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SOME GENERALIZATION OF GRONWALL-BIHARI INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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Abstract

The interest of this paper lies in the estimates of solutions of the three kinds o: Gronwall-Bihari integral inequalities:

(I) $$y(x)\leqslant f(x)+\sum_{i=1}^{n}g_i(x)\int_0^x h_i(s)y(s)ds,$$

(II) $$y(x)\leqslant f(x)+g(x)\psi\left(\int_0^x h(s)w(y(s))\,ds\right)$$

(III) $$y(x)\leqslant f(x)+\sum_{i=1}^{n}g_i(x)\int_0^x h_i(s)y(s)ds+g_{n+1}(x)\psi\left(\int_0^x h_{n+1}(s)w(y(t))ds\right).$$

The results include some modifications and generalizations of the results of D. Willett U. D. Dhongade and Zhang Binggen. Furthermore, applying the conclusion on the abov inequalities to a Volterra integral equation and a differential equation, the authors obtai some new better results.

§0. Introduction

This paper is to obtain estimates of upper bounds of solutions of the follo three types of generalized Gronwall-Bihari integral inequalities:

(I) $$y(x)\leqslant f(x)+\sum_{i=1}^{n}g_i(x)\int_0^x h_i(s)y(s)ds,$$

(II) $$y(x)\leqslant f(x)+g(x)\psi\left(\int_0^x h(s)w(y(s))ds\right),$$

(III) $$y(x)\leqslant f(x)+\sum_{i=1}^{n}g_i(x)\int_0^x h_i(s)y(s)ds+g_{n+1}(x)\psi\left(\int_0^x h_{n+1}(s)w(y(s))ds\right).$$

For Gronwall inequality like (I), [1] and [2] have given some estimates o upper bounds of its solutions; but to obtain the results, not only do they have to im several restrictions on its functions such that the useful scopes are reduced, but the estimates are not sharp. [8] gives an estimate under a general condition, but result is incorrect because of a mistake in induction. Now, we obtain a new re which is better than the above.

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To discuss the inequalities like (II), (III), [2, 3] are under the hypothesis that $u) \in \mathscr{F}$ satisfying:

1) $w(u) > 0$ is nondecreasing for $u \geqslant 0$, $w \in c_0$,

2) $\frac{1}{v} w(u) \leqslant w\left(\frac{u}{v}\right)$ for $u \geqslant 0$, $v > 0$.

t actually, just as proved in [6], the function $w(u)$ satisfying 2) must be a linear ıction, so all results on nonlinear inequalities of those papers are not of any ıaning.

In [4], condition 2) is changed to

$$\frac{1}{v} w(u) \leqslant w_1\left(\frac{u}{v}\right)$$

$u \geqslant 0$, $v \geqslant 1$, where $w_1 > 0$ is nondecreasing function. However, just as indicated in |, its proof is wrong because $f(x) g(x) \geqslant 1$ does not hold.

In [7], condition 2) is modified as to

$$\frac{1}{v} \Omega(u) \leqslant \mu(v) w\left(\frac{u}{v}\right)$$

$u \geqslant 0$, $v > 0$, where $w > 0$ is nondecreasing, $\mu \geqslant 0$, and under that condition the quality

$$u(t) \leqslant a(t) + \sum_{i=1}^{n} \int_0^t f_i(t, s) \Omega(u(s)) ds$$

liscussed. But except for power functions, verification of the above condition and ection of $\mu(v)$ are rather difficult. In fact, Corollary 2 in [7] is untrue because $\mu(v)$ is unsuitable. Furthermore, with an example, we will see, the exactness ts estimate is also not satisfactory.

In [5], results of the inequality (II) and its inverse inequality are obtained. t the requirement that $w(u)$ should be subadditive and submultiplicative is so cting that even the function $w(u) = k + u^\alpha (0 < k < 1, \alpha \in R)$ is not applicable.

In this paper we assume that $w(u)$ satisfies some conditions which are relatively le and can be tested easily, and give some true results on inequalities (II) and I).

§1. Generalization of Linear Integral Inequality

In the sequel, we let $R_+ = [0, +\infty)$. The result of this part is based on the owing well-known lemma.

Lemma 1. *Suppose $f(x)$, $g(x)$ and $h(x)$ are nonnegative and continuous on R_+. If*

$$y(x) \leqslant f(x) + g(x) \int_0^x h(s) y(s) ds, \quad x \in R_+,$$

then

$$y(x)\leqslant f(x)+g(x)\int_0^x h(s)f(s)\exp\left(\int_s^x h(t)g(t)dt\right)ds,\ x\in R_+. \tag{1}$$

Theorem 1. *Suppose $f(x)$, $g_i(x)$ and $h_i(x)$ $(i=1, 2, \cdots, n)$ are nonnegative and continuous on R_+. If*

$$y(x)\leqslant f(x)+\sum_{i=1}^n g_i(x)\int_0^x h_i(s)y(s)ds,\ x\in R_+,$$

then

$$y(x)\leqslant A_n(f),$$

where the function $A_n(f)$ is defined by

$$A_0(u)=u,$$

$$A_{k+1}(u)=A_k(u)+A_k(g_{k+1})\int_0^x h_{k+1}A_k(u)\exp\left(\int_s^x h_{k+1}A_k(g_{k+1})dt\right)ds.$$

To prove Theorem 1 we need the following lemma.

Lemma 2. *Functions $A_k(u)$ $(k=1, 2, \cdots, n)$ defined by (3) satisfy the follo-* *conditions: for $u(x)\geqslant 0$, $v(x)\geqslant 0$,*

1) $A_k(u+v)=A_k(u)+A_k(v)$,

2) $A_k(uv)(x)\leqslant (A_k(u)v)(x)$ *if $v(x)$ is nondecreasing.*

Proof It is obvious that (4) is true for $k=0$. Let (4) be true for $k=i(i=0, n-1)$, i. e.,

$$A_i(u+v)=A_i(u)+A_i(v),\quad A_i(uv)<A_i(u)v,$$

then

$$\begin{aligned}A_{i+1}(u+v)&=A_i(u+v)+A_i(g_{i+1})\int_0^x h_{i+1}A_i(u+v)\exp\left(\int_s^x h_{i+1}A_i(g_{i+1})dt\right)ds\\&=A_i(u)+A_i(g_{i+1})\left(\int_0^x h_{i+1}A_i(u)\exp\left(\int_s^x h_{i+1}A_i(g_{i+1})dt\right)ds\right.\\&\quad+A_i(v)+A_i(g_{i+1})\int_0^x h_{i+1}A_i(v)\exp\left(\int_s^x h_{i+1}A_i(g_{i+1})dt\right)ds\\&=A_{i+1}(u)+A_{i+1}(v),\end{aligned}$$

$$\begin{aligned}A_{i+1}(uv)&=A_i(uv)+A_i(g_{i+1})\int_0^x h_{i+1}A_i(uv)\exp\left(\int_s^x h_{i+1}A_i(g_{i+1})dt\right)ds\\&\leqslant\left[A_i(u)+A_i(g_{i+1})\int_0^x h_{i+1}A_i(u)\exp\left(\int_s^x h_{i+1}A_i(g_{i+1})dt\right)ds\right]v\\&=A_{i+1}(u)v.\end{aligned}$$

This proves that (4) is true for $k=i+1$.

Proof of Theorem 1 The proof is by induction. For $n=1$, (2) becomes (1) hence is true by Lemma 1. Suppose (2) is true for $n=k$ $(1\leqslant k\leqslant n-1)$. Then fr

$$y\leqslant\left(f+g_{k+1}\int_0^x h_{k+1}y\,ds\right)+\sum_{i=1}^k g_i\int_0^x h_i y\,ds,$$

in view of the assumption, we have

$$y \leqslant A_k\left(f + g_{k+1}\int_0^x h_{k+1} y\, ds\right).$$

ecause

$$\int_0^x h_{k+1} y\, ds$$

nondecreasing in x, from (4)

$$y \leqslant A_k(f) + A_k(g_{k+1})\int_0^x h_{k+1} y\, ds.$$

cording to Lemma 1 we obtain

$$y \leqslant A_k(f) + A_k(g_{k+1})\int_0^x h_{k+1} A_k(f) \exp\left(\int_s^x h_{k+1} A_k(g_{k+1})\, dt\right) ds = A_{k+1}(f).$$

e proof is complete.

Remark 1. It is easy to show that the estimate here is better than that of eorem 1 in [1]. Now we prove that under the condition of Theorem 1 in [3], our ults is better too.

According to our result, $y(x) \leqslant A_n(f)$; and according to Theorem 1 in [3],

$$y(x) \leqslant E^n(f).$$

 will show that $A_n(f) \leqslant E^n(f)$.

Clearly, $A_0(f) = E^0(f)$. Suppose $A_k(f) \leqslant E^k(f)$ for $0 \leqslant k \leqslant n-1$. Because $g_i(x) \geqslant 1$ $=1, 2, \cdots, n)$, $f(x)$ and $A_k(u)$ are nondecreasing,

$$A_k(f) \leqslant f A_k(1) \leqslant f A_k(g_{k+1}) \leqslant f E^k(g_{k+1}).$$

$$\begin{aligned}
A_{k+1}(f) &= A_k(f) + A_k(g_{k+1})\int_0^x h_{k+1} A_k(f) \exp\left(\int_s^x h_{k+1} A_k(g_{k+1})\, dt\right) ds \\
&= A_k(f) - A_k(g_{k+1})\int_0^x \frac{A_k(f)}{A_k(g_{k+1})}\, ds \exp\left(\int_s^x h_{k+1} A_k(g_{k+1})\, dt\right) \\
&= A_k(f) - A_k(g_{k+1})\left[\frac{A_k(f)}{A_k(g_{k+1})} \exp\left(\int_s^x h_{k+1} A_k(g_{k+1})\, dt\right)\Big|_0^x\right. \\
&\quad \left. - \int_0^x \exp\left(\int_s^x h_{k+1} A_k(g_{k+1})\, dt\right) d\left(\frac{A_k(f)}{A_k(g_{k+1})}\right)\right] \\
&\leqslant \frac{A_k(f)}{A_k(g_{k+1})}\Big|_{x=0} A_k(g_{k+1}) \exp\left(\int_0^x h_{k+1} A_k(g_{k+1})\, ds\right) \\
&\quad + A_k(g_{k+1}) \exp\left(\int_0^x h_{k+1} A_k(g_{k+1})\, ds\right)\int_0^x d\left(\frac{A_k(f)}{A_k(g_{k+1})}\right) \\
&= A_k(f) \exp\left(\int_0^x h_{k+1} A_k(g_{k+1})\, ds\right) \\
&\leqslant f E^k(g_{k+1}) \exp\left(\int_0^x h_{k+1} E^k(g_{k+1})\, ds\right) = E^{k+1}(f).
\end{aligned}$$

Therefore, $A_n(f) \leqslant E^n(f)$.

To see the differences among the three results we give an example.

Example 1. Let

$$y(x)\leqslant(1+x)+2\int_0^x\frac{1}{1+s}y(s)ds+(1+x)\int_0^x\frac{3+4s}{(1+3s+2s^2)^2}y(s)ds.$$

We have

$$f(x)=g_2(x)=1+x,\ g_1(x)=2,\ h_1(x)=\frac{1}{1+x},\ h_2(x)=\frac{3+4x}{(1+3x+2x^2)^2}.$$

Then

$$A_1(f)=A_1(g_2)=(1+x)+2\int_0^x\exp\left(2\int_s^x\frac{dt}{1+t}\right)ds=1+3x+2x^2.$$

According to (2), we get

$$y(x)\leqslant(1+3x+2x^2)+(1+3x+2x^2)\int_0^x\frac{3+4s}{(1+3s+2s^2)}\exp\left(\int_s^x\frac{3+4t}{1+3t+2t^2}dt\right)ds$$

$$=(1+3x+2x^2)\left[1+(1+3x+2x^2)\left(1-\frac{1}{1+3x+2x^2}\right)\right]$$

$$=(1+3x+2x^2)^2.$$

According to Theorem 1 in [1],

$$D_1W_0=(1+x)+2\left[\exp\left(2\int_0^x\frac{ds}{1+s}\right)\right]\int_0^x ds=1+3x+4x^2+2x^3,$$

$$y(x)\leqslant E_2W_0=D_2(D_1W_0)$$

$$=(1+3x+4x^2+2x^3)\left[1+\exp\left(\int_0^x\frac{3+4s}{(1+3s+2s^2)^2}(1+3s+4s^2+2s^3)ds\right)\right.$$

$$\left.\cdot\int_0^x\frac{3+4s}{(1+3s+2s^2)^2}(1+3s+4s^2+2s^3)\,ds\right].$$

It is easy to see that

$$E_2W_0>(1+3x+4x^2+2x^3)\left[1+\int_0^x\frac{3+4s}{1+3s+2s^2}\exp\left(\int_s^x\frac{3+4t}{1+3t+2t^2}dt\right)ds\right]$$

$$=(1+3x+4x^2+2x^3)(1+3x+2x^2).$$

According to Theorem 1 in [2],

$$E^1(f)=E^1(g_2)=2(1+x)\exp\left(2\int_0^x\frac{ds}{1+s}\right)=2(1+x)^3,$$

$$y(x)\leqslant E^2(f)=2(1+x)^4\exp\left(2\int_0^x\frac{(3+4s)(1+s)^3}{(1+3s+2s^2)^2}ds\right).$$

It is easy to see that

$$E^2(f)\geqslant2(1+x)^4e^{2x}\geqslant(1+3x+2x^2)^2+2(1+x)^4(e^x-x-1).$$

The differences among the estimates are quite lagre.

§ 2. Generalizations of Nonlinear Integral Inequalities

1. The case $w(u)\in\mathscr{F}$

Here we quote a definition of a function class $\mathscr{F}$ given by [6].

Definition. *A function w: $R_+\to R_+$ is said to belong to a function class $\mathscr{F}$ if it* *atisfies the following conditions:*

a) *$w(u)\geqslant 0$ is nondecreasing and continuous on R_+,*

b) $\frac{1}{v}w(u)\leqslant w\left(\frac{u}{v}\right)$ *for* $u\geqslant 0,\ v\geqslant 1.$ (5)

To see a property of $\mathscr{F}$ and its width, we give the following lemma.

Lemma 3. 1) *If $w(u)\in\mathscr{F}$, then $w(u)$ is subadditive;*

2) *If $w(u)$ satisfies a) and is convex on R_+, then $w(u)\in\mathscr{F}$.*

Proof 1) For any $u,\ v\in R_+$, without loss of generality, we assume $v\leqslant u$. Then $=\lambda u,\ 0\leqslant\lambda\leqslant 1$. Because $w(u)\in\mathscr{F}$, $w(u)$ satisfies (5). Hence

$$\frac{1}{1+\lambda}w[(1+\lambda)u]\leqslant w(u),$$

e.,

$$w(u+\lambda u)\leqslant w(u)+\lambda w(u).$$

sing (5) again we obtain

$$w(u+v)\leqslant w(u)+w(v). \tag{6}$$

$w(u)$ is subadditive.

2) Let $w(u)$ is convex on R_+, i. e.,

$$w(\alpha u_1+\beta u_2)\geqslant\alpha w(u_1)+\beta w(u_2)\ \text{for}\ \alpha\geqslant 0,\ \beta\geqslant 0,\ \alpha+\beta=1. \tag{7}$$

t $u_1=u,\ u_2=0$ in (7). Considering $w(0)\geqslant 0$, we get

$$\alpha w(u)\leqslant w(\alpha u)\ \text{for}\ 0\leqslant\alpha\leqslant 1.$$

r fixed $v\geqslant 1$, let

$$\alpha=\frac{1}{v}.$$

nce

$$\frac{1}{v}w(u)\leqslant w\left(\frac{u}{v}\right).$$

erefore $w(u)\in\mathscr{F}$.

Lemma 3 explains that all nonnegative, nondecreasing and convex functions ong to $\mathscr{F}$. In fact, besides those functions, there are many functions in $\mathscr{F}$. For tance

$$w(u)=u\left(1+\sin\frac{\pi}{2(u+1)}\right)$$

not a convex function, but we can prove that it is in $\mathscr{F}$.

Lemma 4. *Suppose $f(x)\geqslant 0$, $g(x)>0$, $h(x)\geqslant 0$ and $y(x)$ are continuous on R_+, $w(u)\in\mathscr{F}$. If*

$$y(x)\leqslant f(x)+g(x)\int_0^\infty h(s)w(y(s))ds,\ x\in R_+, \tag{8}$$

n

$$y(x)\leqslant f(x)+g(x)G^{-1}\left[G\left(\int_0^\infty h(s)\bar{g}(s)w\left(\frac{f(s)}{g(s)}\right)ds\right)+\int_0^\infty h(s)\bar{g}(s)\,ds\right],\ x\in R_+,$$

where

$$G(u)=\int_{u_0}^{u}\frac{ds}{w(s)},\ u_0>0,\ u\geqslant 0,$$

$$\bar{g}(x)=\max\{g(x),\ 1\}.$$

Proof Assume $y(x)$ is a solution of (8), denote $z(x)=\max\{y(x),\ 0\}$, $x\in R_+$, then $z(x)$ is also a solution of (8), i. e.,

$$z(x)\leqslant f(x)+g(x)\int_0^x h(s)w(z(s))ds. \tag{9}$$

At first, consider the case for $g(x)\equiv 1$. Define

$$R(x)=\int_0^x h(s)w(z(s))ds,\ x\in R_+.$$

Then

$$R'(x)=h(x)w(z(x)).$$

From $w(u)\in\mathscr{F}$, we know that

$$R'(x)\leqslant h(x)[w(f(x))+w(R(x))].$$

Integrating both sides from 0 to x, we get

$$R(x)\leqslant\int_0^x h(s)w(f(s))ds+\int_0^x h(s)w(R(s))ds.$$

For any $X\geqslant 0$, when $x\leqslant X$,

$$R(x)\leqslant\int_0^X h(s)w(f(s))\ ds+\int_0^x h(s)w(R(s))ds.$$

Using Bihari's inequality, we have

$$R(x)\leqslant G^{-1}\left[G\left(\int_0^X h(s)w(f(s))\ ds\right)+\int_0^x h(s)ds\ \right].$$

Letting $x=X$ and replacing X by x, we see that

$$R(x)\leqslant G^{-1}\left[G\left(\int_0^x h(s)w(f(s))ds\right)+\int_0^x h(s)ds\ \right]. \quad (10)$$

Therefore

$$z(x)\leqslant f(x)+G^{-1}\left[G\left(\int_0^x h(s)w(f(s))ds\right)+\int_0^x h(s)ds\ \right].$$

When $g(x)\not\equiv 1$, from (9) we have

$$\frac{z(x)}{\bar{g}(x)}\leqslant\frac{f(x)}{\bar{g}(x)}+\int_0^x h(s)\bar{g}(s)w\left(\frac{z(s)}{\bar{g}(s)}\right)ds$$

$$\leqslant\frac{f(x)}{\bar{g}(x)}+\int_0^x h(s)\bar{g}(s)\ w\left(\frac{z(s)}{\bar{g}(s)}\right)ds.$$

Substituting

$$\frac{z(x)}{\bar{g}(x)},\ \ \frac{f(x)}{\bar{g}(x)},\ h(x)\bar{g}(x)$$

for $z(x)$, $f(x)$, $h(x)$ in (10) respectively, we have

$$\frac{z(x)}{\bar{g}(x)}\leqslant\frac{f(x)}{\bar{g}(x)}+G^{-1}\left[G\left(\int_0^x h(s)\bar{g}(s)\ w\left(\frac{f(s)}{\bar{g}(s)}\right)ds\right)+\int_0^x h(s)\bar{g}(s)ds\right]$$

i. e.,

$$z(x)\leqslant f(x)+g(x)G^{-1}\Big[G\Big(\int_0^x h(s)\bar{g}(s)w\Big(\frac{f(s)}{g(s)}\Big)ds\Big)+\int_0^x h(s)\bar{g}(s)ds\Big].$$

loting that $y(x)\leqslant z(x)$, the result inequality becomes true. $w(u)\in\mathscr{F}$ implies that

$$\int_{u_0}^{u}\frac{ds}{w(s)}=\infty.$$

n fact,

$$\frac{w(u)}{u}\leqslant w(1)$$

r $u\geqslant 1$, i. e., $w(u)\leqslant w(1)u$, which follows

$$\int_1^{\infty}\frac{ds}{w(s)}\geqslant\int_1^{\infty}\frac{ds}{w(1)s}=\infty.$$

$$G\Big(\int_0^x h(s)\bar{g}(s)w\Big(\frac{f(s)}{g(s)}\Big)ds\Big)+\int_0^x h(s)\bar{g}(s)ds\in\mathrm{Dom}(G^{-1})$$

r any $x\in R_+$. The proof is complete.

Example 2. Let

$$y(x)\leqslant 4x^2e^x+e^x\int_0^x e^{-s}y^{1/2}(s)ds.$$

hen

$$w(u)=\sqrt{u},\ G(u)=\int_0^u\frac{ds}{\sqrt{s}}=2\sqrt{u},\ G^{-1}(u)=(u/2)^2.$$

ccording to Lemma 4

$$y(x)\leqslant 4x^2e^x+e^xG^{-1}\Big[G\Big(\int_0^x 2s\,ds\Big)+\int_0^x ds\Big]$$

$$=4x^2e^x+e^xG^{-1}[3x]=\frac{25}{4}x^2e^x.$$

it according to Lemma 1 in [7], let

$$w(u)=\sqrt{u},\ \mu(v)=\frac{1}{\sqrt{v}},$$

$$y(x)\leqslant 4x^2e^xw^{-1}\Big\{w(1)+\frac{e^x}{2xe^{x/2}}\int_0^x e^{-s}ds\Big\}$$

$$=4x^2e^x\Big[1+\frac{1}{4x}(e^{x/2}-e^{-x/2})\Big]^2.$$

nce $e^{x/2}-e^{-x/2}>x\,(x>0)$, the former is much more accurate than the latter.

Theorem 2. *Suppose $f(x)$, $g(x)$, $\bar{g}(x)$, $h(x)$, $y(x)$ and $w(u)$ are defined as in* *mma 4, $\psi(u)\geqslant 0$ is nondecreasing, continuous on R_+. If*

$$y(x)\leqslant f(x)+g(x)\psi\Big(\int_0^x h(s)w(y(s))ds\Big),$$

n

$$y(x)\leqslant f(x)+g(x)\psi\Big\{F^{-1}\Big[F\Big(\int_0^x h(s)\bar{g}(s)w\Big(\frac{f(s)}{g(s)}\Big)ds\Big)+\int_0^x h(s)\bar{g}(s)ds\Big]\Big\},\ x\in[0,\ b),$$

where

$$F(u)=\int_{u_0}^{u}\frac{ds}{w(\psi(s))},\ u_0>0,\ u\geqslant 0,$$

$$b=\sup_{x\in R_+}\left\{x:\ F\left(\int_0^x h(s)\bar{g}(s)w\left(\frac{f(s)}{g(s)}\right)ds\right)+\int_0^x h(s)\bar{g}(s)ds\in \mathrm{Dom}(F^{-1})\right\}.$$

The proof is similar to that of Lemma 4, we omit it here.

Theorem 3. *Suppose* $f(x)$, $g_i(x)$ *and* $h_i(x)$ $(i=1, 2, \cdots, n)$ *are defined as in Theorem* 1, $w(u)$, $\psi(u)$ *as in Theorem* 2, $g_{n+1}(x)>0$, $h_{n+1}(x)\geqslant 0$ *are continuous on* R. *If*

$$y(x)\leqslant f(x)+\sum_{i=1}^{n}g_i(x)\int_0^x h_i(s)y(s)ds$$
$$+g_{n+1}(x)\psi\left(\int_0^x h_{n+1}(s)w(y(s))ds\right),$$

then

$$y\leqslant A_n(f)+A_n(g_{n+1})\psi\left\{F^{-1}\left[F\left(\int_0^x h_{n+1}\bar{A}_n(g_{n+1})w\left(\frac{A_n(f)}{A_n(g_{n+1})}\right)ds\right)\right.\right.$$
$$\left.\left.+\int_0^x h_{n+1}\bar{A}_n(g_{n+1})ds\right]\right\},\ x\in[0,b)$$

where

$$F(s)=\int_{u_0}^{u}\frac{ds}{w(\psi(s))},\ u_0>0,\ u\geqslant 0,$$

$A_{n+1}(u)$ *is defined as in Theorem* 1,

$\bar{A}_n(g_{n+1})=\max\{A_n(g_{n+1}), 1\}$, $x\in R_+$,

the determination of b *must make* F^{-1} *have definition.*

Proof To simplify the notation, let

$$\psi(\cdot)=\psi\left(\int_0^x h_{n+1}w(y)ds\right).$$

Therefore

$$y\leqslant(f+g_{n+1}\psi(\cdot))+\sum_{i=1}^{n}g_i\int_0^x h_i y\,ds.$$

According to Theorem 1

$$y\leqslant A_n(f+g_{n+1}\psi(\cdot)).$$

Noting that

$$\psi\left(\int_0^x h_{n+1}w(y)ds\right)$$

is nondecreasing in x, from (4) we get

$$y\leqslant A_n(f)+A_n(g_{n+1})\psi\left(\int_0^x h_{n+1}w(y)\,ds\right).$$

According to Theorem 2, the conclusion holds.

2. The case that $w(u)$ is a concave function

Lemma 5. *Suppose* $w(u)$ *is nonnegative and concave on* R_+, $w(0)=0$. *Then*

$$\frac{1}{v}w(u)\leqslant w\left(\frac{u}{v}\right)\ for\ u\geqslant 0,\ 0<v\leqslant 1.$$

Proof By definition, $w(u)$ is concave, which implies that

$$w(\alpha u_1+\beta u_2)\leqslant \alpha w(u_1)+\beta w(u_2) \text{ for } \alpha\geqslant 0,\ \beta\geqslant 0,\ \alpha+\beta=1.$$

Letting

$$\alpha=v,\ u_1=\frac{u}{v},\ u_2=0,$$

and considering $w(0)=0$, we have

$$w(u)\leqslant vw\left(\frac{u}{v}\right),$$

i. e.,

$$\frac{1}{v}w(u)\leqslant w\left(\frac{u}{v}\right).$$

Lemma 6. *Suppose $f(x)\geqslant 0$, $h(x)\geqslant 0$, $0<g(x)\leqslant 1$, and $y(x)$ are continuous on R_+, $w(0)=0$, w is nonnegative and concave on R_+. If*

$$y(x)\leqslant f(x)+g(x)\int_0^x h(s)w(y(s))ds,\ x\in R_+, \tag{12}$$

then

$$y(x)\leqslant f(x)+g(x)H^{-1}\Big[H\Big(\alpha\int_0^x h(s)g(s)w\Big(\frac{f(s)}{\alpha g(s)}\Big)ds\Big)$$
$$+\beta\int_0^x h(s)g(s)\,ds\Big],\ x\in[0,\ b)$$

for $\alpha>0$, $\beta>0$, $\alpha+\beta=1$, where

$$H(u)=\int_{u_0}^u\frac{ds}{w(s/\beta)},\ u_0>0,\ u\geqslant 0,$$

and the determination of b must make H^{-1} have definition.

Proof Assume that $y(x)$ is a solution of (12). Denote $z(x)=\max\{y(x),\ 0\}$. Then $z(x)$ is also a solution of (12), i. e.,

$$z(x)\leqslant f(x)+g(x)\int_0^x h(s)w(z(s))ds. \tag{13}$$

At first, consider the case for $g(x)\equiv 1$. Define

$$R(x)=\int_0^x h(s)w(z(s))ds.$$

Then

$$R'(x)=h(x)w(z(x))\leqslant h(x)w[f(x)+R(x)].$$

Since $w(u)$ is concave, we have

$$R'(x)=h(x)w\left[\alpha\left(\frac{f(x)}{\alpha}\right)+\beta\left(\frac{R(x)}{\beta}\right)\right]$$
$$\leqslant\alpha h(x)w\left(\frac{f(x)}{\alpha}\right)+\beta h(x)w\left(\frac{R(x)}{\beta}\right)$$

for $\alpha>0$, $\beta>0$, $\alpha+\beta=1$. Integrating both sides from 0 to x, we have

$$R(x)\leqslant\alpha\int_0^x h(s)w\left(\frac{f(s)}{\alpha}\right)ds+\beta\int_0^x h(s)w\left(\frac{R(s)}{\beta}\right)ds.$$

For any $X>0$, when $x\leqslant X$,

$$R(x)\leqslant\alpha\int_0^X h(s)w\Big(\frac{f(s)}{\alpha}\Big)ds+\beta\int_0^x h(s)w\Big(\frac{R(s)}{\beta}\Big)ds.$$

Using Bihari's inequality, we have

$$R(x)\leqslant H^{-1}\Big[H\Big(\alpha\int_0^X h(s)w\Big(\frac{f(s)}{\alpha}\Big)ds\Big)+\beta\int_0^x h(s)ds\Big].$$

Letting $x=X$ and replacing X by x, we see that

$$R(x)\leqslant H^{-1}\Big[H\Big(\alpha\int_0^x h(s)w\Big(\frac{f(s)}{\alpha}\Big)ds\Big)+\beta\int_0^x h(s)ds\Big].$$

Hence

$$z(x)\leqslant f(x)+H^{-1}\Big[H\Big(\alpha\int_0^x h(s)w\Big(\frac{f(s)}{\alpha}\Big)ds\Big)+\beta\int_0^x h(s)ds\Big]. \tag{14}$$

When $g(x)\not\equiv 1$, since $0<g(x)\leqslant 1$, from (11) and (13) we have

$$\frac{z(x)}{g(x)}\geqslant\frac{f(x)}{g(x)}+\int_0^x h(s)g(s)w\Big(\frac{z(s)}{g(s)}\Big)ds.$$

Substituting

$$\frac{z(x)}{g(x)},\quad \frac{f(x)}{g(x)},\quad h(x)g(x)$$

for $z(x)$, $f(x)$, $h(x)$ in (14) respectively, we get

$$\frac{z(x)}{g(x)}\leqslant\frac{f(x)}{g(x)}+H^{-1}\Big[H\Big(\alpha\int_0^x h(s)g(s)w\Big(\frac{f(s)}{\alpha g(s)}\Big)ds+\beta\int_0^x h(s)g(s)\,ds\Big].$$

Noting that $y(x)\leqslant z(x)$, we obtain the result.

Theorem 4. *Suppose $f(x)$, $g(x)$, $h(x)$, $y(x)$ and $w(u)$ are defined as in Lemma 6, $\psi(u)\geqslant 0$ is nondecreasing and continuous on R_+. If*

$$y(x)\leqslant f(x)+g(x)\psi\Big(\int_0^x h(s)w(y(s))ds\Big),$$

then

$$y(x)\leqslant f(x)+g(x)\psi\Big\{I^{-1}\Big[I\Big(\alpha\int_0^x h(s)g(s)w\Big(\frac{f(s)}{\alpha g(s)}\Big)ds\Big)+\beta\int_0^x h(s)g(s)\,ds\Big]\Big\},\quad x\in[0,b)$$

for $\alpha>0$, $\beta>0$, $\alpha+\beta=1$, where

$$I(u)=\int_{u_0}^u\frac{ds}{w(\psi(s)/\beta)},\quad u_0>0,\ u\geqslant 0,$$

and the determination of b must make I^{-1} have definition.

The proof is similar to that of Lemma 6.

Theorem 5. *Suppose $f(x)$, $g_i(x)$, $h_i(x)$ $(i=1, 2, \cdots, n)$ and $A_{n+1}(u)$ are defined as in Theorem 1, $w(u)$, $\psi(u)$ as in Theorem 4, $h_{n+1}(x)\geqslant 0$, $g_{n+1}(x)>0$ are continuous on R_+, and $A_n(g_{n+1}(x))\leqslant 1$. If*

$$y(x)\leqslant f(x)+\sum_{i=1}^n g_i(x)\int_0^x h_i(s)y(s)ds+g_{n+1}(x)\psi\Big(\int_0^x h_{n+1}(s)w(y(s))ds\Big),$$

then

$$y \leqslant A_n(f) + A_n(g_{n+1})\psi\left\{I^{-1}\left[I\left(\alpha\int_0^x h_{n+1}A_n(g_{n+1})w\left(\frac{A_n(f)}{\alpha A_n(g_{n+1})}\right)ds\right) + \beta\int_0^x h_{n+1}A_n(g_{n+1})ds\right]\right\},\ x\in[0, b)$$

for $\alpha>0$, $\beta>0$, $\alpha+\beta=1$, *where* $I(u)$ *and* b *are the same as in Theorem* 4.

The proof is simliar to that of Theorem 3.

Remark 2. The results of Lemma 6, Theorems 4 and 5 can be easily generalized to the case that $g(x)$ or $A_n(g_{n+1}(x))$ is nonnegative and bounded on R_+, we omit them here.

§ 3. Applications of the Above Inequalities

In the following, by way of example of the case $w\in\mathscr{F}$, we explain the applications of the abovo inequalities to Volterra integral equations and differential equations. For the case that $w(u)$ is a concave function we have similar dicussion.

1. An estimate of solutions of Volterra integral equation

Theorem 6. *Consider equation*

$$y(x) = f(x) + \int_0^x k(x, s)y(s)ds,\ x\in R_+. \tag{15}$$

Suppose 1) $k(x, s)$ $(x\geqslant s)$ *are nonnegative and continuous on* $R_+\times R_+$, *and*

$$\frac{\partial k(x, s)}{\partial x} \leqslant \sum_{i=1}^n q_i(x)h_i(s),$$

$$k(x, x)\leqslant m(x);$$

2) $f(x)$, $q_i(x)$, $h_i(x)$ $(i=1, 2, \cdots, n)$ *are defined as in Theorem* 1;

3) $m(x)\geqslant 0$ *is continuous on* R_+.

Then

$$|y(x)| \leqslant A_n(p),\ x\in R_+,$$

where

$$p(x) = f(x) + \int_0^x f(s)m(s)\exp\left(\int_s^x m(t)dt\right)ds,$$

$$g_i(x) = \int_0^x q_i(x)\exp\left(\int_s^x m(t)dt\right)ds,\ i=1, 2, \cdots, n,$$

$A_n(u)$ *is defined as in Theorem* 1.

Proof From (15) we have

$$|y(x)| \leqslant f(x) + \int_0^x k(x, s)|y(s)|ds.$$

By a deduction similar to [2, 619], we obtain

$$|y(x)| \leqslant p(x) + \sum_{i=1}^n g_i(x)\int_0^x h_i(s)|y(s)|ds.$$

The conclusion is true from Theorem 1.

Theorem 7. *Consider equation*

$$y(x)=f(x)+\int_0^x k(x,\ s)y(s)ds+\psi\Big(\int_0^x k^*(x,\ s)w(y(s))ds\Big),\ x\in R_+.$$

Suppose 1) $f(x)\geqslant 0$ *is continuous on* R_+, $w(u)\in\mathscr{F}$, $\psi(u)\geqslant 0$ *is nondecreasing, submultiplicative and continuous on* R_+;

2) $k(x,\ s)$ *is defined as in Theorem* 6;

3) $k^*(x,\ s)$ $(x\geqslant s)$ *is nonnegative and continuous on* $R_+\times R_+$, *and*

$$k^*(x,\ x)=0,\ \frac{\partial k^*(x,\ s)}{\partial x}\leqslant q_{n+1}(x)h_{n+1}(s),$$

where $q_{n+1}(x)$ *and* $h_{n+1}(x)$ *are continuous on* R_+.

Then

$$|y(x)|\leqslant A_n(p)+A_n(g_{n+1})\psi\Big\{F^{-1}\Big[F\Big(\int_0^x h_{n+1}\overline{A}_n(g_{n+1})w\Big(\frac{A_n(p)}{A_n(g_{n+1})}\Big)ds$$
$$+\int_0^x h_{n+1}\overline{A}(g_{n+1})ds\Big]\Big\},\ x\in[0,\ b),$$

where $p(x)$, $g_i(x)$ $(i=1,\ 2,\ \cdots,\ n)$, $A_n(u)$ *and* b *are the same as in Theorem* 6, *and*

$$g_{n+1}(x)=r(x)+\int_0^x r(s)m(s)\exp\Big(\int_s^x m(t)dt\Big)ds,$$

$$r(x)=\psi\Big(\int_0^x q_{n+1}(s)ds\Big),$$

$$\overline{A}_{n+1}(g_{n+1})=\max\{A_{n+}(g_{n+1}),\ 1\},\ x\in R_+.$$

Proof Denote

$$R(x)=\int_0^x k^*(x,\ s)w(y(s))ds.$$

According to [2, p. 626],

$$R(x)\leqslant\int_0^x q_{n+1}(s)ds\int_0^x h_{n+1}(s)w(|y(s)|)ds,$$

$$\psi(R(s))\leqslant r(x)\psi\Big(\int_0^x h_{n+1}(s)w(|y(s)|)ds\Big).$$

Let $T(x)=f(x)+\psi(R(x))$, then

$$T(x)\leqslant f(x)+r(x)\psi\Big(\int_0^x h_{n+1}(s)w(|y(s)|)ds\Big),$$

and

$$|y(x)|\leqslant T(x)+\int_0^x k(x,\ s)|y(s)|ds.$$

According to Theorem 6,

$$|y(x)|\leqslant A_n(\bar{p}),$$

where

$$\bar{p}(x)=T(x)+\int_0^x T(s)m(s)\exp\Big(\int_s^x m(t)dt\Big)ds$$
$$\leqslant\Big[f+\int_0^x fm\exp\Big(\int_s^x m\,dt\Big)ds\Big]$$
$$+\Big[r+\int_0^x rm\exp\Big(\int_s^x m\,dt\Big)ds\Big]\psi\Big(\int_0^x h_{n+1}w(|y|)ds\Big)$$

$$=p(x)+g_{n+1}(x)\psi\left(\int_0^x h_{n+1}(s)w(|y(s)|\,ds\right).$$

Hence from (4)

$$|y(x)|\leqslant A_n(p)+A_n(g_{n+1})\psi\left(\int_0^x h_{n+1}w(|y|)ds\right).$$

Using Theorem 2, we come to the conclusion.

2. An estimate of solutions of a kind of differential equations

Consider equation

$$y'(x)+\alpha(x)y(x)=F(x, y(x)), \tag{16}$$

where y and F are n-dimentional vectors, $\alpha(x)$ is nonnegative and continuous on R_+, F is continuous on $R_+\times R_+$.

Theorem 8. *Suppose F in (16) satisfies*

$$\|F(x, y(x))\|\leqslant\beta(x)\|y(x)\|+\gamma(x)w(\|y(x)\|), \tag{17}$$

where $\|\cdot\|$ denotes a vector norm, $\beta(x)$ and $\gamma(x)$ are nonnegative and continu-ous on R_+. Then all solutions satisfy

$$\|y(x)\|\leqslant\exp\left\{\int_0^x(\beta(s)-\alpha(s))ds\right\}\left\{k+G^{-1}\left[G\left(w(k)\int_0^x\gamma(s)\exp\left(\int_0^s\tau(t)dt\right)ds\right)\right.\right.$$
$$\left.\left.+\int_0^x\gamma(s)\exp\left(\int_0^s\tau(t)ds\right)ds\right]\right\},\quad x\in[0, b), \tag{18}$$

where $k=\|y(0)\|,\ \tau(x)=\max\{\alpha(x), \beta(x)\}.$

Proof From (16) we have

$$\left[\exp\left(\int_0^x\alpha(s)ds\right)y(x)\right]'=\exp\left(\int_0^x\alpha(s)ds\right)F(x, y(x)).$$

$$y(x)=\exp\left(-\int_0^x\alpha(s)ds\right)\left[y(0)+\int_0^x\exp\left(\int_0^s\alpha(t)dt\right)F(s, y(s))ds\right],$$

$$\|y(x)\|\leqslant\exp\left(-\int_0^x\alpha(s)ds\right)\left[k+\int_0^x\exp\left(\int_0^s\alpha(t)dt\right)\right.$$
$$\left.\cdot(\beta(s)\|y(s)\|+\gamma(s)w(\|y(s)\|)ds\right].$$

Let

$$f(x)=k\exp\left(-\int_0^x\alpha(s)ds\right),$$

$$g_1(x)=g_2(x)=\exp\left(-\int_0^x\alpha(s)ds\right),$$

$$h_1(x)=\beta(x)\exp\left(\int_0^x\alpha(s)ds\right),$$

$$h_2(x)=\gamma(x)\exp\left(\int_0^x\alpha(s)ds\right).$$

Then

$$A_1(f)=k\exp\left(-\int_0^x\alpha(s)ds\right)$$
$$+\exp\left(-\int_0^x\alpha(s)ds\right)\int_0^x k\beta(s)\exp\left(\int_s^x\beta(t)dt\right)ds$$
$$=k\exp\left(\int_0^x(\beta(s)-\alpha(s)\right)ds,$$

$$A_1(g_2)=\exp\Big(\int_0^x(\beta(s)-\alpha(s))ds,$$

$$h_2\,\overline{A}_1(g_2)=\gamma(x)\exp\Big(\int_0^x\tau(s)ds\Big),$$

where $\tau(x)=\max\{\alpha(x),\ \beta(x)\}$. According to Theorem 3,

$$\|y(x)\|=k\exp\Big(\int_0^x(\beta(s)-\alpha(s)\Big)ds+\exp\Big(\int_0^x(\beta(s)-\alpha(s))ds$$

$$G^{-1}\Big[G\Big(\int_0^x\gamma(s)\exp\Big(\int_0^s\tau(t)ds\Big)w(k)ds\Big)+\int_0^x\gamma(s)\exp\Big(\int_0^s\tau(t)dt\Big)ds\Big].$$

From Theorem 8, we can earily obtain

Corollary 1. *In addition to the conditions in Theorem* 8, *let*

1) $\int^\infty\tau(x)dx<+\infty$,

2) $\int^\infty\gamma(x)dx<+\infty$.

Then every solution of (15) *is bounded.*

Corollary 2. *In addition to the conditions in Theorem* 8 *and Corollary* 1, *le* $w(0)=0$ *and*

$$\int_0^\delta\frac{ds}{w(s)}=+\infty\ (\delta>0). \qquad (19$$

Then the zero solution of (16) *is stable.*

In fact, from (16), (17), we see that equation (16) has zero solution. Conditio (10) implies $G(u)\to-\infty$, as $u\to0$, i. e., $G^{-1}(u)\to0$, as $u\to-\infty$. Let k be sufficientl small in (18). Considering (19), we can conclude $\|y(x)\|$ must be sufficiently small

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