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THE STRUCTURE OF ORTHOGONAL GROUPS OVER ARBITRARY COMMUTATIVE RINGS**

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Abstract

Let R be an arbitrary commutative ring, and n an integer$\geqslant 3$. It is proved for any ideal J of R that

$$EO_{2n}(R, J)=[EO_{2n}(R), EO_{2n}(J)]=[EO_{2n}(R), EO_{2n}(R, J)]$$
$$=[EO_{2n}(R), O_{2n}(R, J)]=[O_{2n}(R), EO_{2n}(R, J)].$$

In particular, $EO_{2n}(R, J)$ is a normal subgroupof $O_{2n}(R)$. Furthermore, the problem of normal subgroups of $O_{2n}(R)$ has an affirmative solution if and only if $aR\cap \mathrm{Ann}(2)=a^2\mathrm{Ann}(2)$ for each a in R. In particular, if 2 is not a zero divisor in R, then the problem of normal subgroups of $O_{2n}(R)$ has an affirmative solution

The structure of orthogonal groups over fields has been determined.[1,4,5,6, Some results were extended to local rings[7,8,11] and to full rings[12]. In this paper start from elementary orthogonal matrices and, by means of localization and ri extension techniques, determine the structure of orthogonal groups $O_{2n}(R)$ $(n\geqslant$ over arbitrary commutative rings.

§1. Introduction

Throughout this paper R denotes an arbitrary commutative ring (always w identity). The orthogonal group of hyperbolic rank n over R is defined to be

$$O_{2n}(R)=\{g\in GL_{2n}(R)\mid gkg'=k\},$$

where $k=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}$ and g' denotes the transpose of g. By the definition, $g=\begin{pmatrix}A & B'\\ C & D,\end{pmatrix}$ $O_{2n}(R)$ if and only if $AB'+BA'=0$, $CD'+DC'=0$, and $AD'+BC'=I$. In such a ca $g^{-1}=\begin{pmatrix}D' & B'\\ C' & A'\end{pmatrix}$. In particular, if $X=-X'\in M_n(R)$, $A\in GL_n(R)$, then $\begin{pmatrix}I & X\\ 0 & 1\end{pmatrix}$

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$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & A'^{-1} \end{pmatrix}$ are all orthogonal matrices.

Let L denote Ann(2), the annihilator of 2. For $a \in R$, $b \in L$, and $1 \leqslant i, j \leqslant n$, the ollowing matrices are called elementary orthogonal matrices:

$$r_{ij}(a) = I + a(e_{i,\,n+j} - e_{j,\,n+i}),\ i \neq j,$$
$$t_{ij}(a) = I + a(e_{n+i,\,j} - e_{n+j,\,i}),\ i \neq j,$$
$$\sigma_{ij}(a) = I + a(e_{ij} - e_{n+j,\,n+i}),\ i \neq j,$$
$$r_{ii}(b) = I + b e_{i,\,n+i},$$
$$t_{ii}(b) = I + b e_{n+i,\,i},$$

vhere e_{km} denotes the $2n \times 2n$ matrix with 1 in its (k, m)-entry and O elsewhere. Jlearly, $r_{ij}(a)^{-1} = r_{ij}(-a)$, $t_{ij}(a)^{-1} = t_{ij}(-a)$, $\sigma_{ij}(a)^{-1} = \sigma_{ij}(-a)$, $r_{ii}(b)^{-1} = r_{ii}(-b)$, nd $t_{ii}(b)^{-1} = t_{ii}(-b)$.

The group $EO_{2n}(R)$ generated by all elementary orthogonal matrices is called he elementary orthogonal group. Let J be an ideal of R. $EO_{2n}(J)$ denotes the ubgroup of $EO_{2n}(R)$ generated by $r_{ij}(a)$, $t_{ij}(a)$, $\sigma_{ij}(a)$, $r_{ii}(b)$ and $t_{ii}(b)$ for all $i \neq j$, $\in J$, and $b \in J \cap L$. Let $EO_{2n}(R, J)$ be the normal subgroup of $EO_{2n}(R)$ generated y $EO_{2n}(J)$, called the elementary congruence subgroup of level J.

The canonical ring homomorphism $R \to R/J$ induces naturally a group omomorphism λ_J: $O_{2n}(R) \to O_{2n}(R/J)$. Let $O_{2n}(R, J)$ be the inverse image of the enter of $O_{2n}(R/J)$ under λ_J, called the general congruence subgroup of level J. In articular, $O_{2n}(R, O)$ is just the center of $O_{2n}(R)$.

For an invertible matrix $h = (h_{ij})$, define the order of h, Ord(h), to be the ideal eneoated by all $h_{ii} - h_{jj}$ and h_{ij}, $i \neq j$. The order of a subgroup H is Ord$(H) = \sum_{\in H}$ Ord(h). Evidently, if H is a subgroup of $O_{2n}(R)$, then $H \subseteq O_{2n}(R, \text{Ord}(H))$.

Lemma 1. *Denote the commutator* $ghg^{-1}h^{-1}$ *by* $[g, h]$. *Then*

(1) $[r_{ij}(a), t_{km}(b)] = I$ ($k \neq i, j$, *and* $m \neq i, j$),

(2) $[r_{ij}(a), t_{ik}(b)] = \sigma_{jk} - (ab)$ (i, j, *and* k *distinct*),

(3) $[r_{ij}(a), t_{jj}(b)] = \sigma_{ij}(ab) r_{ii}(a^2 b)$ ($i \neq j$),

(4) $[r_{ii}(a), t_{ij}(b)] = \sigma_{ij}(ab) t_{jj}(-ab^2)$ ($i \neq j$),

(5) $\sigma_{ij}(a)\sigma_{ij}(b) = \sigma_{ij}(a+b)$,

(6) $[\sigma_{ij}(a), \sigma_{km}(b)] = I$ ($i \neq m$ *and* $j \neq k$).

(7) $[\sigma_{ij}(a), \sigma_{jk}(b)] = \sigma_{ik}(ab)$ ($i \neq k$),

(8) $t_{ij}(a) t_{ij}(b) = t_{ij}(a+b)$,

(9) $[t_{ij}(a), t_{km}(b)] = I$,

(10) $t_{ij}(a) = t_{ji}(-a)$,

(11) $[t_{ij}(a), \sigma_{km}(b)] = I$ ($k \neq i, j$),

(12) $[t_{ij}(a), \sigma_{jk}(b)] = t_{ik}(ab)$ (i, j, *and* k *distinct*),

(13) $[t_{ij}(a),\ \sigma_{ij}(b)]=I$,

(14) $[t_{ii}(a),\ \sigma_{ik}(b)]=t_{ik}(ab)t_{kk}(-ab^2)$,

(15) $r_{ij}(a)r_{ij}(b)=r_{ij}(a+b)$,

(16) $[r_{ij}(a),\ r_{km}(b)]=I$,

(17) $r_{ij}(a)=r_{ji}(-a)$,

(18) $[r_{ij}(a),\ \sigma_{km}(b)]=I,\ (m\neq i,\ j)$,

(19) $[r_{ij}(a),\ \sigma_{kj}(b)]=r_{ik}(-ab)$ (i, j, *and* k *distinct*),

(20) $r_{ij}(a),\ \sigma_{ij}(b)]=I$,

(21) $[r_{ii}(a),\ \sigma_{ki}(b)]=r_{ik}(-ab)r_{kk}(-ab^2)$;

provided that the elements of the left sides of these equalities have been defined.

Proof By straightforward calculation (cf. [2] and [3]).

Lemma 2. *Let* $n\geqslant 3$, *and* $g\in O_{2n}(R)$. *If* g *commutes with* $r_{ij}(1)$ *for all* $i\neq$ *then* $g=\begin{pmatrix} uI & B \\ 0 & uI\end{pmatrix}$, *where* $u\in R$, $u^2=1$, *and* $B+B'=0$. *If* g *commutes with* $t_{ij}(1)$ *f*
all $i\neq j$, *then* $g=\begin{pmatrix} uI & 0 \\ C & uI\end{pmatrix}$ *where* $u^2=1$ *and* $C+C'=0$. *In particular, the center*
$O_{2n}(R)$ *is* $\{uI\,|\,u^2=1\}$, *and the center of* $EO_{2n}(R)$ *is* $\{uI\,|\,u^2=1\}\cap EO_{2n}(R)$.

Proof By straightforward calculation.

§2. The Congruence Subgroups

In this section we discuss the normality and commutator relations of congruenc subgroups.

For $a\in R$ let $G(R,\ a)$ denote the subgroup of $EO_{2n}(aR)$ generated by $r_{ij}(aR)$ $t_{ij}(aR)$, $\sigma_{ij}(aR)$, $r_{ii}(aL)$ and $t_{ii}(aL)$ for all $i\neq j$.

Lemma 3. *If* $n\geqslant 3$, $a\in R$, *and* g *is an elementary orthogonal matrix, the* $gG(R,\ a^3)g^{-1}\subseteq G(R,\ a)$.

Proof It suffices to prove $ghg^{-1}\in G(R,\ a)$ for any generator h of $G(R,\ a^3)$. Thi can be easily verified by Lemma 1 except the following five cases.

Case 1: $g=r_{ij}(b)$ and $h=t_{ij}(a^3c)$, $i\neq j$, and b, $c\in R$. Then, taking $k\neq i$, j, an using Lemma 1, we have

$$\begin{aligned} ghg^{-1}&=r_{ij}(b)\,[t_{ik}(a),\ \sigma_{kj}(a^2c)]\,r_{ij}(-b)\\ &=[r_{ij}(b)t_{ik}(a)r_{ij}(-b),\ r_{ij}(b)\sigma_{kj}(a^2c)r_{ij}(-b)]\\ &=[\sigma_{jk}(-ab)t_{ik}(a),\ r_{ik}(-a^2bc)\sigma_{kj}(a^2c)]\in G(R,\ a).\end{aligned}$$

Case 2: $g=t_{ij}(b)$ and $h=r_{ij}(a^3c)$, $i\neq j$, and b, $c\in R$. This is similar to Case 1.

Case 3: $g=\sigma_{ij}(b)$ and $h=\sigma_{ji}(a^3c)$, $i\neq j$, and b, $c\in R$. Then, taking $k\neq i$, j, we have

$$\begin{aligned} ghg^{-1} &= \sigma_{ij}(b)[\sigma_{jk}(a), \sigma_{ki}(a^2c)]\sigma_{ij}(-b) \\ &= [\sigma_{ij}(b)\sigma_{jk}(a)\sigma_{ij}(-b), \sigma_{ij}(b)\sigma_{ki}(a^2c)\sigma_{ij}(-b)] \\ &= [\sigma_{ik}(ab)\sigma_{jk}(a), \sigma_{ki}(a^2c)\sigma_{kj}(-a^2bc)] \in G(R, a). \end{aligned}$$

Case 4: $g=\tau_{ii}(b)$ and $h=t_{ii}(a^3c)$, $b, c\in L$. Then, taking $j\neq i$, we have

$$\begin{aligned} ghg^{-1} &= \tau_{ii}(b)t_{ji}(a^2c)[t_{jj}(-ac), \sigma_{ji}(a)]\tau_{ii}(-b) \\ &= \tau_{ii}(b)t_{ji}(a^2c)\tau_{ii}(-b)[\tau_{ii}(b)t_{jj}(-ac)\tau_{ii}(-b), \tau_{ii}(b)\sigma_{ji}(a)\tau_{ii}(-b)] \\ &= \sigma_{ij}(-a^2bc)t_{jj}(-a^4bc^2)t_{ji}(a^2c)[t_{jj}(-ac), \tau_{ij}(-ab)\tau_{jj}(-a^2b)\sigma_{ji}(a)] \\ &\in G(R, a). \end{aligned}$$

Case 5: $g=t_{ii}(b)$ and $h=\tau_{ii}(a^3c)$, $b, c\in L$. This is similar to Case 4.

Lemma 4. *Assume that $n\geqslant 3$, $g\in O_{2n}(R)$, and M is a maximal ideal of R. Then there exists some $s\in R-M$ such that*

$$gG(R, s)g^{-1}\subseteq EO_{2n}(R).$$

Proof Let φ: $O_{2n}(R)\to O_{2n}(R_M)$ be the group homomorphism induced by localization. Since R_M is a local ring, we may write $\varphi(g)=\varepsilon g_1$, where $\varepsilon\in EO_{2n}(R_M)$, and

$$g_1=\begin{pmatrix} I & & & \\ & u & & \\ & & I & \\ & & & u^{-1} \end{pmatrix} \text{ or } \begin{pmatrix} I & & & \\ & 0 & & u \\ & & I & \\ & u^{-1} & & 0 \end{pmatrix}$$

for some unit u in R_M. Assume that ε is a product of m elementary orthogonal matrices in $EO_{2n}(R_M)$. Denote $k=3^m$.

Let $A=R[x, y]$ be the polynomial ring over R in two indeterminates x and y, and let $S^{-1}A$ be the ring of fractions of A with respect to $S=R-M$. Then $S^{-1}A=R_M[x, y]$. We use the same sign φ to denote the canonical homomorphism $O_{2n}(A)\to O_{2n}(S^{-1}A)$.

Consider an elementary orthogonal matrix $h(xy^k)$ in $O_{2n}(A)$, where h is one of the τ_{ij}, t_{ij} and σ_{ij}, $i\neq j$. Then $\varphi(h(xy^k))\in G(S^{-1}A, y^k)$. Obviously, $g_1\varphi(h(xy^k))g_1^{-1}\in G(S^{-1}A, y^k)$. By Lemma 3,

$$\begin{aligned} \varphi(gh(xy^k)g^{-1}) &= \varepsilon g_1\varphi(h(xy^k))g_1^{-1}\varepsilon^{-1}\in \varepsilon G(S^{-1}A, y^k)\varepsilon^{-1} \\ &\subseteq G(S^{-1}A, y)\subseteq EO_{2n}(yS^{-1}A). \end{aligned}$$

Hence, $\varphi(gh(xy^k)g^{-1})$ is a product of a finite number of $\tau_{ml}(yb)$, $t_{ml}(yc)$ and $\sigma_{ml}(yd)$, where the b, c, $d\in S^{-1}A$. For all $h=\tau_{ij}$, t_{ij} and σ_{ij}, $i\neq j$, the number of the b, c and d remains to be finite, and hence, they have a common denominator $s_1\in S$. Replacing y by s_1y, we obtain $\varphi(gh(xs_1^ky^k)g^{-1})\in\varphi(EO_{2n}(yA))$ for all $h=\tau_{ij}$, t_{ij} and σ_{ij}, $i\neq j$. Denote $\varphi(gh(xs_1^ky^k)g^{-1})=\varphi(\tilde{h}(y))$ where $\tilde{h}(y)\in EO_{2n}(yA)$. There exists a common element $s_2\in S$ such that $s_2gh(xs_1^ky^k)g^{-1}=s_2\tilde{h}(y)$ for all $h=\tau_{ij}$, t_{ij} and σ_{ij}, $i\neq j$. Express the two sides of the above equality as matrix polynomials over R in two indeterminates x and y, and compare the coefficients. We derive $gh(xs_1^ks_2^ky^k)g^{-1}=$

$\tilde{h}(s_2y)\in EO_{2n}(R[x, y])$. Take $s'=s_1^ks_2^k$, and let $x=a\in R$, and $y=1$. Then

$$gh(s'R)g^{-1}\subseteq EO_{2n}(R)$$

for all $h=r_{ij}$, t_{ij} and σ_{ij}, $i\neq j$.

Next, let $B=A/(2x)=R[x, y]/(2x)$. Then R can be naturally embedded into B, and $S^{-1}B=R_M[x, y]/(2x)$. Proceeding as above, we may find s_3, $s_4\in S$ such that $gf(xs_3^ks_4^ky^k)g^{-1}\in EO_{2n}(R[x, y]/(2x))$ for all $f=r_{ii}$ and t_{ii}. Take $s''=s_3^ks_4^k$, and let $x=b\in L$, and $y=1$. Then

$$gf(s''L)g^{-1}\subseteq EO_{2n}(R)$$

for all $f=r_{ii}$ and t_{ii}.

Finally, let $s=s's''\in S$. Then $gG(R, s)g^{-1}\subseteq EO_{2n}(R)$ since $sR\subseteq s'R$ and $sL\subseteq s''L$.

Remark. The idea of the proof of Lemma 4 is due to Vaserstein who used similar method to discuss linear groups over rings[14]. In this lemma we improve the idea of Vaserstein.

Theorem 1. *Let $n\geq 3$. Then $EO_{2n}(R)$ is a normal subgroup of $O_{2n}(R)$.*

Proof For any fixed $g\in O_{2n}(R)$, let

$$J=\{c\in R\,|\,gG(R, c)g^{-1}\subseteq EO_{2n}(R)\}.$$

It is easily seen that J is an ideal of R. By Lemma 4, J cannot be contained in any maximal ideal. Thus, $J=R$. This implies that $gEO_{2n}(R)g^{-1}=gG(R, 1)g^{-1}\subseteq EO_{2n}(R)$. Therefore, $EO_{2n}(R)$ is normal in $O_{2n}(R)$ since g is arbitrarily given.

Lemma 5. *Assume that $n\geq 3$, and J is an ideal of R. Then*

$$[EO_{2n}(R),\ \mathrm{Ker}\,\lambda_J]\subseteq EO_{2n}(R, J).$$

Proof Suppose that $h\in EO_{2n}(R)$, and $g\in \mathrm{Ker}\,\lambda_J$, i.e., $g\equiv I \bmod J$.

Let $\tilde{R}=\{(a, b)\in R\times R\,|\,a\equiv b \bmod J\}$, and $\tilde{J}=\{(c, 0)\,|\,c\in J\}$. Then $\tilde{J}$ is an ideal of $\tilde{R}$. Identify

$$\{(g_1, g_2)\in O_{2n}(R)\times O_{2n}(R)\,|\,g_1\equiv g_2 \bmod J\}$$

with $O_{2n}(\tilde{R})$. Let $\tilde{h}=(h, h)\in EO_{2n}(\tilde{R})$, and $\tilde{g}=(g, I)\in \mathrm{Ker}\lambda_{\tilde{J}}$. By Theorem 1, $[\tilde{h}, \tilde{g}]\in EO_{2n}(\tilde{R})\cap \mathrm{Ker}\lambda_{\tilde{J}}$. But $\tilde{R}$ is the semidirect product of $\tilde{J}$ and the subring $\{(a, a)\in R\}$, and hence, $EO_{2n}(\tilde{R})\cap \mathrm{Ker}\,\lambda_{\tilde{J}}=EO_{2n}(\tilde{R}, \tilde{J})$ (cf. [9] and [14]). Therefore, $([h, g], I)=[\tilde{h}, \tilde{g}]\in EO_{2n}(\tilde{R}, \tilde{J})$. This implies that

$$[h, g]\in EO_{2n}(R, J).$$

Theorem 2. *Let R be an arbitrary commutative ring, J an ideal of R, and $n\geq 3$. Then*

$$\begin{aligned}EO_{2n}(R, J)&=[EO_{2n}(R), EO_{2n}(J)]=[EO_{2n}(R), EO_{2n}(R, J)]\\&=[EO_{2n}(R), O_{2n}(R, J)]=[O_{2n}(R), EO_{2n}(R, J)].\end{aligned}$$

In particular, $EO_{2n}(R, J)$ is a normal subgroup of $O_{2n}(R)$.

Proof First observe the following obvious inclusions:

$$[EO_{2n}(R), EO_{2n}(J)]\subseteq [EO_{2n}(R), EO_{2n}(R, J)]\subseteq EO_{2n}(R, J),$$

$$[EO_{2n}(R), EO_{2n}(R, J)]\subseteq [EO_{2n}(R), O_{2n}(R, J)],$$

$$[EO_{2n}(R), EO_{2n}(R, J)]\subseteq [O_{2n}(R), EO_{2n}(R, J)].$$

Thus, it suffices to prove the following three inclusions.

(1) Prove $EO_{2n}(R, J)\subseteq [EO_{2n}(R), EO_{2n}(J)]$.

Since $n\geqslant 3$, $EO_{2n}(J)\subseteq [EO_{2n}(R), EO_{2n}(J)]$ by Lemma 1. For any generator hgh^{-1} of $EO_{2n}(R, J)$, where $h\in EO_{2n}(R)$, and $g\in EO_{2n}(J)$,

$$hgh^{-1}=[h, g]g\in [EO_{2n}(R), EO_{2n}(J)].$$

Hence, $EO_{2n}(R, J)\subseteq [EO_{2n}(R), EO_{2n}(J)]$.

(2) Prove $[EO_{2n}(R), O_{2n}(R, J)]\subseteq EO_{2n}(R, J)$.

Suppose that $h\in EO_{2n}(R)$ and $g\in O_{2n}(R, J)$. By Theorem 1, $[h, g]\in EO_{2n}(R)\cap \operatorname{Ker}\lambda_J$. For the fixed g, define a map

$$\Psi: EO_{2n}(R)\to (EO_{2n}(R)\cap \operatorname{Ker}\lambda_J)/EO_{2n}(R, J)$$

by $h\mapsto [h, g]EO_{2n}(R, J)$. By means of Lemma 5, Ψ is a group homomorphism, and $(EO_{2n}(R)\cap \operatorname{Ker}\lambda_J)/EO_{2n}(R, J)$ is an abelian group. But $EO_{2n}(R)=[EO_{2n}(R), EO_{2n}(R)]$ since $n\geqslant 3$. It follows that Ψ is trivial. Hence $[h, g]\in EO_{2n}(R, J)$ for any $h\in EO_{2n}(R)$ and any $g\in O_{2n}(R, J)$.

(3) Prove $[O_{2n}(R), EO_{2n}(R, J)]\subseteq EO_{2n}(R, J)$.

Suppose that $g\in O_{2n}(R)$ and $h\in EO_{2n}(R, J)$. Use the signs $\tilde R$ and $\tilde J$ as in the proof of Lemma 5. Put $\tilde g=(g, g)\in O_{2n}(\tilde R)$, and $\tilde h=(h, I)\in EO_{2n}(\tilde R, \tilde J)$. Then $[\tilde g, \tilde h]\in EO_{2n}(\tilde R)\cap \operatorname{Ker}\lambda_{\tilde J}$. It follows, from the proof of Lemma 5, that $([g, h], I)=[\tilde g, \tilde h]\in EO_{2n}(\tilde R, \tilde J)$. Therefore, $[g, h]\in EO_{2n}(R, J)$.

§ 3. The Problem of Normal Subgroups

In this section we shall give a necessary and sufficient condition for which the problem of normal subgroups of the orthogonal group has an affirmative solution when $n\geqslant 3$.

Lemma 6. *Assume that $n\geqslant 3$, and g is one of the $r_{ij}(a)$, $t_{ij}(a)$ and $\sigma_{ij}(a)$, $i\neq j$. Let N be the normal subgroup of $EO_{2n}(R)$ generated by g. Then N contains $r_{km}(aR)$, $t_{km}(aR)$, $\sigma_{km}(aR)$, $r_{kk}(a^2L)$ and $t_{kk}(a^2L)$ for all $k\neq m$. Therefore, if $aR\cap L=a^2L$, then $N=EO_{2n}(R, aR)$.*

Proof By Lemma 1.

Lemma 7. *Suppose that $n\geqslant 3$, and $a\in R$ with $aR\cap L\neq a^2L$. Then there exists some $c\in R$ such that the normal subgroup N of $EO_{2n}(R)$ generated by $r_{12}(c)$ is not equal to $EO_{2n}(R, cR)$.*

Proof First note that $aR\cap L\supseteq aL\supseteq a^2L$. When $aR\cap L\neq a^2L$, we may find an element $c\in aR\cap L$ such that $c\notin a^2L$ and $c^2\in a^2L$. Indeed, if $aL\neq a^2L$, then take any $c\in aL-a^2L$. If $aL=a^2L$, take $c\in aR\cap L$ with $c\notin a^2L$, then $c^2\in aL=a^2L$.

Let $J=a^2L$. Since $\lambda_J(EO_{2n}(R))=EO_{2n}(R/J)$, and $\lambda_J(EO_{2n}(R, cR))=EO_{2n}(R/J,$

R/J), it suffices to consider the case $J=0$. Thus, we may assume that $0\neq c\in L$ and $c^2=0$.

Let $K=\left\{\begin{pmatrix} I+cA & cB \\ cC & I+cD \end{pmatrix}\in EO_{2n}(R)\,|\,cB \text{ and } cC \text{ are alternating}\right\}$. By an alternating matrix we mean a matrix $Q=(x_{ij})$ with $x_{ij}=-x_{ji}$ and $x_{ii}=0$ for all i, j. It is not difficult to verify that K is a normal subgroup of $EO_{2n}(R)$. Clearly, $N\subseteq K$, and $r_{11}(c)\notin K$ (cf. [9]).

Definition. *We say that the problem of normal subgroups of the orthogon group $O_{2n}(R)$ has an affirmative solution provided that, for any subgroup H of $O_{2n}(R$ H is normalized by $EO_{2n}(R)$ if and only if there exists a unique ideal J of R such th*

$$EO_{2n}(R,\ J)\subseteq H\subseteq O_{2n}(R,\ J).$$

The ideal J here is actually the order of H.

By Lemma 7 we see that, when $n\geqslant 3$, a necessary condition for the problem normal subgroups of $O_{2n}(R)$ to have an affirmative solution is that $aR\cap L=a^2L$ fo each $a\in R$. Our goal is to prove that this condition is also sufficient.

Lemma 8. *Suppose that $n\geqslant 3$, and H is a non-central subgroup of $O_{2n}(R$ normalized by $EO_{2n}(R)$. Then there exists a maximal ideal M of R such that $\varphi(H)$ is non-central subgroup of $O_{2n}(R_M)$, where φ: $O_{2n}(R)\to O_{2n}(R_M)$ is the homomorphism induced by localization.*

Proof If $\varphi(H)$ is a central subgroup of $O_{2n}(R_M)$ for each $M\in\max(R)$, the $(\mathrm{Ord}(H))_M=\mathrm{Ord}(\varphi(H))=0$ for each $M\in\max(R)$. It would imply $\mathrm{Ord}(H)=0$ sinc $\mathrm{Ord}(H)$ is an R-module. This is a contradiction.

Lemma 9. *In the setting of Lemma 8, $\varphi(H)$ contains some $\sigma_{ij}(b)$, $i\neq j$, $0\neq b\in R_M$.*

Proof Since H is normalized by $EO_{2n}(R)$, $\varphi(H)$ is normalized by $\varphi(EO_{2n}(R))$ By Lemma 8, $\varphi(H)$ contains $g=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ which is non-central in $O_{2n}(R_M)$.

Case 1: Suppose $B=C=0$, i. e., $\varphi(H)$ contains $g=\begin{pmatrix} A & 0 \\ 0 & A'^{-1}\end{pmatrix}$ which is non-central. This case can be easily settled by using the results on linear groups (cf. [10] or [15]).

Case 2: Suppose $B=0$, $C\neq 0$, and A is a scalar matrix, i. e., $g=\begin{pmatrix} uI & 0 \\ C & u^{-1}I\end{pmatrix}$. In such a case, $C+C'=0$. If the diagonal of C has a nonzero entry, say $c_{11}\neq 0$, then $\varphi(H)$ contains

$$[r_{21}(-1),\ [[\sigma_{12}(-1),\ g],\ \sigma_{13}(1)]]=\sigma_{13}(u^{-1}c_{11}).$$

If C has a non-diagonal entry which is not zero, say $c_{12}\neq 0$, then $\varphi(H)$ contains

$$[r_{32}(-1),\ [[\sigma_{13}(-1),\ g],\ \sigma_{21}(1)]]=\sigma_{21}(u^{-1}c_{12}).$$

Case 3: Suppose $B=0$, $C\neq 0$, and A is not a scalar matrix. By Lemma 2, $g=\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ cannot commute with all $t_{ij}(1)$, $i\neq j$, say $gt_{km}(1)\neq t_{km}(1)g$. Then $\varphi(H)$ contains $[g, t_{km}(1)]=\begin{pmatrix} I & 0 \\ C_1 & I \end{pmatrix}$ with $C_1\neq 0$, which satisfies the same conditions as g in Case 2.

Case 4: Suppose $B\neq 0$ but B has a zero column, say $b_1=0$, and $b_2\neq 0$, where b_1 and b_2 are respectively the first and the second columns of B. Denote the first column of D by d_1. Write $t_{12}(1)=\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$. Then $\varphi(H)$ contains

$$g_1=[g, t_{12}(1)]=\begin{pmatrix} I+BXD' & 0 \\ * & * \end{pmatrix}.$$

Claim g_1 is non-central. Otherwise, $I+BXD'=I-b_2d_1'$ is a scalar matrix. Write $b_2d_1'=cI$. Since $b_1=0$, d_1 has at least one element which is invertible in R_M. This implies that b_2 has at least $n-1$ element s which are zero. Hence, $c=0$. Thus, $b_2=0$, contrary to the hypothesis of Case 4. Therefore, g_1 is non-central, and we are reduced to Case 1 or Case 3.

Case 5: Suppose B has not zero columns but there are two columns, say b_1 and b_2, such that $b_1b_2'-b_2b_1'=0$. Write $t_{12}(1)=\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$ again. Then $\varphi(H)$ contains

$$g_1=[g, t_{12}(1)]=\begin{pmatrix} I+BXD' & 0 \\ -X+DXD'-DXB'X & I+DXB' \end{pmatrix}.$$

It is not difficult to show that g_1 is non-central since B has not zero columns. Thus, we are done by Cases 1—3.

General case: $\varphi(H)$ contains $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ which is non-central. By Cases 1—5 we may assume $b_ib_j'-b_jb_i'\neq 0$ for any two columns of B. Then $\varphi(H)$ contains $[g, t_{12}(1)]=\begin{pmatrix} * & b_1b_2'-b_2b_1' \\ * & * \end{pmatrix}$ which is also non-central. Denote $[g, t_{12}(1)]$ by original $g=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $B=xy'-yx'\neq 0$, where $x=(x_1\cdots x_n)'$ and $y=(y_1\cdots y_n)'$. Thus, $b_1=y_1x-x_1y$, $b_2=y_2x-x_2y$, and $b_n=y_nx-x_ny$. By Case 5, we may assume $b_1b_2'-b_2b_1'\neq 0$.

Let $\alpha=x_ny_2-x_2y_n$ and $\beta=x_1y_n-x_ny_1$, and let $s\in R-M$ be a common denominator of α and β. Then $s\alpha$, $s\beta\in R$. Write $t_{1n}(s\alpha)t_{2n}(s\beta)=\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}$. Take

$$g_1=[g, t_{1n}(s\alpha)t_{2n}(s\beta)]=\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}\in\varphi(H).$$

Note that $B_1=BYB'=0$. Thus, if g_1 is non-central, then we are reduced to Cases 1

—3 again. If g_1 is central, then $A_1=I+BYD'$ is a scalar matrix. Note also that the last row of BY is zero, and hence $A_1=I$. This implies $g_1=I$, i.e., $\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix} =\begin{pmatrix} I & 0 \\ Y & I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Thus, $BY=0$. It follows that $\alpha b_1+\beta b_2=0$. Take

$$g_2=[g,\ t_{12}(1)]=\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}\in\varphi(H),$$

where $B_2=b_1b_2'-b_2b_1'\neq 0$ but the last column of B_2 is $\alpha b_1+\beta b_2=0$. Therefore, we ar reduced to Case 4. Thus, Lemma 9 has been proved.

Theorem 3. *If $n\geqslant 3$, and H is a non-central subgroup of $O_{2n}(R)$ normalized b $EO_{2n}(R)$, then H contains some $r_{ij}(a)$, $i\neq j$, $0\neq a\in R$.*

Proof By Lemmas 8 and 9, there is $M\in\max(R)$ such that $\varphi(H)$ contains som $\sigma_{ij}(b)$, $i\neq j$, $0\neq b\in R_M$. Write $b=a/s$ with $a\in R$ and $s\in R-M$. Then, taking $k\neq i$, j, we have $r_{ki}(a)=[r_{kj}(-s),\ \sigma_{ij}(b)]\in\varphi(H)$, i.e., there is $h\in H$ with $\varphi(h)=r_{ki}(a)$. Put $g=r_{ki}(-a)h\in O_{2n}(R)$. Then $\varphi(g)=1$, and hence $ug=uI$ for some $u\in R-M$. Thus, $g\sigma_{ji}(u)=\sigma_{ji}(u)g$, and H contains

$$[h,\ \sigma_{ji}(-u)]=[r_{ki}(a)g,\ \sigma_{ji}(-u)]=[r_{ki}(a),\ \sigma_{ji}(-u)]=r_{kj}(ua).$$

Clearly, $ua\neq 0$.

Theorem 4. *Let R be an arbitrary commutative ring, and $n\geqslant 3$. Then th problem of normal subgroups of $O_{2n}(R)$ has an affirmative solution if and only if $aR\cap L=a^2L$ for each $a\in R$.*

Proof The necessity has been established by Lemma 7. We now prove th sufficiency. Suppose that $aR\cap L=a^2L$ for each $a\in R$, and H is a subgroup of $O_{2n}(R)$ normalized by $EO_{2n}(R)$. If H is a central subgroup of $O_{2n}(R)$, then

$$\{I\}=EO_{2n}(R,\ 0)\subseteq H\subseteq O_{2n}(R,\ 0)=\text{Center}\,(O_{2n}(R)).$$

Now assume that H is non-central. Let $\Omega=\{\text{ideals } K \mid EO_{2n}(R,\ K)\subseteq H\}$, and $J=\sum_{K\in\Omega}K$. Then J is an ideal of R, and $EO_{2n}(R,\ J)\subseteq H$.

Denote $\bar{R}=R/J$, and $\bar{H}=\lambda_J(H)$. Since $\lambda_J(EO_{2n}(R))=EO_{2n}(\bar{R})$, $\bar{H}$ is normalize by $EO_{2n}(\bar{R})$. If $\bar{H}$ is a non-central subgroup of $O_{2n}(\bar{R})$, then by Theorem 3, ther exists some $r_{ij}(\bar{a})\in\bar{H}$, $i\neq j$, $0\neq\bar{a}\in\bar{R}$, i.e., $a\notin J$. Thus, there is $h\in H$ with $\lambda_J(h)=\lambda_J(r_{ij}(a))$. Take $g=r_{ij}(-a)h\in\operatorname{Ker}\lambda_J$. Choose $k\neq i,\ j$. By Lemma 5,

$$[\sigma_{kj}(-1),\ g]\in[EO_{2n}(R),\ \operatorname{Ker}\lambda_J]\subseteq EO_{2n}(R,\ J)\subseteq H.$$

Since H is normalized by $EO_{2n}(R)$, we see that H contains

$$r_{ij}(a)\,[\sigma_{kj}(-1),\ g]\,r_{ij}(-a)\,[h,\ \sigma_{kj}(-1)]=[r_{ij}(a),\ \sigma_{kj}(-1)]=r_{ik}(a).$$

By Lemma 6, $EO_{2n}(R,\ aR)\subseteq H$ since $aR\cap L=a^2L$. It would imply that $aR\subseteq J$ contrary to that $a\notin J$. Hence, $\bar{H}$ must be a central subgroup of $O_{2n}(\bar{R})$. Thus, $\operatorname{Ord}(\bar{H})=0$, i.e., $\operatorname{Ord}(H)\subseteq J$. This shows that

$$EO_{2n}(R,\ J)\subseteq H\subseteq O_{2n}(R,\ J).$$

Evidently, such an ideal J is uniquely determined.

Conversely, suppose that H is a subgroup of $O_{2n}(R)$, and there is an ideal J such that the above inclusions hold. Then by Theorem 2,

$$[EO_{2n}(R), H] \subseteq [EO_{2n}(R), O_{2n}(R, J)] = EO_{2n}(R, J) \subseteq H.$$

This implies that H is normalized by $EO_{2n}(R)$.

Corollary. *Let R be a commutative ring in which 2 is not a zero divisor, and $n \geqslant 3$. Then the problem of normal subgroups of $O_{2n}(R)$ has an affirmative solution.*

Proof If 2 is not a zero divisor, then $L = \mathrm{Ann}(2) = 0$. The result of the corollary is a consequence of Theorem 4.

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