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A NOTE TO THE NEVANLINNA'S FUNDAMENTAL THEOREM

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Abstract

In this paper, the author extends Nevanlinna's second fundamental theorem and establishes the following inequality:

Let
$$p(z, u)=A_\nu(z)u^\nu+A_1(z)u^{\nu-1}+\cdots+A_0(z)$$
be an irreducible two-variable polynomial and $f(z)$ a transcendental entire function, then
$$(\nu-1)T(r, f)<N\left(r, \frac{1}{p(z, f(z))}\right)+S(r, f)$$
with
$$S(r, f)=O(\log(rT(r, f)))\ \text{n. e.}$$
where "n. e." means that the estimation holds for all large r with possibly an exceptional set of finite measure when f is of infinite order.

§1. Introduction

It is well-known that Nevanlinna's second fundamental theorem plays central role in value distributi on theory of meromorphic functions. The theore has been generalized in various forms by many authors. We will give another for of extension to the algebraic functions.

Let u be a ν-valued algebraic function determined by
$$p(z, u)=A_\nu(z)u^\nu+\cdots+A_0(z). \tag{1}$$
Then the set of zeros of $p(z, f(z))$ is coincident wth that of $f(z)-u(z)$ with the sar mutiplicity, where $f(z)-u(z)$ is an algebroid function. We shall prove

Theorem 1. *Let $u(z)$ be an algebraid function determined by* (1) *and $f(z)$ be transcendental entire function. Then we have*
$$(\nu-1)T(r, f)<N\left(r, \frac{1}{p(z, f(z))}\right)+S(r, f)$$
or
$$(\nu-1)T(r, f)<\nu N\left(r, \frac{1}{f-u}\right)+S(r, f),$$
where
$$S(r, f)=O(\log(rT(r, f)))\quad \text{n. e.}$$

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It was conjectured in [3, p. 542] that if f is transcendental entire and g is non-linear entire, then $g(f)$ has infinitely many fixpoints. The conjecturewas also posed when f is mero-morphic [4]. It has beenproved[1] that the conjecture is true when f is meromorphic and g is a polynomial of degree$\geqslant 3$. By Theorem 1 we get

Corollary 1. *If g is a rational function of degree$\geqslant 2$, and f is a transcendental entire function, then $g(f)$ has infinitely many fix-points.*

Proof Let

$$g(u)=\frac{a_0u^n+\cdots+a_n}{b_0u^m+\cdots+b_m}=\frac{g_1(u)}{g_2(u)},$$

$$\nu=\max(n, m)\geqslant 2,$$

where $g_1(u)$ and $g_2(u)$ are relatively prime. Then the set of fix-points of $g(f(z))$ contains the set of zeros of $g_1(f(z))-zg_2(f(z))$.

Since

$$p(z, u)=g_1(u)-zg_2(u)$$

is irreducible as a two-variable polynomial, $p(z, u)=0$ defines u as a ν-valued algebraic function. The corollary follows from Theorem 1.

Remark. When $p(z, u)$ is reducible, it can be written as

$$p(z, u)=p_1(z, u)^{n_1}\cdots p_k(z, u)^{n_k},$$

where p_j's are irreducible and $\deg_u p_j(z, u)=\nu_j$ $(j=1, \cdots, k)$. By the same reasoning as inTheorem 1, we are able to deduce the following inequality

$$(\nu_1+\cdots+\nu_k-1)T(r, f)<N\left(r, \frac{1}{p(z, f(z))}\right)+S(r, f).$$

Theorem 2. *Let $u(z)$ be an algebraic function determined by*

$$p(z, u)=A_\nu(z)u^\nu+\cdots+A_0(z)=0,$$

and f be a transcendental meromorphic function. Then for every $s>0$, we have

$$(\nu-2-s)T(r, f)<\nu N\left(r, \frac{1}{f-u}\right)+s(r, f),$$

where $s(r, f)$ is the same as in Theorem 1.

From Theorem 2 it is easy to derive the following

Corollary 2. *Let $g(z)$ be a rational function of degree$\geqslant 3$ and $f(z)$ a transcendental meromorphic function, then $g(f(z))$ has infinitely many fix-points.*

The fundamental concepts and standard notations of Nevanlinna's theory of meromorphic functions and algebroid functions are employed without explanation (c. f. [2, 5]). And we always use u, w to denote algebroid (or algebraic) fnnctions and α, β, γ to denote constants.

§2. Preliminary Lemmas

Lemma 1[2]. *Let $w=w(z)$ be an algebroid function defined by*

$$B_\mu(z)w^\mu+\cdots+B_1(z)w+B_0(z)=0.$$

Then, for every positive integer k,

$$m\left(r, \frac{w^{(k)}}{w}\right)=O(\log r(T(r, w))) \quad \text{n. e.}$$

Lemma 2. *Let $a_1(z)$, $a_2(z)$, $\cdots$, $a_n(z)$ be meromorphic function in a domain D. Then $a_1(z)$, $\cdots$, $a_n(z)$ are linearly dependent (i. e. there exist α_1, $\cdots$, α_n, not all vanishing, such that*

$$\alpha_1 a_1(z)+\cdots+\alpha_n a_n(z)\equiv 0, \quad \forall z\in D$$

if and only if

$$\Delta(a_1(z), \cdots, a_n(z))\equiv 0, \quad \forall z\in D,$$

where $\Delta(a_1(z), \cdots, a_n(z))$ denote the Wronskian determinant of $a_1(z)$, $\cdots$, $a_n(z)$.

Proof We will use the induction to the number n of the given meromorpl functions.

For $n=1$, the conclusion is trivial.

Suppose that for $n=k-1$ the conclusion is turc. We need to prove the conelusi when $n=k$.

Consider the vector space $F\times F\cdots\times F=F^k$ over the field F of meromorpl functions in the domain D. Since

$$\Delta(a_1(z), \cdots, a_k(z))\equiv\begin{vmatrix} a_1(z) & \cdots & a_k(z) \\ \vdots & & \\ a_1^{(k-1)}(z) & \cdots & a_1^{(k-1)}(z)\end{vmatrix}\equiv 0 \quad (z\in D),$$

there exist $c_1(z)$, $\cdots$, $c_k(z)\in F$, not all vanishing, such that

$$c_1(z)(a_1(z), \cdots, a_1^{(k-1)}(z))^T+\cdots+c_k(z)(a_k(z), \cdots, a_k^{(k-1)}(z))^T\equiv 0 \quad (z\in D),$$

i. e.

$$\begin{cases} c_1(z)a_1(z)+\cdots+c_k(z)a_k(z)\equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_1(z)a_1^{(k-1)}(z)+\cdots+c_k(z)a_k^{(k-1)}(z)\equiv 0. \end{cases} \tag{2.}$$

If there exists some $c_i(z)\equiv 0$, then by the assumption of induction, the conclusi follows. So we may assume

$$c_i(z)\not\equiv 0 \quad (i=1, 2, \cdots, k),$$

then there exists one simply connected domain $D_1\subset D$ such that

$$c_i(z)\neq 0, \infty \quad (\forall z\in D_1, i=1,2, \cdots, k).$$

Differentiating the first $k-1$ equations of (2.1), we get

$$\begin{cases} c_1'(z)a_1(z)+\cdots+c_k'(z)a_k(z)\equiv 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_1'(z)a_1^{(k-2)}(z)+\cdots+c_k'(z)a_k^{(k-2)}(z)\equiv 0. \end{cases} \quad (z\in D_1) \tag{2.}$$

Solving a_k, a_k', $\cdots$, $a_k^{(k-2)}$ from (2.1) and substituting into (2.2), we obtain

$$\left(c_1'-c_1\frac{c_k'}{c_k}\right)(a_1, \cdots, a_1^{(k-2)})+\cdots+\left(c_{k-1}'-c_{k-1}\frac{c_k'}{c_k}\right)(a_{k-1}, \cdots, a_{k-1}^{(k-2)})\equiv 0.$$

If there is at least one $c_i(z)$ such that

$$c_i'(z)-c_i(z)\frac{c_k'(z)}{c_k(z)}\not\equiv 0 \quad (z\in D_1),$$

then by the assumption of induction, we get the conclusion; or else

$$c_i'(z)-c_i(z)\frac{c_k'(z)}{c_k(z)}\equiv 0 \quad (i=1,\ 2,\ \cdots,\ k-1,\ z\in D_1),$$

i. e.

$$\frac{c_i'(z)}{c_i(z)}=\frac{c_k'(z)}{c_k(z)} \quad (i=1,\ 2,\ \cdots,\ k-1,\ z\in D_1),$$

which imply

$$c_i(z)\equiv\gamma_i c_k(z) \quad (i=1,\ 2,\ \cdots,\ k-1,\ z\in D_1) \tag{2.3}$$

with γ_i being non-zero constants.

Substituting (2.3) into the first equatson of (2.1), we get

$$\gamma_1 a_1(z)+\cdots+\gamma_{k-1}h_{k-1}(z)+a_k(z)\equiv 0 \quad (z\in D_1).$$

By the principle of analytic continuation, the above relation remains valid for $z\in D$. And the proof of Lemma 2 is completed.

Lemma 3. *Let $u(z)$ be a ν-valued algebroid function determined by*

$$A_\nu(z)u^\nu+\cdots+A_1(z)u+A_0(z)\equiv 0, \tag{2.4}$$

where the $A(z)$'s are entire functions without common zeros, and $f(z)$ be a transcendental meromorphic function. Then $f(z)-u(z)$ is also a ν-valued algebroid function.

Proof Let $w(z)=f(z)-u(z)$. Replacing u by $f-w$ in (2.4), we get

$$B_\nu(z)w^\nu+\cdots+B_1(z)w+B_0(z)=0, \tag{2.5}$$

where B's are polynomials of A_i and f, so that they are meromorphic functions. Multiplying (2.5) by a suitable entire function we can make the coefficients of w^i's in (2.5) be entire functions without common zeros. Thus w is an at most ν-valued algebroid function. If (2.5) is reducible, then w is a μ-valued $(\mu<\nu)$ algebroid function. Suppose that w is determined by

$$c_\mu(z)w^\mu+\cdots+c_1(z)w+c_0(z)=0. \tag{2.6}$$

If we replace w by $f-u$ in (2.6), then u is an at most μ-valued algebroid function. That is a contradiction, and our conclusion follows.

Lemma 4. *Let matrices $\tilde{A}_{k\times m}$, $\tilde{B}_{k\times n}$, $A_{k\times m}$, $B_{k\times n}$, $S_{m\times m}$, $T_{n\times n}$ satisfy the condition $\tilde{A}=A\cdot S$, $\tilde{B}=B\cdot T$. $(k=m+n)$. Then*

$$\det[\tilde{A}\tilde{B}]=\det[AB]\cdot\det S\cdot\det T,$$

where "det" is the operator of determinant.

Proof The conclusion is deduced simply by

$$[\tilde{A}\tilde{B}]=[AB]\begin{bmatrix} S & 0\\ 0 & T\end{bmatrix}.$$

§ 3. Proofs of the Main Results

By the theory of algebraic function[6], on the cut z-plane $\mathscr{B}$ (suppose that we get one edge) the equation $p(z, u)=0$ defines a ν-valued algebraic function $\{u_1(z), \cdots, u_\nu(z)\}$. We assume that a maximal subset of linear independence of $\{u_1(z), \cdots, u_\nu(z)\}$ $(z\in\mathscr{B})$ is $\{u_1(z), \cdots, u_p(z)\}$. Then in view of Lemma 2

$$\Delta(u_1(z), \cdots, u_p(z))\not\equiv 0 \quad (z\in\mathscr{B}).$$

On $\mathscr{B}$ we define the function

$$L(f):=(-1)^{p+1}\frac{\Delta(f, u_1, \cdots, u_p)}{\Delta(u_1, \cdots, u_p)}. \tag{3.1}$$

Here we suppose that f is a transcendental meromorphic function, though Theorem 1 is just concerning with the entire function.

We claim that $L(f)$ is a simply-valued meromorphic function on the whole z-plane, and

$$L(f)=f^{(p)}+\varphi_1 f^{(p-1)}+\cdots+\varphi_p f, \tag{3.2}$$

where φ_i's are rational functions.

In fact, suppose that $u_1(z), \cdots, u_p(z)$ change to $u_{i_1}(z), \cdots, u_{i_p}(z)$ in turn when $L(f)$ is analytically continued from one edge to the other, where $\{i_1, \cdots, i_p\}\subset\{1, 2, \cdots, n\}$, then $u_{i_1}(z), \cdots, u_{i_p}(z)$ is also linear independent on $\mathscr{B}$. Thus by the choice of $\{u_1, \cdots, u_p\}$ there exists a non-degenerated matrix $S\doteq(\alpha_{ij})_{p\times p}$, such that

$$\begin{bmatrix} u_{i_1}\cdots u_{i_p}\\ u'_{i_1}\cdots u'_{i_p}\\ \vdots \quad \vdots\\ u^{(p)}_{i_1}\cdots u^{(p)}_{i_p}\end{bmatrix}=\begin{bmatrix} u_1\cdots u_p\\ u'_1\cdots u'_p\\ \vdots\quad\vdots\\ u^{(p)}_1\cdots u^{(p)}_p\end{bmatrix}\begin{bmatrix}\alpha_{11}\cdots\alpha_{1p}\\ \vdots\quad\vdots\\ \alpha_{p1}\cdots\alpha_{pp}\end{bmatrix}.$$

In view of Lemma 4 and (3.1), on the other edge of the cut-line, we have

$$L(f)=(-1)^{p+1}\frac{\Delta(f, u_{i_1}, \cdots, u_{i_p})}{\Delta(u_{i_1}, \cdots, u_{i_p})}=(-1)^{p+1}\frac{\Delta(f, u_1, \cdots, u_p)}{\Delta(u_1, \cdots, u_p)},$$

i. e. $L(f)$ is a simply-valued meromorphic function on the whole z-plane. And (3.2) is easily derived from (3.1).

Now, we start from the identical equation

$$\prod_{j=1}^{\nu}\frac{1}{f-u_j}=\sum_{i=1}^{\nu}A_i\frac{L(f)}{f-u_i}\Big/L(f), \tag{3.3}$$

where $\quad A_i=[(u_i-u_1)\cdots(u_i-u_{i-1})(u_i-u_{i+1})\cdots(u_i-u_\nu)]^{-1}.$

Taking logarithm and integrating on $|z|=r$ $(z=re^{i\theta})$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{j=1}^{\nu}\frac{1}{f-u_j}\right|d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\sum_{i=1}^{\nu}A_i\frac{L(f)}{f-u_i}\right|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\left(\sum\frac{A_iL(f)}{f-u_i}\right)^{-1}\right|d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1}{L(f)}\right|d\theta:=I_1-I_2+I_3:=I. \tag{3.4}$$

Using Jenien formula, we get

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{j=1}^{\nu}\frac{1}{f-u_j}\right|d\theta=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{A_\nu(z)}{p(z,f)}\right|d\theta$$

$$=N(r,p(z,f))+N\left(r,\frac{1}{A_\nu}\right)-N\left(r,\frac{1}{p(z,f)}\right)+O(1), \tag{3.5}$$

$$I_1<\frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{\nu}\log^+\left|\frac{L(f)}{f-u_i}\right|d\theta+\sum_{i=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|A_i|\,d\theta+\log\nu$$

$$<\frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{\nu}\log^+\left|\frac{L(f-u_i)}{f-u_i}\right|d\theta+O(\log r).$$

In view of (3.2) and Lemma 3, we have

$$I_1<\frac{1}{2\pi}\int_0^{2\pi}\sum_{i=1}^{\nu}\left\{\log^+\left|\frac{(f-u_i)^{(p)}}{f-u_i}\right|d\theta+\cdots\right.$$

$$\left.+\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{(f-u_i)'}{f-u_i}\right|d\theta\right\}+O(\log r)$$

$$=\nu m\left(r,\frac{(f-u)^{(p)}}{f-u}\right)+\cdots+\nu m\left(r,\frac{(f-u)'}{f-u}\right)+O(\log r), \tag{3.6}$$

$$I_2=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{\prod_{i=1}^{\nu}(f-u_i)}{L(f)}\right|d\theta$$

$$>\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\prod_{i=1}^{\nu}(f-u_i)\right|d\theta-\frac{1}{2\pi}\int_0^{2\pi}\log^+|L(f)|\,d\theta$$

$$=m\left(r,\frac{p(z,f)}{A_\nu}\right)-m(r,L(f))$$

$$>m(r,p(z,f))-m(r,L(f))-O(\log r), \tag{3.7}$$

$$I_3=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\frac{1}{L(f)}\right|d\theta=m\left(r,\frac{1}{L(f)}\right)-m(r,L(f)). \tag{3.8}$$

By (3.6), (3.7), (3.8)

$$I\leq-m(r,p(z,f))+m\left(r,\frac{1}{L(f)}\right)+s_1(r,f), \tag{3.9}$$

where

$$s_1(r,f)=\nu\sum_{k=1}^{p}m\left(r,\frac{(f-u)^{(k)}}{f-u}\right)+O(\log r)$$

and

$$m\left(r,\frac{1}{L(f)}\right)<T\left(r,\frac{1}{L(f)}\right)=m(r,L(f))+N(r,L(f))+O(1)$$

$$<m\left(r,\frac{L(f)}{f}\right)+m(r,f)+N(r,L(f))+O(1). \tag{3.10}$$

In view of (3.2)

$$N(r,L(f))=N(r,f^{(p)})+O(\log r)=N(r,f)+p\overline{N}(r,f)+O(\log r). \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), we obtain

$$I<-m(r, p(z, f))+T(r, f)+p\overline{N}(r, f)+s_2(r, f), \tag{3.12}$$

where

$$s_2(r, f)=\sum_{k=1}^{p} m\left(r, \frac{f^{(k)}}{f}\right)+\nu \sum_{k=1}^{p} m\left(r, \frac{(f-u)^{(k)}}{f-u}\right)+O(\log r).$$

In view of (3.5), (3.12), we get

$$T(r, p(z, f))<T(r, f)+N\left(r, \frac{1}{p(z, f)}\right)+p\overline{N}(r, f)+s_2(r, f)$$

and[7]

$$T(r, p(z, f))=\nu T(r, f)+O(\log r).$$

Therefore

$$(\nu-1)T(r, f)<N\left(r, \frac{1}{p(z, f)}\right)+p\overline{N}(r, f)+s(r, f) \quad (3.$$

where

$$s(r, f)=\sum_{k=1}^{\nu} m\left(r, \frac{f^{(k)}}{f}\right)+\nu \sum_{k=1}^{\nu} m\left(r, \frac{(f-u)^{(k)}}{f-u}\right)+O(\log r).$$

By Lemma 1, $S(r, f)=O(\log(rT(r, f)))$ n. e.

If $f(z)$ is a transcendental entire function, then in view of (3.13), we obt

$$(\nu-1)T(r, f)<N\left(r, \frac{1}{p(z, f)}\right)+S(r, f).$$

Theorem 1 is thus proved.

Proof of Theorem 2 As in the proof of Theorem 1, $u_1(z)$, $u_2(z)$, $\cdots$, $u_\nu(z)$ defined on $\mathscr{B}$. Let $L(s; u_1, \cdots, u_\nu)$ denote the set of all linear combinations of $u_1^{n_1}$, $u_\nu^{n_\nu}$, where n_j's are non-negative integers such that

$$\sum_{j=1}^{\nu} n_j = s.$$

Suppose that $\{b_1(z), \cdots, b_n(z)\}$ is the base of $L(s; u)$ and $\{c_1(z), \cdots, c_k(z)\}$ is the l of $L(s+1; u)$. Let

$$\varphi[f]=\frac{\Delta(c_1, \cdots, c_k, fb_1, \cdots, fb_n)}{\Delta(c_1, \cdots, c_k)\Delta(b_1, \cdots, b_n)}.$$

Then

a) By Lemma 2, $\varphi[f]$ is well defined on $\mathscr{B}$, and $\varphi[f]\not\equiv 0$.

b) By Lemma 4, we are able to prove $\varphi[f]$ is a simply-valued functi Therefore, $\varphi[f]$ is a meromorphic function on the whole plane.

c) $$\varphi[f]=f^n\psi\left[\frac{f'}{f}\right].$$

ψ is a polynomial of $\frac{f^{(i)}}{f}$, with the coefficients being rational function of c_i's and b_i's and their derivatives.

d) $$T(r, \varphi[f])<(n+k)T(r, f)+s(r, f). \tag{3.14}$$

In fact, it is easy to see that

$$\varphi[f]=f^{n+k}\frac{\Delta\left(\frac{c_1}{f},\ \cdots,\ \frac{c_k}{f},\ b_1,\ \cdots,\ b_n\right)}{\Delta(c_2,\ \cdots,c_k)\Delta(b_1,\ \cdots,\ b_n)},$$

hen

$$N(r,\ \varphi[f])<(n+k)N(r,\ f)+O(\log r).$$

3y c)

$$m(r,\ \varphi[f])<nm(r,\ f)+s(r,\ f),$$

o (3.14) follows.

e)

$$\varphi[f-a]=\varphi[f],$$

vhere $a(z)$ is a linear combination of u_i's.

f)

$$\nu m\left(r,\ \frac{1}{f-u}\right)<\frac{1}{n}T\left(r,\ \frac{1}{\varphi[f]}\right)+s(r,\ f). \tag{3.15}$$

Proof Let

$$d_j(z)=\min_{i\neq j}\frac{1}{2}|u_i(z)-u_j(z)|\quad(z=re^{i\theta}),$$

$$d(z)=\min_j d_j(z),$$

$$E_j=\{z;\ |f(z)-u_j(z)|<d(z)\}.$$

.et

$$w_j=f-u_j,\ \psi_j=\psi\left[\frac{w_j'}{w_j}\right].$$

Then in view of e), c),

$$\varphi[w_j]=\varphi[f]=w_j^n\psi_j,$$

o

$$\frac{1}{|f-u_j|^n}=\frac{|\psi_j|}{|\varphi[f]|}$$

nd

$$\log^+(|f-u_j|)^{-1}<\frac{1}{n}\log^+\frac{1}{|\varphi[f]|}+\frac{1}{n}\log^+|\psi_j|\quad(z\in E_j),$$

$$\log^+(|f-u_j|)^{-1}<\log^+\frac{1}{d(z)}\quad(z\notin E_j).$$

ince E_j's are disjoint each other,

$$\sum_{j=1}^{\nu}\log^+\frac{1}{|f-u_j|}<\frac{1}{n}\log^+\frac{1}{|\varphi[f]|}+\frac{1}{n}\sum_{j=1}^{\nu}\log^+|\psi_j|+\nu\log^+\frac{1}{d(z)},$$

$$\log^+\frac{1}{d(z)}\leqslant\sum_{i\neq j}\log^+\frac{1}{|u_i-u_j|}+O(1).$$

y Lemma 3 $f-u$ is a ν-valued algebroid function, so

$$\begin{aligned}\nu m\left(r,\ \frac{1}{f-u}\right)&<\frac{1}{n}m\left(r,\ \frac{1}{\varphi[f]}\right)+s(r,\ f)\\&\leqslant\frac{1}{n}T\left(r,\ \frac{1}{\varphi[f]}\right)+s(r,\ f)\\&=\frac{1}{n}T(r,\ \varphi[f])+s(r,\ f),\end{aligned} \tag{3.16}$$

and (f) is proved.

Since

$$m\left(r, \frac{1}{f-u}\right)=T\left(r, \frac{1}{f-u}\right)-N\left(r, \frac{1}{f-u}\right)$$
$$=T(r, f-u)-N\left(r, \frac{1}{f-u}\right)+O(1)$$
$$=T(r, f)-N\left(r, \frac{1}{f-u}\right)+O(\log r),$$

combining with (3.14) and (3.16), we obtain

$$\nu T(r, f)<\frac{n+k}{n} T(r, f)+\nu N\left(r, \frac{1}{f-u}\right)+s(r, f).$$

By the same argument as in [8], for every $\varepsilon>0$ and sufficiently large s, we hav

$$\frac{k}{n}<1+\varepsilon,$$

so

$$(\nu-2-\varepsilon)T(r, f)<\nu N\left(r, \frac{1}{f-u}\right)+s(r, f).$$

Theorem 2 is proved.

§ 4. Applications

In this section, we will give the necessary and sufficient condition for $g(j$ to have only finitely many fix-points when

$$g(x)=\frac{a_0x^2+a_1x+a_2}{(b_0x+b_1)^2}.$$

If $b_0=0$, without loss of generality, we may assume

$$g(x)=x^2.$$

Then $g(f(z))$ has only finitely many fix-points if and only if

$$f(z)+\sqrt{z},\quad f(z)-\sqrt{z}$$

have finitely many zeros. We consider these two functions on the Riemann su of $\sqrt{z}$. Let $\sqrt{z}=t$. Then

$$f(t^2)+t=\frac{Q(t)e^{\gamma(t)}}{f_1(t^2)}, \qquad ($$

where $f_1(z)$ is the canonical product formed by the poles of $f(z)$, $Q(t)$ is a polync and $\gamma(t)$ is an entire function. Substituting t by $-t$ in (3.17), we obtain

$$f(t^2)-t=\frac{Q(-t)e^{\gamma(-t)}}{f_1(t^2)}. \qquad ($$

Cancelling $f_1(z^2)$ from (3.18) and (3.17), we deduce

$$f(z)=\frac{Q(\sqrt{z})\exp(\gamma(\sqrt{z}))+Q(-\sqrt{z})\exp(\gamma(-\sqrt{z}))}{(Q(\sqrt{z})/\sqrt{z})\exp(\gamma(\sqrt{z}))-(Q(-\sqrt{z})/\sqrt{z})\exp(\gamma(-\sqrt{z}))}$$

nd this is just the conclusion of Theorem 3 in [1].

If $b_0 \neq 0$, an appropriate transform can turn it into the case considered above.

Final Remark. After this paper had just been completed, the author found hat Corollary 2 had been proved in [9] by Osgood and Gross.

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