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EXPONENTIAL STABILITY OF LINEAR SYSTEMS IN BANACH SPACES

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Abstract

In this paper the author proves a new fundamental lemma of Hardy-Lebesgue class $H^2(\sigma)$ and by this lemma obtains some fundamental results of exponential stability of C_0-semigroup of bounded linear operators in Banach spaces. Specially, if $\omega_s=\sup\{\mathrm{Re}\lambda;\ \lambda\in\sigma(A)\}<0$ and $\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant\sigma\}<\infty$, where $\sigma\in(\omega_s,\ 0)$ and A is the infinitesimal generator of a C_0-semigroup e^{tA} in a Banach space X, then (a) $\int_0^\infty e^{-\sigma t}|f(e^{tA}x)|\,dt<\infty$, $\forall f\in X^*$ and $x\in X$; (b) there exists $M>0$ such that $\|e^{tA}x\|\leqslant Ne^{\sigma t}\|Ax\|$, $\forall x\in D(A)$; (c) there exists a Banach space $\hat{X}\supset X$ such that $\|e^{tA}x\|_{\hat{X}}\leqslant e^{\sigma t}\|x\|_{\hat{X}}$, $\forall x\in X$.

§1. Introduction and Main Results

Let X be a Banach space and e^{tA} be a strongly continuous semigroup of bounded linear operators with the infinitesimal generator A in X, briefly called C_0-semigroup[1,2,3]. A C_0-semigroup e^{tA} is called exponentially stable if there exist positive constant numbers M and σ such that $\|e^{tA}\|\leqslant Me^{-\sigma t}$ for $t\geqslant 0$. Let $\omega_0=\omega_0(A)=\lim\limits_{t\to\infty}t^{-1}\ln\|e^{tA}\|$ and $\omega_s=\omega_s(A)=\sup\{\mathrm{Re}\lambda;\ \lambda\in\sigma(A)\}$, where $\mathrm{Re}\lambda$ denotes the real part of a complex number λ and $\sigma(A)$ is the spectrum of a linear operator A. The author[4] has proved an important result to application: a C_0-semigroup e^{tA} in a Hilbert space H is exponentially stable if and only if one of the following conditions holds:

(a) $\omega_s(A)<0$ and there exists $\sigma\in(\omega_s,\ 0)$ such that $\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\,\lambda\geqslant\sigma\}<\infty$;

(b) $\int_0^\infty|(e^{tA}x,\ y)|\,dt<\infty$, $\forall\, x,\ y\in H$.

But this result does not hold in Banach spaces in general (see [3, p. 117] for counterexample). However we can obtain weaker resuits in Banach spaces. Below we describe the main results in present paper.

Definition 1. *A C_0-semigroup e^{tA} in a Banach space X is called weak L_1-stable*

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if there exists $\sigma>0$ *such that* $\int_0^\infty e^{\sigma t}|f(e^{tA}x)|dt<\infty$, $\forall f\in X^*$ *and* $x\in X$.

Theorem 1. *A* C_0*-semigroup* e^{tA} *in a Banach space* X *is weak* L_1*-stable if and only if* $\omega_s(A)<0$ *and there exists* $\sigma\in(\omega_s, 0)$ *such that*

$$M=\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\,\lambda\geqslant\sigma\}<\infty. \tag{1.1}$$

Specially, if (1.1) *holds, then* $\int_0^\infty e^{-\sigma t}|f(e^{tA}x)|dt<\infty$, $\forall f\in X^*$ *and* $x\in X$.

Definition 2. *A* C_0*-semigroup* e^{tA} *in a Banach space* X *is called* A*-exponentially stable on* $D(A)$*, the domain of* A*, if there exist positive constant numbers* M *and* σ *su that* $\|e^{tA}x\|\leqslant Me^{-\sigma t}\|Ax\|, \forall x\in D(A)$ *and* $t\geqslant 0$.

Theorem 2. *If* $\omega_s(A)<0$ *and there exists* $\sigma\in(\omega_s, 0)$ *such that* (1.1) *hol then* C_0*-semigroup* e^{tA} *is* A*-exponentially stable on* $D(A)$ *and there exists* $M_1>0$ *su that* $\|e^{tA}x\|\leqslant M_1e^{\sigma t}\|Ax\|$, $\forall x\in D(A)$ *and* $t\geqslant 0$.

The proof of Theorems 1 and 2 is nontrivial. It needs a fundamental lemr for Hardy–Lebesgue class $H^2(\sigma)$ which will be proved in the second section in tl paper. Function $f(\lambda)$, analytic for $\mathrm{Re}\,\lambda>\sigma$, is said to belong to $H^p(\sigma)$ for sor $p\geqslant 1$ if $\sup\left\{\int_R|f(\tau+i\omega)|^p d\omega;\ \tau>\sigma\right\}<\infty$, where R denotes the real axis.

Theorem 3. *Let* $\omega_s(A)<0$ *and there exists* $\sigma\in(\omega_s, 0)$ *such that* (1.1) *holds a* $f((\lambda-A)^{-2}x)\in H^1(\sigma)$, $\forall f\in X^*$ *and* $x\in X$*. Then* e^{tA} *is exponentially stable.*

We point out that if X is a Hilbert space then the conditions of Theorem3 necessary (see [4]).

Theorem 4. *Let* $\omega_s(A)<0$ *and there exists* $\sigma\in(\omega_s, 0)$ *such that* (1.1) *hol Then there exists a Banach space* $\hat{X}\supset X$ *and a* C_0*-semigroup* $\hat{T}(t)$ *on* $\hat{X}$ *which is extension of* e^{tA} *such that*

$$\|\hat{T}(t)\|_{\hat{X}}\leqslant Me^{\sigma t}\quad for\ t\geqslant 0. \tag{1.}$$

In the last section we will give the proofs of Theorems 1—4.

Finally, combining Theorem 1 and Lemma 3 proved in the last section, obtain the following.

Theorem 5. *Let* A *be the infinitesimal generator of a* C_0*-semigroup* e^{tA} *in Banach space* X*. Then the following conditions are equivalent:*

(a) *there exists* $\sigma>0$ *such that* $\int_0^\infty e^{\sigma t}|f(e^{tA}x)|dt<\infty$, $\forall f\in X^*$ *and* $x\in X$;

(b) $\{\lambda;\ \mathrm{Re}\,\lambda\geqslant 0\}\subset\rho(A)$, *the resolvent set of* A, *and* $\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\,\lambda\geqslant 0\}<$

(c) $\int_0^\infty|f(e^{tA}x)|dt<\infty$, $\forall f\in X^*$ *and* $x\in X$.

§2. A Lemma

In this section we will prove a nontrivial lemma for Hardy-Lebesgue class

$H^2(\sigma)$. As shown in [1, 2], $H^2(\sigma)$ is a Banach space with norm

$$\|f\|_{H^2(\sigma)}=\sup\left\{\left[\int_R |f(\tau+i\omega)|^2 d\omega\right]^{1/2};\ \tau>\sigma\right\} \tag{2.1}$$

and for $f\in H^2(\sigma)$ the boundary function $f(\sigma+i\omega)\in L^2=L^2(R)$ of $f(\lambda)$ exists in the sense that

$$\lim_{\tau\to\sigma}\int_R |f(\sigma+i\omega)-f(\tau+i\omega)|^2 d\omega=0. \tag{2.2}$$

Moreover, with the case for the upper half plane[5], $H^2(\sigma)$ has dual $H^{2*}(\sigma)=L^2/H^2(\sigma)$. Therefore the null element of $H^{2*}(\sigma)$ is $H^2(\sigma)$.

Lemma 1. *Let σ_0 and σ_1 be real numbers with $\sigma_0<\sigma_1$ and the following conditions hold:*

(a) *$f(\lambda)$ is analytic in the right half plane $\mathrm{Re}\lambda>\sigma_0$;*

(b) *$f(\lambda)\in H^2(\sigma_1)$;*

(c) *for some $\sigma\in(\sigma_0,\sigma_1)$, $\lim\limits_{\rho\to\infty}\max\{|f(\sigma+\rho e^{i\theta})|;\ -\pi/2\leqslant\theta\leqslant\pi/2\}=0$.*

Then $f(\lambda)\in H^2(\sigma)$.

Proof Let $g(\lambda)\in H^2(\sigma)$ and $\sigma_2\in[\sigma_1,\infty)$ be arbitrary. Then $g(\lambda)f(\lambda)$ is holomorphic in the right plane $\mathrm{Re}\,\lambda>\sigma$ and $g(\lambda)f(\lambda)\in H^1(\sigma_1)$. Moreover, as shown in [1, § 6.4], for $\tau\in(\sigma,\infty)$, we have

$$|g(\tau+i\omega)|\leqslant[\pi(\tau-\sigma)]^{-1/2}\|g\|_{H^2(\sigma)},$$
$$\lim_{\rho\to\infty}\sup\{|g(\tau+\rho e^{i\theta})|;\ -\pi/2\leqslant\theta\leqslant\pi/2\}=0 \tag{2.3}$$

and

$$\lim_{\tau\to\sigma}\int_R |g(\tau+i\omega)|^2 d\omega=0, \tag{2.4}$$

while from (c) we have

$$M=\sup\{|f(\lambda)|;\ \mathrm{Re}\,\lambda\geqslant\sigma\}<\infty \tag{2.5}$$

and

$$\lim_{\tau\downarrow\infty}\sup\{|f(\tau+i\omega)-f(\sigma+i\omega)|;\ \omega\in R\}=0. \tag{2.6}$$

Given any $\varepsilon>0$, from (c) again, there exists $\omega_\varepsilon>0$ such that

$$\begin{aligned}\mu(f,\varepsilon)&=\sup\{|f(\tau+i\omega)|;\ \tau\in[\sigma,\sigma_2],\ |\omega|\geqslant\omega_\varepsilon\}\\&<[8M\|g\|_{H^2(\sigma)}(\sigma_2-\sigma_0)^{1/2}]^{-1}\pi^{1/2}\varepsilon.\end{aligned} \tag{2.7}$$

Let $\tau\in(\sigma,\sigma_2]$ and $\omega_j\geqslant\omega_\varepsilon$ $(j=1,2)$. We define the contour $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4$, where $\Gamma_1=\{\sigma_2+i\omega;\ -\omega_1\leqslant\omega<\omega_2\}$, $\Gamma_2=\{s+i\omega_2;\ \tau\leqslant s\leqslant\sigma_2\}$, $\Gamma_3=\{\tau+i\omega;\ -\omega_1\leqslant\omega\leqslant\omega_2\}$ and $\Gamma_4=\{s-i\omega_1;\ \tau\leqslant s\leqslant\sigma_2\}$. By the Cauchy theorem. We have

$$\int_\Gamma g(\lambda)f(\lambda)d\lambda=0.$$

Thus we have

$$\left|\int_{-\omega_1}^{\omega_2} g(\tau+i\omega)f(\tau+i\omega)d\omega-\int_{-\omega_1}^{\omega_2} g(\sigma_2+i\omega)f(\sigma_2+i\omega)d\omega\right|$$
$$=\left|\int_{\tau}^{\sigma_2} g(s-i\omega_1)f(s-i\omega_1)\,ds-\int_{\tau}^{\sigma_2} g(s+i\omega_2)f(s+i\omega_2)ds\right|$$
$$\leqslant 2\left|\int_{\tau}^{\sigma_2}[\pi(s-\sigma)]^{-1/2}\mu(f,\ \varepsilon)\,\|g\|_{H^2(\sigma)}ds\right.$$
$$\leqslant 4\pi^{-1/2}\mu(f,\ \varepsilon)\,\|g\|_{H^2(\sigma)}(\sigma_2-\sigma_0)^{1/2}$$
$$<\frac{\varepsilon}{2},\ \text{for } \tau\in(\sigma,\ \sigma_2] \text{ and } \omega_j\geqslant\omega_\varepsilon(j=1,\ 2). \tag{2.8}$$

On the other hand, being (2.2), (2.5) and (2.6), we have

$$\lim_{\tau\downarrow\sigma}\int_R |g(\sigma+i\omega)f(\sigma+i\omega)-g(\tau+i\omega)f(\tau+i\omega)|^2d\omega$$
$$\leqslant 2\lim_{\tau\downarrow\sigma}\int_R|g(\sigma+i\omega)-g(\tau+i\omega)|^2|f(\sigma+i\omega)|^2d\omega$$
$$+2\lim_{\tau\downarrow\sigma}\int_R|g(\tau+i\omega)|^2|f(\sigma+i\omega)-f(\tau+i\omega)|^2d\omega$$
$$\leqslant 2M^2\lim_{\tau\to\sigma}\int_R|g(\sigma+i\omega)-g(\tau+i\omega)|^2d\omega$$
$$+2\lim_{\tau\downarrow\sigma}[\sup\{|f(\tau+i\omega)-f(\sigma+i\omega)|;\ \omega\in R\}]^2\int_R|g(\tau+i\omega)|^2d\omega$$
$$=0.$$

Thus for any fixed $\omega_j\geqslant\omega_\varepsilon$ $(j=1,\ 2)$, because

$$\int_{-\omega_1}^{\omega_2}|g(\sigma+i\omega)f(\sigma+i\omega)-g(\tau+i\omega)f(\tau+i\omega)|d\omega$$
$$\leqslant(\omega_1+\omega_2)^{1/2}\left[\int_R|g(\sigma+i\omega)f(\sigma+i\omega)-g(\tau+i\omega)f(\tau+i\omega)|^2d\omega\right]^{1/2},$$

there exists $\tau=\tau(\omega_1,\ \omega_2)\in(\sigma,\ \sigma_2]$ such that

$$\left|\int_{-\omega_1}^{\omega_2} g(\sigma+i\omega)f(\sigma+i\omega)d\omega-\int_{-\omega_1}^{\omega_2} g(\tau+i\omega)f(\tau+i\omega)d\omega\right|<\frac{\varepsilon}{2}. \tag{2.9}$$

And so, from (2.8) and (2.9), for $\omega_j\geqslant\omega_\varepsilon$ $(j=1,\ 2)$, we have

$$\left|\int_{-\omega_1}^{\omega_2} g(\sigma+i\omega)f(\sigma+i\omega)d\omega-\int_{-\omega_1}^{\omega_2} g(\sigma_2+i\omega)f(\sigma_2+i\omega)d\omega\right|$$
$$\leqslant\left|\int_{-\omega_1}^{\omega_2} g(\sigma+i\omega)f(\sigma+i\omega)\omega-\int_{-\omega_1}^{\omega_2} g(\tau+i\omega)f(\tau+i\omega)d\omega\right|$$
$$+\left|\int_{-\omega_1}^{\omega_2} g(\tau+i\omega)f(\tau+i\omega)d\omega-\int_{-\omega_1}^{\omega_2} g(\sigma_2+i\omega)f(\sigma_2+i\omega)d\omega\right|$$
$$<\varepsilon.$$

Hence, remarking $g(\sigma_2+i\omega)f(\sigma_1+i\omega)\in L^1(R)$ for $\sigma_2\geqslant\sigma_1$, we see that $g(\tau+i\omega)$ $f(\tau+i\omega)$ is integrable on R for $\tau\geqslant\sigma$ and the integral

$$L(g)=\int_R g(\sigma+i\omega)f(\sigma+i\omega)d\omega=\int_R g(\tau+i\omega)f(\tau+i\omega)d\omega$$
$$=\int_R g(\sigma_2+i\omega)f(\sigma_2+i\omega)d\omega$$

is independent of $\tau\in[\sigma, \sigma_2]$. Also since

$$|L(g)|=\left|\int_R g(\sigma_2+i\omega)f(\sigma_2+i\omega)d\omega\right|\leqslant\|g\|_{H^2(\sigma_2)}\|f\|_{H^2(\sigma_2)}$$
$$\leqslant\|g\|_{H^2(\sigma)}\|f\|_{H^2(\sigma_2)},$$

$L(\cdot)$ is a bounded Linear functional on $H^2(\sigma)$. But from (2.4) $L(\cdot)$ is null. Thus $f(\lambda)\in H^2(\sigma)$.

§3. Proofs of Theorems

In this section we give the proof of Theorems 1—4. Below, we assume that A is the infinitesimal generator of a C_0-semigroup e^{tA} in a Banach space X. Let $\omega_0=\omega_0(A)$ and $\omega_s=\omega_s(A)$ be the same as in the introduction.

Lemma 2[4]. *Let $\sigma>\omega_s$ such that $\sup\{\|\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant\sigma\}<\infty$. Then for any $x\in X$, $\lim\limits_{|\omega|\to\infty}\|(\tau+i\omega-A)^{-1}x\|=0$ uniformly for $\tau\in[\sigma,\sigma_1]$, where σ_1 is an arbitrary real number wi h $\sigma_1>\sigma$.*

Lemma 3. *Let σ be a real number such that the et $\{\lambda;\ \mathrm{Re}\lambda\geqslant\sigma\}\subset\rho(A)$ and $M=\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant\sigma\}<\infty$. Then here exists $\sigma_1<\sigma$ such that the set $\{\lambda;\ \mathrm{Re}\lambda\geqslant\sigma_1\}\subset\rho(A)$ and $\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant\sigma_1\}<\infty$.*

Proof Let $0<\delta<(2M)^{-1}$ and $\sigma_1=\sigma-\delta$. For $\lambda=\tau+i\omega$, $\sigma_1\leqslant\tau\leqslant\sigma$, $\omega\in R$, being $\|(\tau-\sigma)(\sigma+i\omega-A)^{-1}\|\leqslant\delta M\leqslant1/2$, series $\sum\limits_{n=0}^{\infty}(\sigma-\tau)^n(\sigma+i\omega-A)^{-(n+1)}$ is convergent absolutely. Therefore $\lambda\in\rho(A)$ and we have $(\lambda-A)^{-1}=\sum\limits_{n=0}^{\infty}(\sigma-\tau)^n(\sigma+i\omega-A)^{-(n+1)}$ and $\|(\lambda-A)^{-1}\|\leqslant\sum\limits_{n=0}^{\infty}(\sigma-\tau)^nM^{n+1}\leqslant2M$.

Lemma 4. *Let $\sigma>\omega_s(A)$ such that $M=\sup\{\|\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant\sigma\}<\infty$. Then for any $x\in X$ and $f\in X^*$ we have $f((\lambda-A)^{-1}x)\in H^2(\sigma)$.*

Proof Given any $x\in X$ and $f\in X^*$, $f((\lambda-A)^{-1}x)$ is analytic for $\mathrm{Re}\lambda>\omega_s(A)$. Let $\sigma_1>\max\{\omega_0(A),\sigma\}$ and $\varepsilon\in(0,\sigma_1-\omega_0)$. Then there exists $M_\varepsilon\geqslant1$ such that $\|e^{tA}\|\leqslant Me^{(\omega_0+\varepsilon)t}$ for $t\geqslant0$. Thus for $\lambda=\tau+i\omega$ with $\tau\geqslant\sigma_1$ we have

$$\|e^{-\lambda t}f(e^{tA}x)\|\leqslant M_\varepsilon\|f\|\|x\|\cdot e^{-(\tau-\omega_0-\varepsilon)t}\ \text{for}\ t\geqslant0.$$

By Plancherel's theorem, for

$$f((\tau+i\omega-A)^{-1}x)=\int_0^\infty e^{-i\omega t}e^{-\tau t}f(e^{tA}x)dt,$$

we have

$$\int_R|f((\tau+i\omega-A)^{-1}x)|^2d\omega=2\pi\int_0^\infty|f(e^{tA}x)|^2e^{-2\tau t}dt$$
$$\leqslant2\pi\int_0^\infty M_\varepsilon^2\|f\|^2\|x\|^2e^{-2(\tau-\omega_0-\varepsilon)t}dt\leqslant\frac{2\pi M_\varepsilon^2\|f\|^2\|x\|^2}{2(\tau-\omega_0-\varepsilon)}$$
$$\leqslant\frac{\pi M_\varepsilon^2\|f\|^2\|x\|^2}{(\sigma_1-\omega_0-\varepsilon)}.$$

Therefore $f((\lambda-A)^{-1}x)\in H^2(\sigma_1)$. Also by Lemma 2 and noting $\|(\lambda-A)^{-1}\|\leqslant M_s\cdot(\mathrm{Re}\lambda-\omega_0-s)^{-1}$ for $\mathrm{Re}\,\lambda>\omega_0+s$ by Hille-Yosida's theorem, we have

$$\lim_{\rho\to\infty}\max\{|f((\sigma+\rho e^{i\theta}-A)^{-1}x)|;\ -\pi/2\leqslant\theta\leqslant\pi/2\}=0.$$

So by Lemma 1 $f((\lambda-A)^{-1}x)\in H^2(\sigma)$.

Lemma 5. *Let $\omega_s(A)<0$ and there exists $\sigma\in(\omega_s, 0)$ such that* (1.1) *holds. Then we have*

$$\text{(a)}\quad e^{tA}x=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma-i\omega_1}^{\sigma+i\omega_1}e^{\lambda t}(\lambda-A)^{-1}x\,d\lambda,\ \textit{for } x\in D(A),\ t\geqslant0; \quad (3.1)$$

$$\text{(b)}\quad e^{tA}x=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma-i\omega_1}^{\sigma+i\omega_1}e^{\lambda t}\lambda^{-m}(\lambda-A)^{-1}A^m x\,d\lambda,\ \textit{for } x\in D(A^m),$$

$$m=1,\ 2,\ \cdots,\ t\geqslant0. \quad (3.2)$$

Proof (a) Taking $\sigma_2>\max\{0,\ \omega_0(A)\}$, for any $x\in D(A)$, by the inver[se] formula in [6, p. 261], we have

$$e^{tA}x=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma_2-i\omega_1}^{\sigma_2+i\omega_1}e^{\lambda t}(\lambda-A)^{-1}x\,d\lambda,\ \text{for }\geqslant0. \quad (3.3)$$

Fer any fixed $t\geqslant0$, $e^{\lambda t}(\lambda-A)^{-1}x$ is analytic in the domain $\{\lambda;\ \mathrm{Re}\,\lambda>\omega_s(A)\}$ $\|e^{\lambda t}(\lambda-A)^{-1}x\|\leqslant e^{\sigma_2 t}\|(\lambda-A)^{-1}x\|$, and so by Lemma 2

$$\lim_{|\omega|\to\infty}\max\{\|e^{\lambda t}(\lambda-A)^{-1}x\|;\ \sigma\leqslant\mathrm{Re}\,\lambda\leqslant\sigma_2\}=0.$$

Thus as shown in [6, p. 250], from (3.3), we see that (3.1) holds.

(b) Let $x\in D(A^m)$, $m=1,\ 2,\ \cdots$. Since $(\lambda-A)^{-1}x=\lambda^{-1}[x+(\lambda-A)^{-1}Ax]=\lambda^{-1}x+\lambda^{-2}Ax+\cdots+\lambda^{-m}A^{(m-1)}x+\lambda^{-m}(\lambda-A)^{-1}A^m x$ and for $\sigma<0$,

$$\lim_{\omega_1\to\infty}\int_{\sigma-i\omega_1}^{\sigma+i\omega_1}e^{\lambda t}\lambda^{-j}A^{(j-1)}x\,d\lambda=0\ \ (j=1,\ 2,\ \cdots,\ m),$$

we see that (3.2) holds from (3.1).

Proof of Theorem 1 Let $\sigma>0$ such that $\int_0^\infty e^{\sigma t}|f(e^{tA}x)|dt<\infty$, for $f\in X^*$ and [$x\in$] X. By Baire's category argument, there exists a constant $M>0$ such that

$$\int_0^\infty e^{\sigma t}|f(e^{tA}x)|dt\leqslant M\ \|f\|\,\|x\|,\ \text{for } f\in X^* \text{ and } x\in X. \quad (3.4)$$

For $\mathrm{Re}\,\lambda\geqslant-\sigma$ we define linear operator $R(\lambda)$ by

$$f(R(\lambda)x)=\int_0^\infty e^{-\lambda t}f(e^{tA}x)dt,\ \text{or } f\in X^* \text{ and } x\in X. \quad (3.5)$$

From (3.4), $R(\lambda)\in B(X)$, the Banach algebra of all bounded linear oper[ators] defined on X into X, and we can easilly check $R(\lambda)=(\lambda-A)^{-1}$. Therefore $\lambda\in\rho(A)$ and from (3.4) we have

$$\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant-\sigma\}\leqslant M. \quad (3.6)$$

Also by lemma 3 we have $\omega_s(A)<-\sigma$.

Conversely, if $\omega_s(A)<-\sigma$ and (3.6) holds. By Lemma 3, there exists $\sigma_0\in(\sigma,-\omega_s)$ such that $\sup\{\|(\lambda-A)^{-1}\|;\ \mathrm{Re}\lambda\geqslant-\sigma_0\}<\infty$. Thus for any $f\in X^*$ and $x\in X$, by Lemma 4, we have $f((\lambda-A)^{-1}x)\in H^2(-\sigma_0)$. By Paley-Weiner theorem[2, p.163], there exists a $g(t)\in L^2(0,\infty)$ such that in the strong topology of $L^2(0,\infty)$

$$g(t)=\frac{1}{2\pi}\lim_{\omega_1\to\infty}\int_{-\omega_1}^{\omega_1} e^{i\omega t}f((-\sigma_0-i\omega-A)^{-1}x)d\omega \tag{3.7}$$

and for $\operatorname{Re}\lambda>0$

$$f((\lambda-\sigma_0-A)^{-1}x)=\int_0^\infty e^{-\lambda t}g(t)dt.$$

Moreover, by Plancherel's theorem

$$\int_0^\infty |g(t)|^2dt=\int_{-\infty}^\infty |f((-\sigma_0+i\omega-A)^{-1}x)|^2d\omega.$$

But for $\operatorname{Re}\lambda>\omega_0+\sigma_0$,

$$f((\lambda-\sigma_0-A)^{-1}x)=\int_0^\infty e^{-\lambda t}e^{\sigma_0 t}f(e^{tA}x)dt,$$

and so by the uniqueness of Fourier transform, we have

$$e^{\sigma_0 t}f(e^{tA}x)=g(t),\ \text{for almost everywhere } t\in[0,\infty). \tag{3.8}$$

Therefore

$$\int_0^\infty e^{2\sigma_0 t}|f(e^{tA}x)|^2dt=\int_{-\infty}^\infty |f((-\sigma_0+i\omega-A)^{-1}x)|^2d\omega.$$

Thus

$$\int_0^\infty e^{\sigma t}|f(e^{tA}x)|dt=\int_0^\infty e^{-(\sigma_0-\sigma)t}e^{\sigma_0 t}|f(e^{tA}x)|dt$$
$$\leqslant\left\{\int_0^\infty e^{-2(\sigma_0-\sigma)t}dt\right\}^{1/2}\left\{\int_0^\infty e^{2\sigma_0 t}|f(e^{tA}x)|^2dt\right\}^{1/2}<\infty.$$

Proof of Theorem 2 Let $f\in X^*$ and $x\in X$. By Lemma 4, $f((\sigma+i\omega-A)^{-1}x)\in L^2(R)$. Thus from (3.2) we have

$$e^{-\sigma t}|f(e^{tA}A^{-1}x)|=\frac{1}{2\pi}\lim_{\omega_1\to\infty}e^{-\sigma t}\left|\int_{\sigma-i\omega_1}^{\sigma+i\omega_1}e^{\lambda t}\lambda^{-1}f((\lambda-A)^{-1}x)d\lambda\right|$$
$$=\frac{1}{2\pi}\lim_{\omega_1\to\infty}\left|\int_{-\omega_1}^{\omega_1}e^{i\omega t}(\sigma+i\omega)^{-1}f((\sigma+i\omega-A)^{-1}x)d\omega\right|$$
$$\leqslant\frac{1}{2\pi}\int_R(\sigma^2+\omega^2)^{-1/2}|f((\sigma+i\omega-A)^{-1}x)|d\omega<\infty.$$

Thus, by the resonance theorem, there exists $M>0$ such that

$$\|e^{-\sigma t}e^{tA}A^{-1}\|\leqslant M,\ \text{for } t\geqslant 0.$$

We show that Theorem 2 is the best possible result because even if $\omega_s(A)<0$ and there exists $\sigma\in(\omega_s,0)$ such that (1.1) holds, $\|e^{tA}\|$ can be unbounded, see [3, p. 7] for example.

Proof of Theorem 3 Let $f\in X^*$, $x\in X$ and $\sigma_0\in(\sigma,0)$. From (3.7) and (3.8), in the strong topology of $L^2(0,\infty)$, we have

$$f(e^{tA}x)=\frac{1}{2\pi}\lim_{\omega_1\to\infty}\int_{-\omega_1}^{\omega_1}e^{i\omega t}e^{\sigma_0 t}f((\sigma_0+i\omega-A)^{-1}x)d\omega$$
$$=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma_0-i\omega_1}^{\sigma_0+i\omega_1}e^{\lambda t}f((\lambda-A)^{-1}x)d\lambda.$$

But, for $t>0$, by partial integration and Lemma 1, we have

$$\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma_0-i\omega_1}^{\sigma_0+i\omega_1} e^{\lambda t}f((\lambda-A)^{-1}x)d\lambda$$
$$=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}[t^{-1}e^{\lambda t}f((\lambda-A)^{-1}x)]_{\sigma_0-i\omega_1}^{\sigma_0+i\omega_1}$$
$$+\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma_0-i\omega_1}^{\sigma_0+i\omega_1}\frac{e^{\lambda t}}{t}f((\lambda-A)^{-2}x)d\lambda$$
$$=\frac{1}{2\pi i}\lim_{\omega_1\to\infty}\int_{\sigma_0-i\omega_1}^{\sigma_0+i\omega_1}\frac{e^{\lambda t}}{t}f((\lambda-A)^{-2}x)d\lambda$$
$$=\frac{1}{2\pi}\int_R\frac{1}{t}e^{\sigma_0 t}e^{i\omega t}f((\sigma_0-i\omega-A)^{-2}x)d\omega.$$

The above integral is absolutely convergent, since $f((\lambda-A)^{-2}x)\in H^1(\sigma)$. Thus almost everywhere $t\in(0,\infty)$,

$$f(e^{tA}x)=\frac{1}{2\pi}\int_R\frac{1}{t}e^{\sigma_0 t}e^{i\omega s}f((\sigma_0+i\omega-A)^{-2}x)d\omega.$$

Especially for $t\geqslant 1$, by the continuity of $e^{tA}x$, we have

$$e^{-\sigma_0 t}|f(e^{tA}x)|\leqslant\frac{1}{2\pi}\int_R|f((\sigma_0+i\omega-A)^{-2}x)|d\omega<\infty.$$

By the resonance theorem, there exists $M>0$ such that $\|e^{-\sigma_0 t}e^{tA}\|\leqslant M$ for $t\geqslant 1$. $\eta=\max\{\|e^{tA}\|;\ 0\leqslant t\leqslant 1\}$, then $\|e^{tA}\|\leqslant(M+\eta e^{-\sigma_0})e^{\sigma_0 t}$ for $t\geqslant 0$.

Finally, we give the proof of Theorom 4.

Proof of Theorem 4 Let $\omega_s(A)<0$ and there exists $\sigma\in(\omega_s, 0)$ such that (holds. By Theorem 1 and (3.4), we have

$$\int_0^\infty e^{-\sigma t}|f(e^{tA}x)|dt\leqslant M\|f\|\|x\|,\ \text{for } x\in X \text{ and } f\in X^*. \qquad ($$

We now define

$$\|x\|_1=\sup\left\{\int_0^\infty e^{-\sigma t}|f(e^{tA}x)|dt;\ f\|\leqslant 1\right\}.$$

Clearly, $\|\cdot\|_1$ is a norm on X and $\|x\|_1\leqslant M\|x\|$ for $x\in X$ from (3.9). Moreover,

$$\|e^{tA}x\|_1=\sup\left\{\int_0^\infty e^{-\sigma s}|f(e^{sA}x)|ds;\ \|f\|\leqslant 1\right\}$$
$$=e^{\sigma t}\sup{}^{\infty}{}_{t}e^{-\sigma\tau}|f(e^{\tau A}x)|d\tau;\ \|f\|\leqslant 1\}$$
$$\leqslant e^{\sigma}\|{}^{t}x\|_1,\ \text{for } x\in X \text{ and } t\geqslant 0. \qquad (3$$

Let $\hat{X}$ be the completion of $(X, \|\cdot\|_1)$ and $\hat{T}(t)$ be the extension of e^{tA} on $\hat{X}$. $\hat{T}(t)$ is a C_0-semigroup on $\hat{X}$ and from (3.10) we see that (1.2) holds.

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