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PROPERTIES IN THE LARGE OF QUADRATIC SYSTEMS IN THE PLANE

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Abstract

The author proves that any quadratic system in the plane can be changed into the quadratic system (E_1) or (E_2) by means of linear transformations, and then gives a necessary and sufficient condition for the systems (E_1) and (E_2) to be bounded for $t\geqslant 0$ and to have precisely one monotone unbounded orbit for $t\geqslant 0$ respectively.

§ 0. Introduction

A complete and detailed investigation on quadratic systems can be found in monograph[1]. The distribution of limit cycles and the global structure of orbits (including a class of bounded quadratic systems) are especially considered in [1]. For general bounded quadratic systems, the following results were obtained in [2].

The quadratic system

$$\dot{x}=a_{11}x+a_{12}y+P(x, y),\ \dot{y}=a_{21}x+a_{22}y+Q(x, y),$$

where P, Q are homogeneous quadratic polynomials of x and y, is bounded if it is affine-equivalent to

$$\dot{x}=a_{11}x,\ \dot{y}=a_{21}x+a_{22}y+xy$$

with $a_{11}<0$, $a_{22}\leqslant 0$, or

$$\dot{x}=a_{11}x+a_{12}y+y^2,\ \dot{y}=a_{22}y$$

with $a_{11}\leqslant 0$, $a_{22}\leqslant 0$, $a_{11}+a_{22}<0$, or

$$\dot{x}=a_{11}x+a_{12}y+y^2,\ \dot{y}=a_{21}x+a_{22}y-xy+cy^2$$

with one of the following

(i) $a_{11}<0$, $|c|<2$;

(ii) $a_{11}=a_{21}=0$;

(iii) $a_{11}=0$, $a_{12}+a_{21}=0$, $a_{21}\neq 0$, $a_{22}+ca_{21}\leqslant 0$.

But it is difficult to use the above result directly to determine if a given quadratic system is bounded. According to Ye's classification[1] a class of quadratic systems are treated uniformly in [3]. Precisely, the system

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$$\dot{x}=-y+\delta x+lx^2+mxy+ny^2,\ \dot{y}=x(1+ax+by) \tag{III}$$

is bounded for $t\geqslant 0$ iff one of the following holds:

(1) $n=0,\ (b-1)^2+4ma<0,\ mb<0$;

(2) $n=0,\ (b-1)^2+4ma<0,\ b=m+a=0,\ m(m\delta+1)\leqslant 0,\ m\neq 0$;

(3) $n=0,\ b=1,\ ab>0$.

The above result contains a mistake, and it is corrected partly in [4]. In his lecture notes on quadratic systems (Strasburg, 1983) my adviser professor Yeh Yianqian gave the following example

$$\dot{x}=y+\delta x+mxy+ny^2,\ \dot{y}=x(1-mx-ny)$$

which is bounded iff $\delta\leqslant 0$. In fact, we have $\frac{dV}{dt}=\delta x^2$ along the orbits of the abc system, where $V=x^2+y^2$. This shows that "$n=0$" is not necessary for (III) to bounded.

An equivalence between the results of [2] and [3] is considered in [5], a then the above result of [3] is modified as follows.

A quadratic system with a nondegenerate singular point of index $+1$ whi does not lie on an integral line is bounded iff it is affine-equivalent to (III) w: one of the above conditions (1) and (2).

But for general quadratic systems, the problem when they are bounded l not been solved so far. One of the purposes of the present paper is to solve t: problem. Our main results are listed in section 1 below.

§1. Main Results

We will prove in section 2 that any quadratic system with a singular point the form

$$\dot{x}=\sum_{0<i+j<2}a_{ij}x^iy^j,\ \dot{y}=\sum_{0<i+j<2}b_{ij}x^iy^j \tag{(}$$

is affine-equivalent to

$$\dot{x}=-y+ax^2+bxy,\ \dot{y}=sx+\delta y+lx^2+mxy+ny^2, \tag{E}$$

where $s=0$, 1 or -1, or

$$\dot{x}=cx+ax^2+bxy,\ \dot{y}=\lambda x+\delta y+lx^2+mxy+ny^2. \tag{E}$$

Obviously, (E_2) has no limit cycles around the origin, and (E_1) has limit cyc around the origin only if $s<0$.

We first introduce the following definition.

Definition. *Let γ^+ be a non-singular positive semi-orbit of (E). If there is straight line L such that the intersection $\gamma^+\cap L$ is countably infinite, then the or bit called positively oscillating, otherwise, it is called positively monotone. If a monotone orbit is positinely unbounded, then it is said to be monotone unbounded.*

Throughout this paper we always disouss the boundedness and the unboundedness for positive time, i.e., for $t\geqslant 0$.

For system (E_1) we have the following theorems.

Theorem 1.1. *Quadratic system* (E_1) *is bounded iff one of the following conditions holds:*

(i) $n=0,\ a+m=0,\ a^2+bl=0,\ b^2+a(a-b\delta)=0,\ ab<0,\ \varepsilon=1$;

(ii) $n=0,\ \varepsilon=a=m=l=0,\ \delta<0,\ b\neq 0$;

(iii) $n=0,\ b(b\delta+m)<0,\ (a-m)^2+4bl<0$;

(iv) $n=b\delta+m=0,\ \varepsilon b+l+a\delta=0,\ \varepsilon ab\leqslant 0,\ (a-m)^2+4bl<0,\ \varepsilon\neq 0$.

Theorem 1.2. *System* (E_1) *has precisely one monotone unbounded orbit iff one of the following holds:*

(i) $n(b-n)>0,\ (a-m)^2<4(n-b)l,\ nb>0$;

(ii) $n=b\delta+m=0,\ b(\varepsilon b+l+a\delta)>0,\ (a-m)^2+4bl<0$;

(iii) $n=b,\ m=a,\ bl<0$.

For system (E_2) we have the following theorems.

Theorem 1.3 *System* (E_2) *is bounded iff one of the following holds:*

(i) $b=0,\ a=n=c=m=0,\ \delta<0$;

(ii) $b=0,\ a=n=0,\ c<0,\ \delta\leqslant 0$;

(iii) $b\neq 0,\ n=0,\ am=bl,\ a+m=0,\ a\delta+mc=\lambda b,\ c\leqslant 0,\ \delta<0$;

(iv) $b\neq 0,\ n=0,\ \delta\leqslant 0,\ (a-m)^2+4bl<0$.

Theorem 1.4. *System* (E_2) *has precisely one monotone unbounded orbit iff one of the following holds:*

(i) $n=b,\ m=a,\ bl<0$;

(ii) $b(b-n)>0,\ nb>0,\ (a-m)^2<4l(n-b)$.

Applying Theorfms 1.1 and 1.2 to the system

$$\dot{x}=-y+\delta x+lx^2+mxy+ny^2,\quad \dot{y}=x(1+ax+by), \tag{III}$$

we obtain the following theorem.

Theorem 1.5. (a) *System* $(III)_{a=0}$ *is bounded iff*

(i) $l=0,\ b(m-b\delta)\geqslant 0,\ m^2+4nb<0$, *or*

(ii) $l=0,\ m=b\delta,\ b+n=0,\ m^2+4nb<0$;

system $(III)_{a=0}$ *has precisely one monotone unbounded orbit iff*

(i) $l(b-l)>0,\ m^2<4n(l-b)$, *or*

(ii) $l=0,\ m=b\delta,\ b(n+b)>0,\ m^2+4nb<0$, *or*

(iii) $m=0,\ b=l,\ nb<0$.

(b) *system* $(III)_{n=0}$ *is bounded iff one of the following holds:*

(i) $bl=ma,\ b+l=0,\ a+b\delta=m,\ mb<0$,

(ii) $(b-l)^2+4ma<0,\ mb<0$,

(iii) $(b-l)^2+4ma<0,\ b=m+a=0,\ m(l+m\delta)\leqslant 0,\ m\neq 0$;

system $(\mathrm{III})_{n=0}$ *has precisely one monotone unbounded orbit iff* $b=0$, $m(m+a)>0$, $l^2+4ma<0$.

§ 2. Preliminaries

Consider the general form of quadratic systems

$$\dot{x}=\sum_{0\leq i+j\leq 2} a_{ij}x^i y^j,\quad \dot{y}=\sum_{0\leq i+j\leq 2} b_{ij}x^i y^j. \tag{E}$$

If (E) has no singular points, it is certainly unbounded. If (E) has at least one singular point, then by a translation we can make $a_{00}=b_{00}=0$. If $b_{20}=0$, by exchanging x and y we can make $a_{02}=0$. If $b_{20}\neq 0$, the following cubic equation

$$-b_{20}u^3+(a_{20}-b_{11})u^2+(a_{11}-b_{02})u+a_{02}=0$$

has a real root u. By means of the transformation $x_1=x-uy$, $y_1=y$ we can make new $a_{02}=0$. Thus without loss of generality we can suppose $a_{00}=b_{00}=a_{02}=0$ in (E). If $a_{10}=0$, then (E) has the form

$$\dot{x}=ry+ax^2+bxy,\quad \dot{y}=sx+\delta y+lx^2+mxy+ny^2. \tag{2.1}$$

If $a_{10}\neq 0$, $a_{01}\neq 0$, then we can use the transformation $x_2=x$, $y_2=x+a_{01}/a_{10}y$ to transform (E) into the form of (2.1). If $a_{01}=0$, then (E) has the form

$$\dot{x}=cx+ax^2+bxy,\quad \dot{y}=\lambda x+\delta y+lx^2+mxy+ny^2. \tag{2.2}$$

We can suppose $r\neq 0$ in (2.1) (otherwise, it is contained in (2.2)), and thus, without loss of generality, $r=-1$. If $s<0$ (>0), we can also make $s=-1(=1)$ by the transformation $x_1=|s|^{1/2}x$, $t_1=|s|^{1/2}t$. Summarizing the above discussion we then get the following proposition.

Proposition 2.1. *Any quadratic system with a singular point is affine-equivalent to*

$$\dot{x}=-y+ax^2+bxy,\quad \dot{y}=sx+\delta y+lx^2+mxy+ny^2, \tag{E_1}$$

where $s=0$, 1 *or* -1, *or*

$$\dot{x}=cx+ax^2+bxy,\quad \dot{y}=\lambda x+\delta y+lx^2+mxy+ny^2, \tag{E_2}$$

and the orbit orientation is preserved.

From [6] or [1], we can prove the following proposition.

Proposition 2.2. *Let*

$$\varphi(x)=(a^2n+b^2l-amb)x^3+(am+sb^2-ab\delta-2bl)x^2+(l-2bs+a\delta)x+s,$$

$$\psi(x)=(ab+mb-2na)x^2+(b\delta-2a-m)x-\delta.$$

Then (E_1) *is equivalent to*

$$\dot{x}=y-F(x),\quad \dot{y}=-g(x) \tag{2.3}$$

and

$$\dot{x}=y-F_1(x),\quad \dot{y}=-g_1(x) \tag{2.3_1}$$

respectively on the regions $1-bx>0$ *and* $1-bx<0$, *where*

$$F(x)=\int_0^x f(x)dx,\quad f(x)=\begin{cases} e^{nx}\psi(x), & b=0\\ \psi(x)(1-bx)^{-2-n/b}, & b\neq 0,\end{cases}$$

$$F_1(x)=\int_{2/b}^x f_1(x)dx,\ f_1(x)=-\psi(x)(bx-1)^{-2-n/b},$$

$$g(x)=\begin{cases} x\varphi(x)e^{2nx}, & b=0,\\ x\varphi(x)(1-bx)^{-3-2n/b}, & g_1(x)=-x\varphi(x)(bx-1)^{-3-2n/b}, b\neq 0.\end{cases}$$

Proof Put $y_1=-y+ax^2+bxy$. Then from (E_1)

$$\dot{x}=y_1,\ \dot{y}_1=-\psi_0(x)-\psi_1(x)y_1-\psi_2(x)y_1^2,$$

where

$$\psi_0(x)=x\varphi(x)/(1-bx),\ \psi_1(x)=\psi(x)/(1-bx),\ \psi_2(x)=(n+b)/(1-bx).$$

Also set $Y=u(x)y_1+v(x)$ where u, v are to be determined. Then

$$\dot{Y}=(u'(x)-u(x)\psi_2(x))y_1^2+(v'(x)-u(x)\psi_1(x))y_1-u(x)\psi_0.$$

Now we require that u and v satisfy

$$u'(x)=u(x)\psi_2(x),\ v'(x)=u(x)\psi_1(x).$$

Note that $\psi_i(x)$, $i=1, 2, 3$, are not defined for $1-bx=0$. We can choose u, v as follows:

$$u(x)=\begin{cases}\exp\int_0^x (n+b)/(1-bx)dx \text{ for } 1-bx>0,\\ \exp\int_{b2/}^x (n+b)/(1-bx)dx \text{ for } 1-bx<0,\end{cases}$$

$$v(x)=\begin{cases}\int_0^x u(x)\psi_1(x)dx \text{ for } 1-bx>0,\\ \int_{2/b}^x u(x)\psi_1(x)dx \text{ for } 1-bx<0.\end{cases}$$

Then we obtain

$$\dot{x}=(Y-v(x))/u(x),\ \dot{Y}=-u(x)\psi_0(x),\ 1-bx\neq 0.$$

By making the time change $d\tau=dt/u(x)$ the proposition follows.

Remark 2.1. From the above proof it is easy to see that (E_1) has monotone unbounded orbits on the region $1-bx\geqslant 1$ $(\leqslant -1)$ iff (2.3) $((2.3)_1)$ has monotone unbounded orbits on the same region.

For (E_2) we have a similar proposition if $b\neq 0$.

In order to discuss (E_1) and (E_2) we first consider the general system of the form

$$\dot{x}=y-F(x),\ \dot{y}=-g(x), \tag{2.4}$$

Suppose that F, $g\in C^1(x_1, x_2)$, $0<x_1<x_2\leqslant\infty$.

In the following we will abbreviate monotone unbounded orbits to MU orbits.

Lemma 2.1. *If there exists $x_0\in(x_1, x_2)$ such that $g(x)<0$ for $x\in(x_0, x_2)$ then* (2.4) *has a group of MU orbits on the strip $x_0<x<x_2$ as long as $x_2-g(x_2-0)=\infty$.*

Proof Easily by analizing the vector field defined by (2.4).

The following lemma is a special case of Theorem 2.1[4].

Lemma 2.2. *If there exists* $x_0\in(x_1, x_2)$ *such that*

(1) $G(x_2-0)=\infty$, *where* $G(x)=\int_{x_0}^{x} g(x)dx$;

(2) $g(x)>0$, $F(x)\leqslant-\sqrt{8G(x)}+(G(x))^{\alpha}$ *for* $x\in(x_0, x_2)$, $0<\alpha<1/2$, *then* (2.4) *has a group of MU orbits on the strip* $x_0<x<x_2$.

Lemma 2.3. *Let* $x_2=\infty$. *If there exist* $M>0$, $x_0>x_1$ *such that* $g(x)>0$, $F(x)\leqslant-\sqrt{8G(x)}+M$ *for* $x>x_0$ *then* (2.4) *has a group of MU orbits on* $x>x_0$. *In particular if* $G(\infty)<\infty$, $\limsup_{x\to\infty} F(x)<\infty$ *then the conclusion holds.*

Proof We can suppose $M=0$ (otherwise, by means of $y_1=y+M$). Then th function $y=-(2G(x))^{1/2}$ is an MU solution of the equation

$$\frac{dy}{dx}=\frac{-g(x)}{y+\sqrt{8G(x)}},\quad x>x_0.$$

Then the lemma follows from the comparision theorem.

A generalization of Lemma 2.3 can be found in [7].

Remark 2.2. *If* $F, g\in C^1(x_2', x_1')$, $-\infty\leqslant x_2'<x_1'<0$, *then we have result analogous to the above lemmas.*

§ 3. The Proofs

We mainly prove the results about the system

$$\dot{x}=-y+ax^2+bxy,\quad \dot{y}=\varepsilon x+\delta y+lx^2+mxy+ny^2, \tag{E$_1$}$$

where $\varepsilon=0$, 1 or -1.

Proposition 3.1. *If* $b=0$, *then* (E_1) *has a group of MU orbits.*

Proof By Proposition 2.2 and Remark 2.1 we need only to prove the system

$$\dot{x}=y-F(x),\quad \dot{y}=-g(x), \tag{3.1}$$

where

$$F(x)=\int_0^x f(x)dx,\ f(x)=-(2anx^2+(m+2a)x+\delta)e^{nx},$$

$$g(x)=x(a^2nx^3+amx^2+(l+a\delta)x+\varepsilon)e^{2nx},$$

has a group of MU orbits. For $n\neq0$ the proposition follows from Lemma 2.3 and Remark 2.2. Let $n=0$. If $am<0$ or $am=0$, $l+a\delta\neq0$, then the propo-sition follows from Lemma 2.1 and Remark 2.2. If $am=0$, $l+a\delta=0$, then $2a+m\neq0$, and the proposition follows from Lemma 2.1 for $\varepsilon=-1$, from Lemma 2.2 and Remark 2.2 for $\varepsilon=1$, and is easy to prove for $\varepsilon=0$. If $am>0$ and $2a\neq m$, then

$$\lim_{x\to\pm\infty}\frac{F(x)}{\sqrt{G(x)}}=-\frac{2a+m}{\sqrt{am}}\begin{cases}<-\sqrt{8}, & a>0,\\ >\sqrt{8}, & a<0,\end{cases}$$

where $G(x)=\int_0^x g(x)dx$, in this case we use Lemma 2.2 and Remark 2.2. If $2a=m$, we can take $1/4<\alpha<1/2$, and $x_0>0$ such that if $a<0$, $F(x)\geqslant\sqrt{8G(x)}-(G(x))^\alpha$ for $x<-x_0$;

if $a>0$, $F(x)\leqslant-\sqrt{8G(x)}+(G(x))^\alpha$ for $x>x_0$.

n fact, for example, $a<0$, we have for $1/4<\alpha<1/2$

$$\lim_{x\to-\infty}\frac{F(x)}{\sqrt{G(x)}}=\sqrt{8},\quad \lim_{x\to-\infty}\frac{F^2-8G}{2FG^\alpha}=0,\quad \lim_{x\to-\infty}\frac{G^\alpha}{2F}=0.$$

It follows that $F+G^\alpha>\sqrt{8G}$ for $x<0$, $|x|\gg1$. The proof is completed.

Now we consider the case $b\neq0$. By means of $x_1=-bx$, $y_1=-by$ we can let $b=-1$, and therefore consider

$$\dot{x}=-y+ax^2-xy,\ \dot{y}=\varepsilon x+\delta y+lx^2+mxy+ny^2. \tag{3.2}$$

The systems corresponding to (2.3) and $(2.3)_1$ are

$$\dot{x}=y-F(x),\ \dot{y}=-g(x),\ x>-1 \tag{3.3}$$

and

$$\dot{x}=y-F_1(x),\ \dot{y}=-g_1(x),\ x<-1, \tag{3.3$_1$}$$

where

$$F(x)=\int_0^x f(x)dx,\ f(x)=\psi(x)(1+x)^{n-2},\ g(x)=x\varphi(x)(1+x)^{2n-3},$$

$$F_1(x)=\int_{-2}^x f_1(x)dx,\ f_1(x)=-\psi(x)(-x-1)^{n-2},\ g_1(x)=-x\varphi(x)(-x-1)^{2n-3},$$

$$\varphi(x)=Ax^3+A_1x^2+(1+2\varepsilon+a\delta)x+\varepsilon,\ \psi(x)=Bx^2+B_1x-\delta,$$

$$A=a^2n+l+am,\ A_1=am+\varepsilon+2\delta+2e,$$

$$B=-(a+m+2an),\ B_1=-(\delta+2a+m).$$

Let

$$G(x)=\int_0^x g(x)dx,\quad G_1(x)=\int_{-2}^x g_1(x)dx.$$

Proposition 3.2. *Let $A=0$. If (3.2) does not have a group of MU orbits (in other words, if the number of MU orbits are at most countable), then one of the following holds:*

(i) $a+m+2an=0$, $\varepsilon+a(a+\delta)=0$, $l=a^2(n+1)$, $\varepsilon+a^2n>0, a>0$, $n\geqslant0$;

(ii) $\varepsilon=a=m=l=0$, $\delta>0$, $n\geqslant0$.

Proof By Remark 2.1 system (3.3) ($(3.3)_1$) does not have a group of MU orbits on $x\geqslant0$ ($\leqslant-2$). It follows from Lemma 2.1 and Remark 2.2 that $A_1=0$, and hence $\varphi(x)=(\varepsilon+a^2n)x+\varepsilon$. Thus we also have $\varepsilon+a^2n\geqslant0$, and if $\varepsilon+a^2n=0$ then $\varepsilon=0$. For the case $\varepsilon+a^2n>0$ it is direct that

$$G(x),\ G_1(x)\sim\begin{cases}\dfrac{\varepsilon+a^2n}{2n}|x|^{2n}, & n\neq0,\\ \varepsilon\ln|x|, & n=0\end{cases},$$

$$F(x),\ F_1(x)\sim\begin{cases}\dfrac{B}{n+1}|x|^{n+1}, & n+1\neq0,\\ B\ln|x|, & n+1=0\end{cases}$$

as $|x|\gg1$ (where $u(x)\sim cv(x)$ as $|x|\gg1$ means that $\lim\limits_{x\to\pm\infty}u(x)/v(x)=c$). Thus

Lemma 2.3 and Remark 2.2 give $n\geqslant 0$, and we have $B=0$ by Lemma 2.2 and Remark 2.2. From $B=A=A_1=0$ we easily see that $l=a^2(n+1)$, $\varepsilon+a(a+\delta)=0$, $\delta+a-2an\neq 0$. Hence

$$F(x)\sim\begin{cases}-\dfrac{\delta+a-2an}{n}x^n, & n\neq 0\\ -(a+\delta)\ln x, & n=0\end{cases}\quad \text{as } x\gg 1,$$

and then

$$\lim_{x\to+\infty}\frac{F(x)}{\sqrt{G(x)}}=\begin{cases}-\dfrac{\sqrt{2}(\delta+a-2an)}{\sqrt{n(\varepsilon+a^2n)}}=\mu, & n>0,\\ -\infty, & n=0,\quad a+\delta>0,\\ +\infty, & n=0,\quad a+\delta<0.\end{cases}$$

By Lemma 2.2 it is easy to see that $\delta+a-2an<0$. Again combining the fact tl $\varepsilon+a^2n>0$, $\varepsilon+a(a+\delta)=0$ we have $a>0$. For the case $\varepsilon+a^2n=0$, we have $\varepsilon=0$ a $an=0$, and thus $g=0$, $g_1=0$. In a similar manner we have $n+1\geqslant 0$, $B=0$, $B_1>$ And it follows from the fact that $A=A_1=0$ and by Lemma 2.3 that $l=a=m=$ $n\geqslant 0$, $\delta<0$. Thus the proof is finished.

Proposition 3.3. *Let $A>0$. If (3.2) does not have a group of MU orbits, the*

(i) $n+1>0$, $(a-m)^2<4l(n+1)$; *or*

(ii) $n+1=0$, $m=a$, $l>0$.

Proof. We have now

$$G(x),\ G_1(x)\sim\begin{cases}\dfrac{A}{2n+2}|x|^{2n+2}, & n+1\neq 0\\ A\ln|x|, & n+1=0\end{cases}\quad \text{as } |x|\gg 1,$$

while F, F_1 are as before. Then Lemma 2.3 gives $n+1\geqslant 0$. If $n+1=0$, then fr Lemma 2.2 $B=0$, thus $m=a$, $l>0$, and (ii) follows. If $n+1>0$, then

$$\lim_{x\to+\infty}\frac{F(x)}{\sqrt{G(x)}}=\lim_{x\to-\infty}\frac{F_1(x)}{\sqrt{G_1(x)}}=\frac{\sqrt{2}B}{\sqrt{(n+1)A}}=\tilde{\mu}.$$

It follows from Lemma 2.2 and Remark 2.2 that $|\tilde{\mu}|<\sqrt{8}$, i. e., $B^2<4A(n+$ which is equivalent to $(m-a)^2<4l(n+1)$. Thus (i) follows, and the proof completed.

We will see later that under the condition of Proposition 3.3 system (3. may have only one MU orbit, and have a group of unbounded spiral orbits in t same time.

Next we use Poincare transformations to analize the behavior of orbits ne infinite singular points.

Proposition 3.4. *Suppose one of the conditions* (i) *and* (ii) *of Proposition* 3.2 *holds. Then* (3.2) *has precisely two infinite singular points, one lies on y-axis, the other on the integral line $y=ax$.* (3.2) *is bounded if $n=0$, and has a group of MU* *bits if $n\neq0$, in this case (3.2) has another integral line $y=ax-\frac{a+\delta}{n}$.*

Proof First suppose (i) holds. Put $x=v/z$, $y=1/z$, $d\tau=dt/z$, then, from (3.2),

$$\begin{aligned}\frac{dv}{d\tau}&=-(n+1)v+(a-m)v^2-lv^3-z(1+\delta v+\varepsilon v^2),\\ \frac{dz}{d\tau}&=-z(n+mv+lv^2+\delta z+\varepsilon vz),\end{aligned}\tag{3.4}$$

$n>0$, then (3.4) has a stable node at the origin, in this case (3.4) has a group of U orbits. If $n=0$, then by putting $x=z$, $y=v$ and $t=-\tau$ we have from (3.4)

$$\begin{aligned}\frac{dx}{dt}&=x(my+\delta x+ly^2+\varepsilon xy)=P(x,y),\\ \frac{dy}{dt}&=y+x+(m-a)y^2+ly^3+x(\delta y+\varepsilon y^2)=Q(x,y).\end{aligned}\tag{3.5}$$

We can get from $Q(x,y)=0$

$$y=y(x)=-x+(a-m+\delta)x^2+\cdots=-x+(\delta+2a)x^2+\cdots.$$

Then $P(x,y(x))=(a+\delta)x^2+\cdots$. Note that $a+\delta<0$ from $n=0$. By Theorem 1[[8], Chapter 2], the origin is a saddle-node point of (3.5), and we have Figure 1(a).

In order to discuss the infinite singular points not lying on y-axis, put $x=1/z$, $y=u/z$, $d\tau=dt/z$. Then we have from (3.2)

$$\begin{aligned}\frac{du}{d\tau}&=(n+1)u^2+(m-a)u+l+z(\varepsilon+\delta u+u^2),\\ \frac{dz}{d\tau}&=z(uz+u-a).\end{aligned}\tag{3.6}$$

From condition (i) we have

$$(n+1)u^2+(m-a)u+l=(n+1)(u-a)^2.$$

Thus (3.6) has only one singular point $(a,0)$ on the line $z=0$.

Set $v=u-a$, then from (3.6)

$$\frac{dV}{d\tau}=v((n+1)v+(\delta+2a)z+zv),\quad \frac{dz}{d\tau}=z(az+v+zv).\tag{3.6$'$}$$

Hence the line $v=0$ gives the integral line $y=ax$ of (3.2). The derivative of the function $V=z-kv$ along the orbits of (3.6)$'$ at $V=0$ is

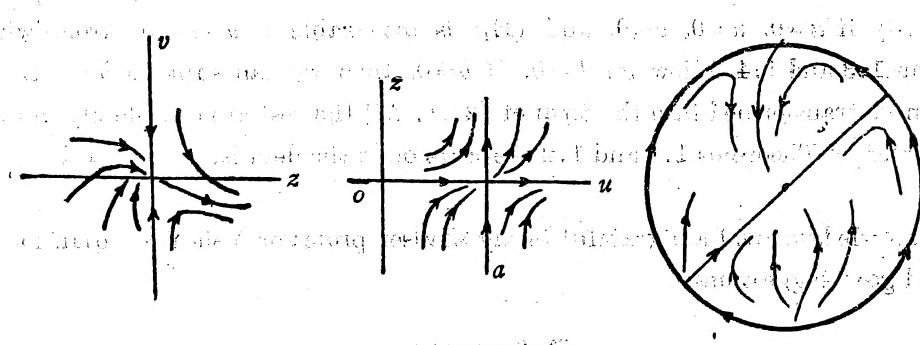
$$\left.\frac{dV}{d\tau}\right|_{V=0}=-(a+\delta)kv^2\left(k+\frac{n}{a+\delta}\right).$$

Thus the value $k=-n/(a+\delta)$ gives another integral line $y=ax-\frac{a+\delta}{n}$ of (3.2) if $n\neq0$. For $n=0$ we have $a+\delta<0$, and then $\left.\frac{dV}{d\tau}\right|_{V=0}>0$ for $v\neq0$. Hence Figure 1(b) follows. We get Figure 1(c) from Figures 1(a)(b).

Suppose now the condition (ii) holds. If $n>0$, then (3.2) has a group of MU orbits. If $n=0$, then from (3.2)

$$\dot{x}=-y-xy,\ \dot{y}=\delta y,$$

which is bounded. The proof is finished.



(a) (b) (c)

Fig. 1

Proposition 3.5. *Suppose one of the conditions* (i) *and* (ii) *of Proposition* 3 *holds. Then* (3.2) *has only one infinite singular point, which lies on y-axis.* (3.2) *bounced iff* $n=0$, $\delta<m$ *or* $n=\delta-m=0$, $l+a\delta=\varepsilon\neq 0$, *and* $\varepsilon a\geqslant 0$. *It has only one* M *orbit iff* $n<0$ *or* $n=\delta-m=0$, $l+a\delta<\varepsilon$.

Proof Obviously, (3.2) has only one singular point at infinity, and it lies y-axis. Let (i) hold first. If $n<0$, then (3.4) has a saddle point at the origin; this case (3.2) has only one MU orbit. If $n>0$, (3.2) has a group of MU orbits before. Let $n=0$, and consider (3.5). A simple computation gives

$$P(x,\ y(x))=(\delta-m)x^2+(l-\varepsilon+m(a-m+\delta))x^3+Dx^4+\cdots,$$

where $D=-a\varepsilon$ when $\delta-m=0$ and $l-\varepsilon+am=0$. From Theorem 7.1[8] we ha Figure 1(a) for $\delta<m$, or $\delta=m$, $l-\varepsilon+a\delta=0$ and $\varepsilon a>0$, and the following Figure 2 for $\delta>m$, or $\delta=m$, $l-\varepsilon+a\delta=0$ and $\varepsilon a<0$. If $\delta=m$, $l-\varepsilon+a\delta\neq 0$, then the origin is a node (saddle) point of (3.5) for $l-\varepsilon+a\delta>0$ (<0). Now let $\delta=m$, $l-\varepsilon+a\delta=0$ and $\varepsilon a=0$. If $a=0$, then $l=\varepsilon=1$, $|\delta|<2$, and thus from (3.2)

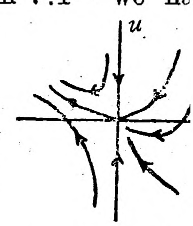


Fig. 2

$$\dot{x}=(1+x)(-y),\ \dot{y}=(1+x)(x+\delta y),$$

which is bounded. If $\varepsilon=0$, then $l=\varepsilon-a\delta=-am$, which contradicts the inequal $(m-a)^2<41$.

If (ii) holds, then the origin is a saddle of (3.5); in this case (3.2) has o one MU orbit. The proof is ended.

Now Theorems 1.1 and 1.2 follow easily from the above propositions a Lemma 2.1.

As an application, consider

$$\dot{x}=-y+\delta x+lx^2+mxy+ny^2,\ \dot{y}=x(1+ax+by). \qquad \text{(I}$$

If $a=0$ $(n=0)$, then (III) can be changed into the form of (E_1) by putting $x_1=$ $y_1=-x$ $(x_1=x,\ y_1=y-\delta x)$. Then Theorem 1.5 follows from Theorems 1.1 and 1

We finally give an outline of the proofs of Theorems 1.3 and 1.4.

If $b=0$ in (E_2), then from the form of (E_2) it does not have a group of MU orbits only if $a=0$, $n=0$, $c\leqslant 0$, and (E_2) is integrable if $a=n=0$, from which Theorems 1.3 and 1.4 follow for $b=0$. If $b\neq 0$, then we can suppose $b=-1$, and (E_2) can be transformed into the form of (2.4). All the rest are completely similar to the proofs of Theorems 1.1 and 1.2, here we omit the details.

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