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PIECEWISE SMOOTH SOLUTIONS OF SEMILINEAR HYPERBOLIC SYSTEMS IN HIGHER SPACE DIMENSION

CHEN SHUXING (陈恕行)*

Abstract

This paper discusses piecewise smooth solutions for semilinear hyperbolic systems in multi-dimensional space. In the class of piecewise smooth functions the author proves the existence and uniqueness of local solutions to the Cauchy problem for 3×3 hyperbolic system. Besides, it is also proved that when two characteristic surfaces bearing weak singularities intersect, the solution will still be piecewise smooth and the weak singularities will propagate along all characteristic surfaces.

§1. Introduction

Recently, propagation and interaction of progressing waves for nonlinea equations attracts many mathematicians' attention. Usually, there are three ways t describe the progressing waves. The first one is by conormal distributions. The theor on such progressing waves is developed by J. M. Bony, S. Alinhao, R. Melrose et a (see [1—6]). The second one is by striated or stratified waves, introduced and studie by J. Rauch & M. Reed in [7, 8]. The third one is by piecewise smooth solution which is studied in [9, 10] for one space-dimensional case, and studied in [11, 12 for 2×2 systems in higher space-dimensional case. Using functions with jum discontinuities to discribe singularities of solutions is a classical way, but the proble on propagation and interaction of such progressing waves for hyperbolic systems wit more equations is still open. In this paperwe are going to deal with this problen We restrict our discussion to semilinear strictly and symmetric hyperbolic system Existense and uniqueness for 3×3 systems and discontinuous data are establishe and a clear understanding on propagation and interaction of piecewise smoot progressing waves is obtained. These results are known only for 2×2 systems (se [11]). In latter case the interaction of progressing waves does not appear.

Let us consider

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* Institute of Mathematics, Fudan University, Shanghai, China.

$$Pu \equiv \left(A_1\partial_t + A_2\partial_x + \sum_{j=3}^{n} A_j\partial_{y_j}\right)u = F(t, x, y, u), \tag{1.1}$$

where P is a hermitian symmetric hyperbolic $k\times k$ operator of first order, F and all coefficients are smooth. Suppose that the Cauchy data are given on $t=0$, which are piecewise smooth with jump only over a submanifold $\sigma\subset\{t=0\}$ with dimension $n-2$. Let Σ_1, Σ_2 and Σ_3 be the characteristic surfaces of the system through σ. The main results in this paper are

Theorem 1. *For each point $p\in\sigma$, there exists a unique local piecewise solution u of the Cauchy problem for (1.1), where u has jump only over $\Sigma_1\cup\Sigma_2\cup\Sigma_3$. Moreover, if the initial data are smooth at a point $p_0\in\sigma$, then the solution is also smooth on the characteristic curves on Σ_1, Σ_2 and Σ_3 passing through p.*

Theorem 2. *Suppose that u is a bounded solution of (1.1) in Ω, which is divided into $\Omega^\pm$ by a space-like mani-fold Σ, and Ω^+ is in the domain of determinacy for Ω^-. Suppose that u is piecewise smooth in Ω^-, Σ_1 and Σ_2 are characteristic hypersurfaces for P, which intersect transversally at σ in Ω^+, Σ_3 is a forward characteristic hypersurface through σ.*

1) *If in Ω^- the solution u has jump only over Σ_1, then u is piecewise smooth in Ω with jump only over Σ_1.*

2) *If in Ω^- the solution u has jump only over $\Sigma_1\cup\Sigma_2$, then u is piecewise smooth in Ω with jump only over Σ_1, Σ_2 and Σ_3.*

In order to prove Theorems 1 and 2, we reduce the Cauchy problem of (1.1) to Goursat problem in a domain with edge σ. By the property of finite influence domain for hyperbolic system, we may assume that F in (1.1) and Cauchy data vanish outside a bounded domain, and we may restrict ourselves to discussion in a neighbourhood of the origin. In § 2, by changing the coordinates (1.1) is reduced to a simpler form. In § 3 we compute all jumps of u and its direvatives at the origin, then in § 4 we compute all jumps along Σ_1 and Σ_2 (if Σ_3 is in between Σ_1 and Σ_2), where we use the transport equations established in [11]. In § 5 by folding the domain between Σ_2 and Σ_3 to the domain between Σ_1 and Σ_3, we establish apriori estimates for piecewise solutions of (1.1), and then we complete the proof of Theorems 1 and 2 in § 6.

§ 2. Preliminaries

Assume that the equation of σ on $t=0$ is $x=\varphi(y)$, and the equation Σ_i is

$$x=\psi_i(t, y) \tag{2.1}$$

with $\psi_i(0, y)=\varphi(y)$ for $i=1, 2, 3$. Obviously, all ψ_i are smooth functions, and all $\partial_y^\beta\psi_i(t, y)$ tend to 0, when $t\to 0$. The strictly hyperbolisity of the systems allow us to assume $\psi_2<\psi_3<\psi_1$ and

$$\partial_t\psi_2(0,\ y)<\partial_t\psi_3(0,\ y)<\partial_t\psi_1(0,\ y).$$

Near the origin, the characteristic surfaces through σ divide the whole domain to several parts I, II, III_1 and III_2 (see Fig. 1). Assume that the initial data only have jump on σ. Then by the property of finite influence domain for hyperbolic system, we may obtain smooth solution in I, II, which is determined completely by the initial data on $t=0$ for $x>0$ and $x<0$ respectively. Therefore, we only need to consider a Goursat problem in the domain III with boundary value on Σ_1 and Σ which could be determined by initial data.

For the convenience of computation, we introduce some transformations to flatte the surface Σ_1, Σ_2 and Σ_3. First, by a coordinate transformation, we may assun $\varphi=0$, and the equation of Σ_1 (resp. Σ_2) is $x=t$ (resp. $x=-t$) (see [11]). Denote tl equation of Σ_3 by $x=f(t,\ y)$, we have $|f'|<1$. Therefore, let

$$x'=\frac{\sqrt{2}}{2}(t+x),\quad t'=\frac{\sqrt{2}}{2}(t-x)$$

(here and later, we keep the coordinates y unchanged, and omit them in tl expression of transformation), Σ_1, Σ_2 and Σ_3 are expressed by $t'=0$, $x'=0$ and

$$F(t',\ x',\ y)\equiv\frac{\sqrt{2}}{2}(x'-t')-f\Big(\frac{\sqrt{2}}{2}(t'+x'),\ y\Big)=0. \qquad (2.2$$

Since

$$\frac{\partial F}{\partial x'}=\frac{\sqrt{2}}{2}-f'_t\,\frac{\sqrt{2}}{2}>0,$$

the equation of Σ_3 can be written in the form

$$x'=g(t',\ y). \qquad (2.3$$

Then by the transformation

$$t''=t',\ x''=\frac{t'}{g(t',\ y)}\,x'$$

Σ_1, Σ_2, Σ_3 become $t''=0$, $x''=0$, $t''=x''$. Furthermore, by

$$t'''=\frac{\sqrt{2}}{2}(t''+x''),\quad x'''=\frac{\sqrt{2}}{2}(-t''+x''),$$

these surfaces are changed to

$$t'''=x''',\ t'''=-x''',\ t'''=0.$$

Let us still denote x''', t''' by x, t, the surfaces Σ_1, Σ_2, Σ_3, will be $t=x$, $t=-$ $t=0$ respectively. After these transformations, the original system has the form

$$X_l u_l+\sum_{i=1}^{3}\sum_{j=1}^{n} b_{i l v_j} u_i=f_l,\ l=1,\ 2,\ 3, \qquad (2.4$$

where $X_l=\partial_t+\lambda_l\partial_x$, and $\lambda_1=1$ on Σ_1, $\lambda_2=-1$ on Σ_2, $\lambda_3=0$ on Σ_3. Without loss genelarity we may assume the initial plane is still $t=0$, because outside the domain III, the solution has been determined. In this way we have flattened the characteristic surfaces, this is illustrated by Fig. 2.

In the sequal we denote $\mathrm{III}_\tau=\mathrm{III}\cap\{t\leqslant\tau\}$, $\Sigma_{i\tau}=\Sigma_i\cap\{t\leqslant\tau\}$, $\Pi_\tau=\{t=\tau\}$, $\Pi_{i\tau}=\mathrm{III}_i\cap\Pi_\tau$, and denote by $u|_{\Sigma^+_{1,2}}$ $(u|_{\Sigma^-_{1,2}})$ the limit of u on $\Sigma_{1,2}$ from above (below), by $u|_{\Sigma_3^l}$ $(u|_{\Sigma_3^r})$ the limit of of u on Σ_3 from left (right).

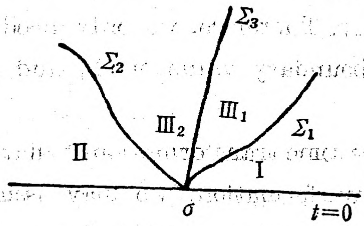

Fig. 1

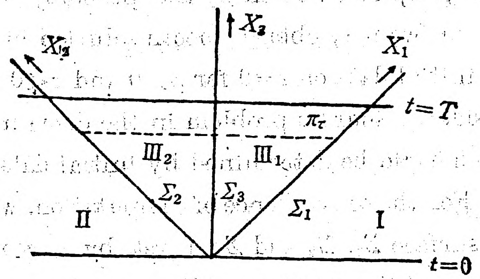

Fig. 2

§3. Jump at the Origin

Suppose that u is a piecewise smooth solution of (2.4) near the origin, and the initial data have jump only over $x=0$. The purpose of this section is to determine all jumps of u and its derivative over Σ_1, Σ_2, Σ_3 along $x=t=0$. Certainly, it is enough to give a procedure to compute them at the origin.

Let $\{\ \}_{\Sigma_l}$ denote the jump on Σ_l, and $\{\ \}_{\Sigma_l^0}$ denote the value of $\{\ \}_{\Sigma_l}$ at $t=0$, $[\ ]_0$ denote the jump of initial data at $x=0$. We will compute $\{X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}\partial_y^\beta u_i\}_{\Sigma_l^0}$ by induction. In the process of computing these quantities, we often uso "known" to express the terms, which have been obtained by previous step.

1) $|\alpha|=0$.

Noticing that X_l is transversal to $\Sigma_{l'}$ if $l\neq l'$, we have $\{u_l\}_{\Sigma^0_{l'}}=0$ by the system. By differentiating with respect to y, we know $\{\partial_y^\beta u_l\}_{\Sigma^0_{l'}}=0$. Hence we can express other jumps of u by jumps of the initial data:

$$\{\partial_y^\beta u_l\}_{\Sigma_3^0}=[\partial_y^\beta u_l^0]_0.$$

2) $|\alpha|=1$.

In this step we need todetermine all $\{X_i\partial^\beta u_l\}_{\Sigma^0_{l'}}$, this form contains 27 quantities even for $\beta=0$.

Since X_l is tangential to Σ_l, we know

$$\{X_{l'}\partial^\beta u_l\}_{\Sigma^0_{l'}}=X_{l'}\{\partial^\beta u_l\}_{\Sigma^0_{l'}}$$

from 1) immediately.

By the system itself, $\{X_l u_l\}_{\Sigma_{l'}}$ can be expressed by the quantities estimated in 1).

Since anyone in $\{X_1, X_2, X_3\}$ can be linearly expressed by other two vectors, we have for l, l', l'', being differential from each other,

$$\{X_{l''}u_l\}_{\Sigma_l^0} = C_1\{X_l u_l\}_{\Sigma_l^0} + C_2\{X_{l'}u_l\}_{\Sigma_l^0} = \text{known}.$$

As for the terms with the form $\{X_{l'}u_l\}_{\Sigma_l^0}$ for $l \neq l'$, we have

$$\begin{aligned}\{X_2 u_l\}_{\Sigma_1^0} &= [X_2 u_l|_{t=0}]_{x=-0}^{x=+0} - \{X_2 u_l\}_{\Sigma_2^0} - \{X_2 u_l\}_{\Sigma_3^0} \\ &= [(\lambda_1 - \lambda_2)u_{1x}^0]_0 - [b_{1ij}\partial_{y_j} u_i^0]_0 + [f_1^0]_0 - \{X_2 u_1\}_{\Sigma_2^0} - \{X_2 u_1\}_{\Sigma_3^0} \\ &= \text{known},\end{aligned}$$

and similar calculation for others.

It is obvious that $\partial_y^\beta\{X_l u_l\}_{\Sigma_l^0}$ are known. Because $[\partial_y, X_l]$ can be expressed X_1, X_2, X_3 with smooth coefficients, it is easy to compute all terms $\{X_l \partial_y^\beta u_l\}_{\Sigma_l^0}$ induction with respect to $|\beta|$.

3) When all jumps $\{X^\alpha \partial_y^\beta u_l\}_{\Sigma_l^0}$ with $|\alpha| \leqslant m$ are determined, we can comp $\{X^\alpha \partial_y^\beta u_l\}_{\Sigma_l^0}$ with $|\alpha| = m+1$ as in step 2).

Because of $[X, X^\alpha]$ ($|\alpha| = m$) = combination of differential operators of α'-order ($|\alpha|' \leqslant m$), by the hypothesis of induction and the system itself we have

$$\{X_l(X^\alpha u_l)\}_{\Sigma_l^0} = \{X^\alpha(X_l u_l)\}_{\Sigma_l^0} + \{[X_l, X^\alpha]u_l\}_{\Sigma_l^0}.$$

For $l \neq l'$, noticing anyone of X_1, X_2, X_3 can be expressed by other two, we ha

$$\begin{aligned}\{X_{l'}X^\alpha u_l\}_{\Sigma_l^0} &= \{aX_{l'}^{m+1}u_l\}_{\Sigma_l^0} + \{X^{\alpha_1}X_l u_l\}_{\Sigma_l^0} + \text{lower order terms} \\ &= aX_{l'}^{m+1}\{u_l\}_{\Sigma_l^0} + \text{known} = 0 + \text{known}.\end{aligned}$$

Hence for different l, l', l'', the jump $\{X_{l''}X^\alpha u_{l'}\}_{\Sigma_l^0}$ are also known. Using the init data once more, for different l, l', l'',

$$\{X_{l'}X^\alpha u_l\}_{\Sigma_l^0} = [X_{l'}X^\alpha u_l|_{t=0}]_0 - \{X_{l'}X^\alpha u_l\}_{\Sigma_{l'}^0} - \{X_{l'}X^\alpha u_l\}_{\Sigma_{l''}^0} = \text{known}.$$

Similar to step 2), we can determine all jumps $\{X^\alpha \partial^\beta u_l\}_{\Sigma_l^0}$ for $|\alpha| = m+1$, $|$ $< \infty$ by induction with respect to $|\beta|$.

Therefore, all jumps of derivatives of u over Σ_1, Σ_2, Σ_3 at the origin can determined by induction with respect to $|\alpha|$.

§4. Jumps over Characteristic Surfaces

In this section we determine jumps of u and its derivatives over the characterist surfaces Σ_1, Σ_2. From them the value of all components of u and its derivaties on Σ can be obtained. Here we still compute these quantities by induction.

First, by the system itself,

$$\{u_1\}_{\Sigma_2} = \{u_3\}_{\Sigma_2} = \{u_2\}_{\Sigma_1} = \{u_3\}_{\Sigma_1} = 0.$$

Then along Σ_2, by substracting the equation

$$X_2u_2+\Sigma b_{2ij}\partial_{y_j}u_i=f_2 \tag{4.1}$$

 lower side from the equation at upper side, we obtain

$$X_2\{u_2\}_{\Sigma_1}+\Sigma b_{22j}\partial_{y_j}\{u_2\}_{\Sigma_1}$$
$$=f_2(t,\ x,\ y,\ \{u_2\}_{\Sigma_1}+u_2|_{\Sigma_1^-})-f_2(t,\ x,\ y,\ u_2|_{\Sigma_1^-}). \tag{4.2}$$

his is a nonlinear equation of first order for the variable $\{u_2\}_{\Sigma_1}$ along Σ_2 with itial datum $\{u_2\}_{\Sigma_2^0}$, it can be solved by elementary integration. Therefore, $\{u_2\}_{\Sigma_1}$ known, and so is $\{u_1\}_{\Sigma_1}$. Because all ∂_{y_j} are tangential to $\Sigma_{1,2}$ on $\Sigma_{1,2}$, $\{\partial_y^\beta u_l\}_{\Sigma_{1,2}}$ are nown.

Suppose that all $\{X^\alpha\partial_y^\beta u_l\}_{\Sigma_{1,2}}$ for $|\alpha|\leqslant m$ and any β are known. We can etermine them for $|\alpha|=m+1$ and any β as follows.

By the hypothesis of induction

$$\{X_lX^\alpha u_{l'}\}_{\Sigma_l}=X_l\{X^\alpha u_{l'}\}_{\Sigma_l}=\text{known}\quad (l=1,\ 2,\ l'=1,\ 2,\ 3).$$

y the system itself, for $l=1,\ 2$ and $l'=1,\ 2,\ 3$

$$\{X_{l'}X^\alpha u_{l'}\}_{\Sigma_l}=\{X^\alpha X_{l'}u_{l'}\}_{\Sigma_l}+\{[X_{l'},\ X^\alpha]u_{l'}\}_{\Sigma_l}$$
$$=\{X^\alpha(\Sigma b_{l'ij}\partial_{y_j}u_i)\}_{\Sigma_l}+\{[X_{l'},\ X^\alpha]u_{l'}\}_{\Sigma_l}=\text{known}. \tag{4.3}$$

y the independence of X_1, X_2, X_3, we known $\{X_2X^\alpha u_3\}_{\Sigma_1}$, $\{X_3X^\alpha u_2\}_{\Sigma_1}$, $\{X_3X^\alpha u_1\}_{\Sigma_1}$, $X_1X^\alpha u_3\}_{\Sigma_1}$. Acting X_2X^α on the first equation in the system and then taking the ump over Σ_1, we have

$$\{X_2X^\alpha X_1u_1\}_{\Sigma_1}+\sum_{i,j}\{X_2X^\alpha b_{1ij}\partial_{y_j}u_i\}_{\Sigma_1}=\{X_2X^\alpha f_1\}_{\Sigma_1} \tag{4.4}$$

$$X_1\{X_2X^\alpha u_1\}_{\Sigma_1}+\{[X_2X^\alpha,\ X_1]\,u_1\}_{\Sigma_1}+\sum_j b_{11j}\partial_{y_j}\{X_2X^\alpha u_1\}_{\Sigma_1}$$
$$+\sum_j\{[X_2X^\alpha,\ b_{11j}\partial_{y_j}]u_1\}_{\Sigma_1}+\sum_{i\neq 1}\sum_j\{X_2X^\alpha b_{1ij}\partial_{y_j}u_i\}_{\Sigma_1}$$
$$=\{X_2X^\alpha f_1\}_{\Sigma_1}$$

$$X_1\{X_2X^\alpha u_1\}_{\Sigma_1}+\sum_j b'_j\partial_{y_j}\{X_2X^\alpha u_1\}_{\Sigma_1}+\{X^{\alpha+1}u_1\}_{\Sigma_1}=\text{known}. \tag{4.5}$$

ince $\{X_1^{m+1}u_1\}_{\Sigma_1}$ is known, the system composed of equations with the form (4.5) for ll α ($|\alpha|=m$) is the one of first order with the same principal part for unknown ariables $\{X_2X^\alpha u_1\}$. Therefore, using the initial data at the origin, $\{X_2X^\alpha u_1\}_{\Sigma_1}$ ith $|\alpha|=m$ can be determined locally. In the same way, all $\{X_1X^\alpha u_2\}_{\Sigma_2}$ are known. umming up above considerations we obtain all quantities $\{X^\alpha u_l\}_{\Sigma_{1,2}}$ with $|\alpha|=m+1$ ocally. Finally, in view of the fact that all ∂_{y_j} are tangential to $\Sigma_{1,2}$ on $\Sigma_{1,2}$, we now all $\{X^\alpha\partial_y^\beta u_l\}_{\Sigma_{1,2}}$ with $|\alpha|=m+1$.

By induction, all jumps of u and its derivatives over are known, hence the oundary value of u and its derivatives on $\Sigma_{1,2}^+$ are obtained.

§ 5. Apriori Estimate

Now we are going to estimate u in domain III. Let $\partial^{s,s'}u$ be $\partial^s_{t,x,y}\partial_t^{s'}(x\partial_x)^r\partial_y^\beta u$ with

$\alpha+r+|\beta|=s'$, $\|u\|_{H^{s,s'}}$ be

$$(\sum_{r\le s,\, r'\le s'} \|\partial^{r,r'}u\|^2)^{1/2}$$

for s, $s'\in N$. Moreover, we denote

$$\|u\|_{B^s}=\sum_{r\le s/2}\|u\|_{H^{r,s-2r}},$$
$$\|u\|_{\hat{B}^s}=\sum_{r\le s/2}\|u\|_{H^{r+1,s-2r-1}}. \tag{5.1}$$

Lemma 1. *If $s>n$, then the space B^s is invariant under C^∞ composition.*

Proof According to the embedding theorem, $H^{r,s-2r}\subset L^\infty$ if $r\geqslant 1$ and $s-r>n$ Therefore, $B^s\subset L^\infty$ if $s>n/2+1$. By the proposition 5.8 in [6], $H^{0,s}\cap L^\infty$ is invari under nonlinear composition. By the results in § 1 of [13], $H^{r,s-2r}$ is invari under nonlinear composition, if $r+s-2r>n/2$ and $r>1/2$. Therefore, if $s>n$, $H^{r,s-2r}$ with $r\geqslant 1$ and $H^{0,s}\cap L^\infty$ are invariant, it means that B^s is also invari under nonlinear composition.

Lemma 2. *There is $\tau_0>0$ such that for any $\tau<\tau_0$, the piecewise smooth compactly supported solution u in III_τ satisfies*

$$\|u_3\|_{B^s(\mathrm{III}_{1\tau})}+\|u_3\|_{B^s(\mathrm{III}_{2\tau})}+\|u_{1,2}\|_{\hat{B}^s(\mathrm{III}_{1\tau})}+\|u_{1,2}\|_{\hat{B}^s(\mathrm{III}_{2\tau})}$$
$$\leqslant C_s\sum_{r\le s}\|\partial^r u\|_{H^{s-r}(\Sigma_1^{1\tau}\cup\Sigma_1^{2\tau})}, \tag{5}$$

where $s>n$.

Proof Let us begin with folding III_2 to III_1 by transformation $x\to -x$. Set

$$u_{i+3}(x,\ t)=u_i(-x,\ t)\quad (i=1,\ 2,\ 3),$$

we have

$$\tilde{A}U+\sum_{j=3}^{n}\tilde{B}_j\partial_{y_j}U=F\quad \text{in } \mathrm{III}_1, \tag{5}$$

where $U=(u_1,\ u_2,\ u_3,\ u_4,\ u_5,\ u_6)^T$, $A=\mathrm{diag}(X_1,\ X_2,\ X_3,\ X_1',\ X_2',\ X_3')$ with

$$X_i'=\partial_t-\lambda_i(t,\ -x,\ y)\partial_x$$

for $i=1,\ 2,\ 3$, $\tilde{B}_j=\mathrm{diag}(B_j,\ B_j)$ are still symmetric, $F=(f_1,\ f_2,\ f_3,\ f_1',\ f_2',\ f_3')^T$ w $f'(t,\ x,\ U)=f_i(t,\ -x,\ u_4,\ u_5,\ u_6)$ for $i=1,\ 2,\ 3$. On the boundary Σ_2, we know components of U and their derivatives, but on the boundary Σ_3, we only have conditions

$$u_4-u_1=0,\ u_5-u_2=0. \tag{5}$$

Multiplying (5.3) by U and then integrating in III_1, we can obtain

$$\int_{\mathrm{II}_{1\tau}} U^2\,dx\,dy\leqslant\int_{\mathrm{III}_{1\tau}}(U^2+F^2)\,dt\,dx\,dy+\frac{\sqrt{2}}{2}\int_{\Sigma_1^{1\tau}}U^2\,dS$$
$$+\int_{\Sigma_{3\tau}}(-u_1^2+u_2^2+u_4^2-u_5^2)\,dS.$$

The last term vanishes by the boundary conditions, thus by Gronwall's inequal

$$\int_{\mathrm{II}_{1\tau}} U^2\,dx\,dy\leqslant C\left(\int_{\mathrm{III}_{1\tau}}F^2\,dt\,dx\,dy+\int_{\Sigma_1^{1\tau}}U^2\,dS\right). \tag{5.5}$$

ice ∂_t, $x\partial_x$, ∂_{y_j} are tangential to Σ_3, acting the operator $\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta$ with

$$\alpha+\gamma+|\beta|\leqslant s$$

(5.2) we obtain

$$(\widetilde{A}+\sum_j\widetilde{B}_j\partial_{y_j})\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta U = [\widetilde{A}+\sum_j\widetilde{B}_j\partial_{y_j},\ \partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta]U+\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta F. \tag{5.6}$$

ticing $\partial_t(x\partial_x)^\gamma\partial_y^\beta U$ on Σ_1 is already known, and on Σ_3

$$\begin{aligned}\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta u_4-\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta u_1=0,\\ \partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta u_5-\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta u_2=0,\end{aligned} \tag{5.7}$$

en in the same way as above we obtain

$$\int_{\Pi_{1\tau}}(\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta U)^2dx\,dy\leqslant C\Big(\int_{\mathrm{III}_{1\tau}}F_1^2\,dt\,dx\,dy+\int_{\Sigma_{1\sigma}^+}(\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta U)^2dS\Big) \tag{5.8}$$

iere

$$F_1^2=([\widetilde{A}+\sum_j\widetilde{B}_j\partial_{y_j},\ \partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta]U)^2+(\partial_t^\alpha(x\partial_x)^\gamma\partial_y^\beta F)^2. \tag{5.9}$$

In view of the form of the operator

$$\widetilde{A}+\sum_j\widetilde{B}_j\partial_{y_j},$$

e know

$$(\widetilde{A}+\sum_j B_j\partial_{y_j})U$$

ntains tangential differentials of $u_{3,6}$ with respect to Σ_3 and differential of $u_{1,2,4,5}$ ong the direction transversal to Σ_3. The latter can be expressed by tangential fferentials of U by means of system (2.4). Therefore, we have

$$\int_{\mathrm{III}_{1\tau}}F_1^2\,dt\,dx\,dy\leqslant\|F\|^2_{H^{0,s}(\mathrm{III}_{1,\tau})}+\|U\|^2_{H^{0,s}(\mathrm{III}_{1,\tau})},$$

hich yields

$$\|\partial^{0,s}U\|^2_{\Pi_{1\tau}}\leqslant C(\|F\|^2_{H^{0,s}(\mathrm{III}_{1,\tau})}+\|U\|^2_{H^{0,s}(\Sigma_{1,\tau}^+)}),$$

id by virture of the transversity of X_1, X_1', X_2, X_2' to Σ_3,

$$\begin{aligned}&\|\partial^{1,s-1}u_{1,2,4,5}\|^2_{\Pi_{1\tau}}+\|\partial^{0,s}u_{3,6}\|^2_{\Pi_{1\tau}}\\ &\leqslant C(\|F\|^2_{H^{0,s}(\mathrm{III}_{1,\tau})}+\|U\|^2_{H^{0,s}(\Sigma_{1,\tau}^+)}).\end{aligned} \tag{5.10}$$

For fixed s, let us prove the inequality

$$\begin{aligned}&\sum_{r\leqslant s/2}(\|\partial^{r,s-2r}u_{3,6}\|^2_{\Pi_{1\tau}}+\|\partial^{r+2,s-2r-2}u_{1,2,4,5}\|^2_{\Pi_{1\tau}})\\ &\leqslant C_s(\|F\|^2_{B^s(\mathrm{III}_{1,\tau})}+\|\partial^s U\|^2_{\Sigma_{1,\tau}^+})\end{aligned} \tag{5.11}$$

y induction with respect to r first. Obviously, (5.10) shows the validity of (5.10) r $r=0$. Now suppose (5.11) is valid for some r with $s/2-1\geqslant r\geqslant 0$, we prove its ılidity for $r+1$. Acting $L=X_2^{r+1}\partial^\alpha(x\partial_x)^\gamma\partial_y^\beta$ with $\alpha+\gamma+|\beta|\leqslant s-2r-2$ on the third juation in (2.4), we have

$$\begin{aligned}&X_3Lu_3+b_{33j}\partial_{y_j}Lu_3+[L,\ X_3]u_3+[L,\ b_{33j}\partial_{y_j}]u_3\\ &+Lb_{31j}\partial_{y_j}u_1+Lb_{32j}\partial_{y_j}u_2=L f_3,\end{aligned}$$

or

$$X_3 Lu_3 + b_{33j}\partial_{y_j} Lu_3 + aLu_3 = g, \tag{5.12}$$

where

$$\|g\|_{L^2(\mathrm{III}_{1\tau})} \leq C(\|f_3\|_{H^{r+1,s-2r-2}(\mathrm{III}_{1\tau})} + \|u_{1,2}\|_{H^{r+1,s-2r-1}(\mathrm{III}_{1\tau})}).$$

Applying the energy method as above showed to (5.12), we have

$$\|Lu_3\|^2_{\mathrm{II}_{1\tau}} \leq C(\|f_3\|^2_{H^{r+1,s-2r-2}(\mathrm{III}_{1\tau})} + \|u_{1,2}\|^2_{H^{r+1,s-2r-1}(\mathrm{III}_{1\tau})} + \|\partial^{r+1,s-2r-1} u_3\|^2_{\Sigma_{1\tau}}). \tag{5.13}$$

Obviously, for u_6 in $\mathrm{III}_{1\tau}$ (or for u_3 in $\mathrm{III}_{2\tau}$) the estimate with the same form is valid, hence

$$\|\partial^{r+1,s-2r-2} u_{3,6}\|^2_{\mathrm{II}_{1\tau}} \leq C(\|F\|^2_{B^s(\mathrm{III}_{1\tau})} + \|\partial^s U\|^2_{\Sigma_{1\tau}}).$$

Thus in view of the transversity of X_1, X_2 with respect to Σ_3, by using the first equations in (2.4), we can estimate $\|\partial^{r+2,s-2r-3} u_{1,2,4,5}\|^2_{\mathrm{II}_{1\tau}}$ by the right hand sid (5.11). Hence (5.11) is proved by induction.

Having established the estimate (5.11), we only need the fact that $\|F\|^2_{B^s(\mathrm{III}_1}$ dominated by $\|U\|^2_{B^s(\mathrm{III}_{1\tau})}$ in order to obtain Lemma 2. The fact holds due to Lem 1, therefore, substituting it into (5.11) implies the validity of (5.2) immediat The proof is then complete.

Remark. If we denote

$$\|u\|^2_{s(\tau)} = \|u\|^2_{H^s(\mathrm{I}\cap(t=\tau))} + \|u\|^2_{H^s(\mathrm{II}\cap(t=\tau))} + \|u\|^2_{H^s(\mathrm{II}_{1\tau})} + \|u\|^2_{H^s(\mathrm{II}_{2\tau})}, \tag{5.}$$

the combining (5.2) with the estimate of U in domains I, II, we have

$$\|u\|_{s(\tau)} \leq g_s(\|u^0\|_{2s+n+2}) \tag{5.}$$

for all piecewise smooth solutions u of (2.4) with initial data $u|_{t=0} = u^0$.

§ 6. Existence and Propagation of Singularities

After establishing the apriori estimate (5.2) or (5.15)), we can prove Theor 1 on existence and uniqueness of local piecewise solution as in [11]. Here we o give the sketch. Substituting derivatives $\partial_{y_j} u$ in (2.4) by

$$\frac{1}{2h}(u(t, x, y+he_j) - u(t, x, y-he_j)),$$

tho system is then changed to a strictly hyperbolic system L_h with parameter h in space-dimensional case. Since the increase of the unknown variables does not ca any new trouble, we can use (5.2) to obtain a uniform estimate for the solutior $L_h u = f$ with the same initial data. By choosing a convergent subsequence and pass to limit, we obtain the existence of the solution for the Goursat problem. Combin the consideration in the domain I and II, we obtain the local existence for Cau problem. The uniqueness comes from the apriori estimate obviously, then Theo 1 is proved.

For Theorem 2, we only give the proof of the conclusion 2). Choose a space-like hypersurface $\tilde{\pi}$ containing $\Sigma_1 \cap \Sigma_2$. By a coordinate transformation, we may assume

$\tilde{\sigma}$ is on $t=0$. It is showed in [11] that the solution u is piecewise smooth up to). Then by Theorem 1, u is piecewise smooth in Ω except $\Sigma_1\cup\Sigma_2\cup\Sigma_3$, and only r$\Sigma_1$, Σ_2 Σ_3, the solution u or its derivatives are allowed to have jump. This derives conclusion in Theorem 2.

The conclusion in Theorem 2 is a version of interaction of progressing waves in ›ewise category of Bony's corresponding conclusion for conormal distributions.

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