G_{R(X_0,3r)}(qu)(X)$ is continuous in $R(X_0, 3r)$ by (3.15)'. By (3.15), we know that $u(X)$ is continuous in $R(X_0, 3r)$.

As is similar to (3.15) and (3.15)' by regarding q asz ero outside $\overline{R(X_0, 2r)}$, we have for any $X\in R(X_0, 2r)$,

$$u(X)=P_{R(X_0,2r)}u(X)+G_{R(X_0,2r)}(qu)(X), \tag{3.16}$$

$$G_{R(X_0,3r)}(qu)(X)=V_{R(X_0,2r)}(qu)(X)-P_{R(X_0,2r)}\varphi(X), \tag{3.16$'$}$$

where $\varphi(X)$ is defined by (3.14) corresponding to $R(X_0, 2r)$.

Since qu is continuous in $\overline{R(X_0, 2r)}$, by Lemma 3.2, $\frac{\partial}{\partial x_i}V_{R(X_0,2r)}(qu)(x, t)$ exists and is continuous in $(x, t)\in R(X_0, 2r)$, $i=1, \cdots, d$. By (3.16) and (3.16)' we know that $\frac{\partial}{\partial x_i}u(X)$ exists and is continuous in $R(X_0, 2r)$, $i=1, \cdots, d$. Hence $u(x, t)$ is Hölder continuous in x uniformly for t in $\overline{R(X_0, r)}$. Consequently, qu has the same property as u in $\overline{R(X_0, r)}$.

For the same reason, with the convention that $q=0$ outside $\overline{R(X_0, r)}$, for any $X\in R(X_0, r)$, we have

$$u(X)=P_{R(X_0,r)}u(X)+G_{R(X_0,r)}(qu)(X), \tag{3.17}$$

$$G_{R(X_0,r)}(qu)((X)=V_{R(X_0,r)}(qu)(X)-P_{R(X_0,r)}\varphi(X), \tag{3.17$'$}$$

where $\varphi(X)$ has the clear meaning.

By Lemma 3.3, we have for any $X\in R(X_0, r)$,

$$\frac{1}{2}\Delta V_{R(X_0,r)}(qu)(x, t)+(qu)(x, t)=\frac{\partial}{\partial t}V_{R(X_0,r)}(qu)(x, t).$$

It follows from this and (3.17), (3.17)' that

$$\frac{1}{2}\Delta u(X)+(qu)(X)=\frac{\partial}{\partial t}u(X), \quad X\in R(X_0, r).$$

Hence u is a solution on $R(X_0, r)$. Since X_0 is arbitrary, we know that u is a solution of (3.1) on D.

Step 4. By Theorem 3.1, we know that $u(X)$ has the boundary property as desired, the whole proof is now completed.

For a more general functional class of q, we have the following

Theorem 3.3. *Suppose that $q \in K_d$ is locally bounded on D and f is bounded measurable on ∂D. Then $u(X) = E^X\{e_q(\tau_D) f(X_{\tau_D})\}$ is continuous in D.*

Proof For any $X_0 \in D$, there exists an $r > 0$, such that $\overline{R(X_0, r)} \subset D$ and q is bounded on $\overline{R(X_0, r)}$. For any $X \in R(X_0, r)$, by the strong Makov property, we have

$$u(X) = E^X\{e_q(\tau_{R(X,r)}) u(X\tau_{R(X,r)})\}. \tag{3.18}$$

Since $u(X)$ is bounded on $\overline{D}$ and hence on $\partial R(X_0, r)$, it follows from (3.18) th it is sufficient to prove the theorem by supposing that q is bounded on $\overline{D}$ and D an interval. But then the conclusion follows from the proof of Theorem 3.2.

In contrast to Theorem 3.2, the following result gives a probabilis repesentation and a martingale characterization of the solutions for diffusi equations.

Theorem 3.4. *Let $q \in K_d$, if the function $u(X)$ in $C^{2,1}(D)$ satisfies*

$$\frac{1}{2} \Delta u(X) + (qu)(X) = \frac{\partial u(X)}{\partial s}, \quad X = (x, s) \in D. \tag{3.1}$$

Then for every domain E, $\overline{E} \subset D$, the following two conditions are satisfied

(1) $u(X) = E^X\{e_q(t \wedge \tau_E) u(X_{t \wedge \tau_E})\}, \ X \in D,$ (3.2

(2) *For any $X \in D$, $\{e_q(t \wedge \tau_E) u(X_{t \wedge \tau_E}), \mathscr{F}_t\}_{t \geq 0}$ forms a P^X-martingale.*

Furthermore, (1) *and* (2) *are equivalent.*

Proof Now suppose that $u(X)$ satisfies (3.19) on D, we prove (1).

If $X = (x, s) \in D \setminus \overline{E}$, then $P^X\{\tau_E = 0\} = 1$, hence (1) is true. Thus we have o to consider the case that $X \in \overline{E}$. Define a function $F(X, y)$ on $D \times R^1$ by

$$F(X, y) = yu(X), \quad X = (x, s) \in D, \ y \in R^1. \tag{3.}$$

Then

$$\frac{\partial F(X, y)}{\partial x_i} = y \frac{\partial u(X)}{\partial x_i}, \quad \frac{\partial^2 F(X, y)}{\partial x_i^2} = y \frac{\partial^2 u(X)}{\partial x_i^2},$$

$$\frac{\partial F(X, y)}{\partial s} = y \frac{\partial u(X)}{\partial s}, \quad \frac{\partial F(X, y)}{\partial y} = u(X), \tag{3.}$$

$$X = ((x_1, \cdots, x_d), s) \in D, y \in R^1, \ i = 1, \cdots, d.$$

For the SSTBM $\{X_t, \mathscr{F}_t\}_{t \geq 0} = \{[x_t, s-t], \mathscr{F}_t\}_{t \geq 0}$ from $X = (x, s)$, we set

$$M^i(t) = \{x_i^{t \wedge \tau_E}\}_{t \geq 0}, \ i = 1, \cdots, d.,$$

$$V^0(t) = \{s - \wedge t\tau_E\}_{t \geq 0}, \ V^1(t) = \{e_q(t \wedge \tau_E)\}_{t \geq 0}, \tag{3.}$$

$$N(t) = (M^1(t), \cdots, M^d(t), V^0(t), V^1(t)).$$

Then $M^i(t)$ is a continuous L^2-martingale with respect to $\{\mathscr{F}_t\}_{t \geq 0}$ and $V^0(t)$, V^1 are $\mathscr{F}_t$-adapted processes which are of bounded variations on each finite interval, and so $N(t)$ is a $\{\mathscr{F}_t\}_{t \geq 0}$ semimartingale. Since $\langle M^i, M^j \rangle_r = \langle x^i, x^j \rangle r \wedge \tau_E = \delta_{ij}(r \wedge \tau_E)$, using Ito's formula[8] for $F(X, y)$ and $N(t)$, we obtain

$$e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})-u(X)$$
$$=\sum_{i=1}^{d}\int_0^t e_q(r)\frac{\partial u}{\partial x_i}(X_r)I_{(r<\tau_E)}dx_r^i-\int_0^t e_q(r)\frac{\partial u}{\partial s}(X_r)I_{(r<\tau_E)}dr$$
$$+\int_0^t u(X_r)q(X_r)e_q(r)I_{(r<\tau_E)}dr+\frac{1}{2}\int_0^t e_q(r)\Delta u(X_r)I_{(r<\tau_E)}dr.$$

In view of the assumption (3.19), the above equality reduces to

$$e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})-u(X)$$
$$=\sum_{i=1}^{d}\int_0^t e_q(r)\frac{\partial u}{\partial x_i}(X_r)I_{(r<\tau_E)}dx_r^i=\overline{M}(t),\quad \text{say.} \tag{3.24}$$

By the conclusion in [8], $\{\overline{M}(t),\ \mathscr{F}_t\}_{t\geq 0}$ is a P^X-martingale with $\overline{M}(0)=0$. Then from (3.24), we obtain

$$u(X)=E^X\{e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})\}.$$

Hence (1) is true for $X\in D$.

To complete the proof, it is enough to show that (1)⇒(2).

Now suppose that (1) holds. We only consider $X=(x,s)\in\overline{E}$. Since $\overline{E}$ is bounded, $u(X_{t\wedge\tau_E})$ is bounded P^X-a.s. Hence we can easily show that $e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})$ is P^X-integrable. For any $0\leq s<t<\infty$, by the strong Markov property, we have

$$E^X[e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})\,|\,\mathscr{F}_s]=E^X[e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})\,|\,\mathscr{F}_{s\wedge\tau_E}]$$
$$=e_q(s\wedge\tau_E)E^{X(s\wedge\tau_E)}\{e_q((t-s)\wedge\tau_E)u(X_{(s-t)\wedge\tau_E})\}$$
$$=e_q(s\wedge\tau_E)u(X_{s\wedge\tau_E})$$

which implies (2).

Next result deepens (1) in the above theorem.

Theorem 3.5. *Let $q\in K_d$ and suppose that $u(X)$ in $C^{2,1}(D)$ satisfies* (3.19). *Then for every domain E, $\overline{E}\subset D$, we have*

$$u(X)=E^X\{e_q(\tau_E)u(X_{\tau_E})\},\ X\in D.$$

Proof We only consider $X\in\overline{E}$. By Theorem 3.4(1), we have

$$u(X)=E^X\{e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})\}.$$

Since $\overline{E}$ is bounded, $u(X_{t\wedge\tau_E})$ is bounded P^X-a. s.. Hence, $\exists M>0$ s. t.

$$|e_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})|\leq Me_{|q|}(t\wedge\tau_E)\leq Me_{|q|}(\tau_E).$$

It follows from Proposition 2.2 that $E^X\{e_{|q|}(\tau_E)\}<\infty$. Hence by dominated convergence theorem, we have

$$u(X)=\lim_{t\uparrow\infty}E^Xe_q(t\wedge\tau_E)u(X_{t\wedge\tau_E})=E^X\{e_q(\tau_E)u(X_{\tau_E})\}.$$

We now proceed to show the uniqueness of the solution.

Theorem 3.6. *Let $q\in K_d$. Suppose that $u(X)\in C^{2,1}(D)$ is bounded and satisfies*

$$\frac{1}{2}\Delta u(X)+(qu)(X)=\frac{\partial u(X)}{\partial s},\quad X=(x,s)\in D.$$

Furthermore, we assume

$$\lim_{D\ni X\to Z}u(X)=f(Z),$$

where f is a bounded measurable function on ∂D and Z is any regular boundary point of D. For any $X \in D$, assume that $P^X\{X_{\tau_D}$ is regular for $\partial D\}=1$. Then we have

$$u(X)=E^X\{e_q(\tau_D)f(X_{\tau_D})\}, \quad X\in D.$$

Proof Choose a sequence of domains $\{D_n\}$ such that $\bar{D}_n \subset D_{n+1} \subset \bar{D}_{n+1} \subset D$ and $D_n \uparrow D$. Then $\tau_{D_n} \uparrow \tau_D$, P^X-a. s. $\forall X \in D$. We are sure that for any $X\in D$, $\{e(\tau_{D_n})u(X_{\tau_{D_n}}), \mathscr{F}_{\tau_{D_n}}\}_{n\geq 1}$ is a P^X-martingale. In fact, for any $n\geqslant 2$, by the strong Markov property, we have

$$\begin{aligned}&E^X\{e_q(\tau_{D_n})u(X_{\tau_{D_n}})\,|\,\mathscr{F}_{\tau_{D_{n-1}}}\}\\&\quad=e_q(\tau_{D_{n-1}})E^X[\{e_q(\tau_{D_n})u(X_{\tau_{D_n}})\}\theta_{\tau_D}\,|\,\mathscr{F}_{\tau_{D_{n-1}}}]\\&\quad=e_q(\tau_{D_{n-1}})E^{X_{\tau_{D_{n-1}}}}\{e_q(\tau_{D_n})u(X_{\tau_{D_n}})\}=e_q(\tau_{D_{n-1}})u(X_{\tau_{D_{n-1}}}),\end{aligned}$$

where in the last equation wehave used Theorem 3.5. By the assumption, u bounded, say by M. Hence by Proposition 2.2, we obtain

$$\begin{aligned}&\sup_n E^X\{|e_q(\tau_{D_n})u(X_{\tau_{D_n}})|^2\}\\&\leqslant M^2\sup_n E^X\{e_{2|q|}(\tau_{D_n})\}\leqslant M^2E^X\{e_{2|q|}(\tau_D)\}<\infty.\end{aligned}$$

We know from Theorem 3.5 that for any $n\geqslant 1$

$$u(X)=E^X\{e_q(\tau_{D_n})u(X_{\tau_{D_n}})\}, \quad X\in D.$$

It follows from the martingale convergence theorem (see [9] Theorem 3.11) that

$$u(X)=\lim_{n\to\infty}E^X\{e_q(\tau_{D_n})u(X_{\tau_{D_n}})\}=E^X\{e_q(\tau_D)f(X_{\tau_D})\}.$$

Putting Theorems 3.1, 3.2 and 3.6 together, we finally obtain the following

Theorem 3.7. *Suppose that q is the same as in Theorem 3.2, and f is bounded measurale on ∂D and continuous at every regular boundary point of D and for any $X\in D$, $P^X\{X_{\tau_D}$ is regular for $\partial D\}=1$. Then the Dirichlet problem (3.1) and (3.2) has a unique bounded solution.*

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