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ERGODIC THEOREMS FOR LINEAR GROWTH PROCESSES WITH DIFFUSION***

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Abstract

In this paper a simple class of the infinite dimensional reaction-diffusion processes—— the linear growth processes with diffusion is studied. This paper is devoted to the ergodicity of these processes. Tne exact value of parameters at which the change of phase occurs is given, the set of all translation invariant invariant measures and the corresponding domain of attraction for each translation invariant invariant measure are described.

§1. Introduction

The reaction-diffusion processes were proposed firstly bey G. Nicolis and Prigogine[10], Haken[5] and others, andwere studied by them. Since 1979, Yan Shijian et al, have systematically studied these processes. Up to now, they have solved a lot of problems about the existence uniqueness[3,13,14,16].

The linear growth process with diffusion studied in this paper is a class of the reaction-diffusion processes. In this paper we study the ergodicity of the linear growth processes with diffusion which is useful for studying the ergodicity of other reaction-diffusion processes.

Let S be a finite or a countable set, one may think of each $u\in S$ as a container which can contain arbitrarily finite particles. Suppose each particle in the container $u\in S$ can independently split from one to two at rate λ_1, and can die independently at rate λ_2, here the evolution of the particles in each container is like the linear growth model proposed by Feller[4]. Furthermore, suppose that in each container there is a source of particles which produces particles at rate λ_3, and finally suppose that for each $u\in S$ there is an exponential clock with parameter one and when the clock in the container u rings, a particle from u goes independently to v with probability $p(u, v)$, here $P=(p(u, v))$ is a transition probability matrix on S. This is the model studied in this paper.

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For the sake of simplicity, we always assume that P is irreducible and satisfies the Liggett condition:

$$\sup_x \sum_y p(y, x) = K < \infty. \tag{1.1}$$

As mentioned in [9], in this case there exists a positive function $\alpha(\cdot)$ on S with

$$\sum_x \alpha(x) < \infty$$

and constant $M > 0$ such that

$$\sum_y p(x, y)\alpha(y) \leqslant M\alpha(x) \tag{1.2}$$

for all $x \in S$, where M is defined to be the smallest of such constant. In order to avoid infinite many particles coming to a fixed container in finite time, it is necessary to impose some restrictions on the initial configuration of the process. We take

$$\mathscr{E}_\alpha = \{\eta \in E: \|\eta\| = \sum_x \eta(x)\alpha(x) < \infty\} \tag{1.3}$$

as the configuration space in stead of $E = Z_+^S$, $Z_+ = \{0, 1, 2, \cdots\}$. Let $\mathscr{B}(\mathscr{E}_\alpha)$ be the smallest σ-algebra on $\mathscr{E}_\alpha$ relative to which all mappings $\eta \to \eta(x)$, $x \in S$ are measurable. Let $\mathscr{L}(\mathscr{E}_\alpha)$ be the class of Lipschitz functions on $\mathscr{E}_\alpha$. Those are the ones for which there is a constant C such that

$$|f(\eta) - f(\zeta)| \leqslant C\|\eta - \zeta\|_\alpha$$

for all $\eta, \zeta \in \mathscr{E}_\alpha$, where

$$\|\eta - \zeta\| = \sum_x |\eta(x) - \zeta(x)|\alpha(x),$$

L(f) is defined to be the smallest of such constant.

The formal expression for the generator of the semigroup of our process is as follows: for $f \in \mathscr{L}(\mathscr{E}_\alpha)$,

$$\begin{aligned} \Omega f(\eta) = & \sum_x (\lambda_1\eta(x) + \lambda_3)[f(\eta + e_x) - f(\eta)] \\ & + \sum_x \lambda_2\eta(x)[f(\eta - e_x) - f(\eta)] \\ & + \sum_x \eta(x) \sum_y p(x, y)[f(\eta - e_x + e_y) - f(\eta)], \end{aligned} \tag{1.4}$$

where e_x is the configuration which is zero everywhere but one at x.

The existence uniqueness theorem of Markov processes with the generator (1.4) was given by the authors[16] (also see [3]). Those Markov processes denoted by η_t, $t \geqslant 0$ are called the linear growth processes with diffusion.

For the sake of simplicity, we focus our attention on the case being not any source, i. e. $\lambda_3 = 0$. In this case θ ($\theta(x) = 0$ for all $x \in S$) is a absorbing state of the processes, and using the coupling technique, we can prove the statement that if the process with parameters λ_1 and λ_2 is ergodic, then the process with parameters λ_1' and λ_2' satisfying $\lambda_1' \leqslant \lambda_1$ and $\lambda_2' \geqslant \lambda_2$ is ergodic as well. Therefore there may exist change of phase for our process with parameters λ_1 and λ_2. The following theorem illustrates this situation.

Theorem 1.5 (trivial case). *Let S be a finite set and $\lambda_3=0$. $p(t, \eta, \zeta)$ denotes the transition function for the process η_t, $t\geqslant 0$ with parameters λ_1 and λ_2. If $0<\lambda_1\leqslant\lambda_2$, then η_t, $t\geqslant 0$ is ergodic, i. e.*

$$\lim_{t\to\infty} p(t, \eta, \zeta)=0$$

for all $\eta\in E$ and $\zeta\neq 0$, and $p(t, 0, 0)=1$; If $\lambda_2<\lambda_1$, then η_t, $t\geqslant 0$ is nonergodic.

This theorem suggests the results we want to get for infinite S. Unfortunately, the similar results do not hold generally. In fact, if the initial configuration of the process is particularly unbounded, then the influence of it to the ergodicity of the process is negligible. The critical value to the ergodicity of the process is quite different from that for the case of finite S. An example is given in section 4 to explain it. In order to avoid this case, we furthermore restrict the process to the space which is called the minimal. configurotion space as follows. For arbitrarily fixed $x_0\in S$, define

$$\alpha_M(x)=\sum_{n=0}^{\infty} 1/M^n p^{(n)}(x, x_0),\ M>K.$$

Using (1.1), we can prove that

$$\sum_y p(x, y)\alpha_M(y)\leqslant M\alpha_M(x) \tag{1.6}$$

for all $x\in S$, and

$$\sum_x \alpha_M(x)<\infty,\ M>K. \tag{1.7}$$

Define

$$\mathscr{E}_M=\{\eta\in E: \sum_x \eta(x)\alpha_M(x)<\infty\}. \tag{1.8}$$

Clearly $\alpha_M(x)$ is decreasing for all $x\in S$, when M increases. Hence $\mathscr{E}_M$, $M>K$, is nondecreasing set sequence when M incrérses. Let

$$\mathscr{E}=\bigcap_{M>K}\mathscr{E}_M. \tag{1.9}$$

Clearly $\mathscr{E}$ includes all bounded configurations by (1.7). Let

$$\|\eta-\zeta\|_M=\sum_x|\eta(x)-\zeta(x)|\alpha_M(x),\ \text{for } \eta, \zeta\in\mathscr{E}_M, \tag{1.10}$$

and

$$\rho(\eta, \zeta)=\sum_{n=1}^{\infty} 1/2^n\|\eta-\zeta\|_{K+1/n}/(1+\|\eta-\zeta\|_{K+1/n}), \tag{1.11}$$

for $\eta, \zeta\in\mathscr{E}$. It is easy to check that $\rho(\cdot, \cdot)$ is a metric and $(\mathscr{E}, \rho)$ is a Polish space. Let $P(\mathscr{E})$ be the set of all probability measures on $\mathscr{E}$. The sot of all bounded cylinder functions on $\mathscr{E}$ is denoted by $\mathscr{F}(\mathscr{E})$.

Definition 1.12. *The process $\{\eta_t\}$ is said to be ergodic, if there exists a $\mu\in P(\mathscr{E})$ such that for all $\nu\in P(\mathscr{E})$, $f\in\mathscr{F}(\mathscr{E})$,*

$$\lim_{t\to\infty} \nu P(t)f=\mu f.$$

Write $\lim \nu P(t)=\mu$. Here $P(t)$ is the semigroup determined by generator Ω defined by (1.4).

One of the main theorems of this paper is

Theorem 1.13. (i) *If* $\lambda_1-\lambda_2+K-1<0$, *then* $\{\eta_t\}$ *is ergodic*; (ii) *If* $\lambda_1-\lambda_2>0$, *and P satisfies*

$$\inf_{t>0}\sum_u p(t, u, x)=B>0, \text{ for some } x, \tag{1.14}$$

where

$$p(t, u, x)=\sum_{n=0}^{\infty}\frac{t^n}{n!}e^{-t}p^{(n)}(u, x), \tag{1.15}$$

then $\{\eta_t\}$ *is nonergodic.*

Remark. If P is doubly stochastic, then $K=1=B$. It follows that the proc $\{\eta_t\}$is ergodic if $\lambda_1-\lambda_2<0$; and the process $\{\eta_t\}$ is nonergodic if $\lambda_1-\lambda_2>0$.

Besides, when the K is less than 1, Theorem 1.13 (i) means it is possible tł the process is ergodic even if $\lambda_1>\lambda_2$. By comparison with the ergodicity for the proc when S is finite, this means that diffusion influences the ergodicity for the proc heavily.

Suppose $S=Z^d$, ($d\geqslant 1$) and P to be random walk on Z^d, i. e. $p(x, y)=p(0, y-$ for any $x, y\in Z^d$. In this case (have supposed $\lambda_3=0$) Ω is translation invariant, so the semigroup $P(t)$. Therefore we return to study the translation invariant ca: Let $\mathscr{S}$ be the set of all probability measures on $\mathscr{E}$ which are translation invariaı and let $\mathscr{I}$ be the set of invariant measures for the process $\{\eta_t\}$ and $P_i(\mathscr{E})=\{\mu\in P(\mathscr{E}$ $\int\|\eta\|^i_M\mu(d\eta)<\infty, \forall M>K\}$, $i=1, 2$. Since randon walk must be doubly stochast: according to Theorem 1.13 the process $\{\eta_t\}$ eventually diet if $\lambda_1<\lambda_2$. Hence on invariant measure is δ_θ; If $\lambda_1>\lambda_2$, then the process $\{\eta_t\}$ is nonergodic, but there not any nontrivial ($\neq\delta_\theta$) translation invariant invariant measure in $P_1(\mathscr{E})$. Th the interesting case to study is $\lambda^1=\lambda_2$. For the translation invariant case, we have

Theorem 1.16. *Assume that* $S=Z^d$, $p(x, y)=p(0, y-x)$ *for any* $x, y\in S$, *a* $\lambda_1=\lambda_2$, $\lambda_3=0$; *and the symmetrized chain* $\bar{P}=(\bar{p}(x, y))$ *of P it transient, where* $\bar{p}($ $y)=1/2[p(x, y)+p(y, x)]$. *Then*

(i) *For each* $\rho>0$, *there exists a unique* $\nu_\rho\in\mathscr{S}\subset\mathscr{I}$. *which satisfies*

$$\int\eta(x)\nu_\rho(d\eta)=\rho, \tag{1.1}$$

$$\int\eta(x)\eta(y)\nu_\rho(d\eta)=\rho(\rho+\delta_{xy})+2\lambda_1\int_0^\infty\bar{p}(s, x, y)ds, \tag{1.1}$$

where $\bar{p}(s, x, y)$ *is the Q-process with Q-matrix* $2[\bar{p}-I]$;

(ii) *Suppose* $\mu\in\mathscr{I}\cap\mathscr{S}\cap P_1(\mathscr{E})$, *then there is a probability measure* λ *on* $[0, \infty$ *such that*

$$\mu=\int_0^\infty\nu_\rho\lambda(d\rho);$$

(iii) *If* $\mu\in\mathscr{S}_e$, *the set of all translation invariant ergodic probability measurcs,* *and*

$$\int \eta(x)\mu(d\eta)=\rho<\infty,$$

then $\lim \mu P(t)=\nu_\rho$; *Moreover, if* $\mu\in\mathscr{S}\cap P_2(\mathscr{E})$, *then for all* $x, y\in S$,

$$\lim_{t\to\infty}\int \eta(x)\eta(y)\mu P(t)(d\eta)=\int \eta(x)\eta(y)\nu_\rho(d\eta);$$

If $\mu\in\mathscr{S}$, *but*

$$\int \eta(x)\mu(d\eta)=\infty,$$

then

$$\lim_{k\to\infty}\lim_{t\to\infty}\mu P(t)\eta\colon \eta(x)\geq k\}>0.$$

For the case having particle source, we have

Theorem 1.19. (i) *If* $\lambda_1-\lambda_2+K-1<0$, *and* $\lambda_3>0$, *then* $\{\eta_t\}$ *is ergodic; Here the unique invariant measure is* $\mu_0=\lim \delta_\theta P(t)$;

(ii) *If* $\lambda_1-\lambda_2\geq 0$ *and* $\lambda_3>0$, *and* P *satisfies: for some* x,

$$\inf_{t>0}\sum_u p(t, u, x)=B>0, \tag{1.20}$$

and

$$\sup_{t>0}\sum_u p(t, u, x)=A<\infty, \tag{1.21}$$

then $\{\eta_t\}$ *is nonergodic.*

In order to give some impression to readers, we give an example in section 4, it means that the ergodicity of the same transition mechanics on different configuration space is different.

§ 2. First and Second Moment

The first and second moment of the process at time t play an important role in proving the main theorems. In order to compute the first and second moment, we want to use the method of constructing the semigroup of the process, i. e. we firstly compute the first and second moment for finite S, then by taking limit we get the expressions for countable S[16]. In virtue of the construction of the semigroup, take a finite subset sequence Λ_n, $n\geq 1$ of S such that $\Lambda_n\uparrow S$, let

$$p_n(x, y)=\begin{cases} p(x, y), & x, y\in\Lambda_n,\ x\neq y,\\ p(x, y)+\sum\limits_{z\notin\Lambda_n} p(x, z), & x=y\in\Lambda_n,\\ 1, & x=y\notin\Lambda_n,\\ 0, & \text{otherwise,}\end{cases}$$

and let

$$\begin{aligned}\Omega_n f(\eta)=&\sum_{x\in\Lambda_n}(\lambda_1\eta(x)+\lambda_3)[f(\eta+e_x)-f(\eta)]\\ &+\sum_{x\in\Lambda_n}\lambda_2\eta(x)[f(\eta-e_x)-f(\eta)]\\ &+\sum_{x\in\Lambda_n}\eta(x)\sum_y p_n(x, y)[f(\eta-e_x+e_y)-f(\eta)].\end{aligned}$$

Let $P_n(t)$ denote the semigroup determined by Ω_n, then

$$P(t)f(\eta)=\lim_{n\to\infty} P_n(t)f(\eta) \text{ for } f\in\mathscr{L}_a,\ \eta\in\mathscr{E}_a.$$

For arbitrarily fixed x, $y\in S$, define the functions f_x, f_{xy} on $\mathscr{E}_a$ respectively as follows.

$$f_x(\eta)=\eta(x),\ \eta\in\mathscr{E}_a;\ f_{xy}(\eta)=\eta(x)(\eta(y)-\delta_{xy}),\ \eta\in\mathscr{E}_a.$$

Lemma 2.1. *For arbitrary $x\in S$,*

$$P(t)f_x(\eta)=e^{(\lambda_1-\lambda_2)t}\sum_u \eta(u)p(t,u,x)$$
$$+\lambda_3\int_0^t e^{(\lambda_1-\lambda_2)s}\sum_u p(s,u,x)ds. \tag{2.}$$

Proof We firstly prove (2.2) for the case of $|S|<\infty$. Because $f_x\in\mathscr{L}_a$. we ha

$$\frac{d}{dt}P(t)f_x(\eta)$$
$$=P(t)\Omega f_x(\eta)=(\lambda_1-\lambda_2-1)P(t)f_x(\eta)+\sum_u P(t)f_u(\eta)p(u,x)+\lambda_3.$$

Let $u_t(x,\eta)=P(t)f_x(\eta)$, $x\in S$, then

$$\begin{cases}\dfrac{d}{dt}u_t(x,\eta)=(\lambda_1-\lambda_2-1)u_t(x,\eta)+\sum_y u_t(y,\eta)p(y,x)+\lambda_3,\\ u_0(x,\eta)=\eta(x),\ x\in S.\end{cases} \tag{2.}$$

By the theory of the mininal nonnegative solutions ([7]. Chap. 3), the system equations

$$\frac{d}{dt}u_t(x,\eta)=(\lambda_1-\lambda_2-1)u_t(x,\eta)+\sum_y u_t(y,\eta)p(y,x),$$
$$u_0(x,\eta)=\eta(x),\ x\in S$$

has a unique solution $e^{(\lambda_1-\lambda_2)t}\sum_y \eta(y)p(t, \ ,x)$. Thus (2.3) has unique solution

$$u_t(x,\eta)=e^{(\lambda_1-\lambda_2)t}\sum_y (y)p(t,y,x)$$
$$+\lambda_3\int_0^t e^{(\lambda_1-\lambda_2)(t-s)}\sum_y p(t-s,y,x)ds.$$

Substituting s for $t-s$ in the second term above, we get (2.2) for finite S.

Now let S be countable. Because $f_x\in\mathscr{L}_a$, we have

$$P(t)f_x(\eta)=\lim P_n(t)f_x(\eta).$$

By the result proved above

$$P_n(t)f_x(\eta)=e^{(\lambda_1-\lambda_2)t}\sum_{y\in\Lambda_n}\eta(y)p_n(t,y,x)$$
$$+\lambda_3\int_0^t e^{(\lambda_1-\lambda_2)s}\sum_y p_n(s,y,x)ds.$$

$$p_n(t,y,x)\leq e^t p(t,y,x) \quad\text{and}\quad p_n(t,y,x)\to p(t,y,x) \text{ as } n\to\infty.$$

By the dominated convergence theorem, (2.2) follows from the above equation.

Now we discuss the computation of the second moment of η_t at time t.

Lemma 2.4. *Let S be finite, then*

$$P(t)f_{xy}(\eta)=e^{2(\lambda_1-\lambda_2)t}\sum_{u,v}p(t,u,x)p(t,v,y)f_{uv}(\eta)$$
$$+\lambda_3\sum_{u,v}\int_0^t[P(s)f_u(\eta)+P(s)f_v(\eta)]e^{2(\lambda_1-\lambda_2)(t-s)}$$
$$\times p(t-s,u,x)p(t-s,v,y)ds$$
$$+2\lambda_1\sum_u\int_0^t P(s)f_u(\eta)e^{2(\lambda_1-\lambda_2)(t-s)}p(t-s,u,x)p(t-s,u,y)\,ds. \quad (2.5)$$

Proof By definition (1.4) of Ω, and noting that S is finite

$$\Omega f_{xy}(\eta)=2(\lambda_1-\lambda_2)f_{xy}(\eta)+2\delta_{xy}\lambda_1 f_x(\eta)+\lambda_3(f_x(\eta)+f_y(\eta))$$
$$+\sum_u\{f_{uy}(\eta)[p(u,x)-\delta_{ux}]+f_{ux}(\eta)[p(u,y)-\delta_{uy}]\}.$$

Thus

$$\frac{d}{dt}P(t)f_{xy}(\eta)=P(t)\Omega f_{xy}(\eta)$$
$$=2(\lambda_1-\lambda_2)P(t)f_{xy}(\eta)+(\lambda_1\delta_{xy}+\lambda_3)[P(t)f_x(\eta)+P(t)f_y(\eta)]$$
$$+\sum_u\{P(t)f_{uy}(\eta)[p(u,x)-\delta_{ux}]+P(t)f_{ux}(\eta)[p(u,y)-\delta_{uy}]\}.$$

A straight check demonstrates that the right hand side of (2.5) satisfies the above system of equations. By uniqueness of its solution, (2.5) holds.

Lemma 2.6. *For arbitrary $x, y\in S$,*

$$\lim_{n\to\infty}P_n(t)f_{xy}(\eta)=e^{2(\lambda_1-\lambda_2)t}\sum_{u,v}p(t,u,x)p(t,v,y)f_{uv}(\eta)$$
$$+\lambda_3\sum_{u,v}\int_0^t[P(s)f_u(\eta)+P(s)f_v(\eta)]e^{2(\lambda_1-\lambda_2)(t-s)}$$
$$\times p(t-s,u,x)p(t-s,v,y)dt$$
$$+2\lambda_1\sum_u\int_0^t P(s)f_u(\eta)e^{2(\lambda_1-\lambda_2)(t-s)}$$
$$\times p(t-s,u,x)p(t-s,u,y)ds. \quad (2.7)$$

Proof When $x\in\Lambda_n, u\notin\Lambda_n$, $p(t,u,x)=0$. By Lemma 2.4

$$P_n(t)f_{xy}(\eta)=e^{2(\lambda_1-\lambda_2)t}\sum_{u,v}p_n(t,u,x)p_n(t,v,y)f_{uv}(\eta)$$
$$+\lambda_3\sum_{u,v}\int_0^t[P_n(s)f_u(\eta)+P_n(s)f_v(\eta)]e^{2(\lambda_1-\lambda_2)(t-s)}$$
$$\times p_n(t-s,u,x)p_n(t-s,v,y)ds$$
$$+2\lambda_1\sum_u\int_0^t P_n(s)f_u(\eta)e^{2(\lambda_1-\lambda_2)(t-s)}$$
$$\times p_n(t-s,u,x)p_n(t-s,u,y)\,ds,$$

but $p_n(t,u,x)\leqslant e^t p(t,u,x)$ and $P_n(t,u,x)\to p(t,u,x)$, as $n\to\infty$. $P_n(t)f_u(\eta)\leqslant e^t P(t)f_u(\eta)$ and $P_n(t)f_u(\eta)\to P(t)f_u(\eta)$, as $n\to\infty$. Let $n\to\infty$, by dominated convergence theorem, we get (2.7).

In order to get the expression of the second moment, naturally, we expect to prove

$$\lim_{n\to\infty}P_n(t)f_{xy}(\eta)=P(t)f_{xy}(\eta).$$

But $f_{xy}\notin\mathscr{L}_\alpha$. Therefore we have to prove that this equality holds for larger function class which contains the function f_{xy} than $\mathscr{L}_\alpha$. Let

$$\mathscr{L}_2=\{f:\ \mathscr{E}_\alpha\to R^1,\ |f(\eta)-f(\zeta)| \leqslant L_2(f)[\|\eta-\zeta\|+\sum_{x,y}|f_{xy}(\eta)-f_{xy}(\zeta)|\alpha(x)\alpha(y) \text{ for all } \eta, \zeta\in\mathscr{E}_\alpha \text{ and } \eta\leqslant\zeta \text{ or } \eta\geqslant\zeta\}.$$

It is easy to check that $\Omega f(\eta)$ is well defined for $f\in\mathscr{L}_2$, and

$$|\Omega f(\eta)|\leqslant(\lambda_1+\lambda_2+M+1)\|\eta\|(2\|\eta\|+1).$$

Lemma 2.8. *Let S be a finite set and $f\in\mathscr{L}_2$, then $P(t)f\in\mathscr{L}_2$.*

Proof We prove the conclusion only for η^1, $\eta^2\in\mathscr{E}_\alpha$ and $\eta^1\geqslant\eta^2$. The proof fo $\eta^1\leqslant\eta^2$ is similar. Using coupling argument, there exist two processes η_t^i, $i=1, 2$ defined on the same probability space with initial configurations η^i, $i=1,$ respectively such that $\eta_t^1\geqslant\eta_t^2$ for all $t\geqslant0$. The expectation operator on thi probability space is denoted by E. Then

$$\begin{aligned}&|P(t)f(\eta^1)-P(t)f(\eta^2)|\\&=|Ef(\eta_t^1)-Ef(\eta_t^2)|\leqslant E|f(\eta_t^1)-f(\eta_t^2)|\\&\leqslant L_2(f)E\{\sum_x(\eta_t^1(x)-\eta_t^2(x))\alpha(x)\\&\quad+\sum_{x,y}(f_{xy}(\eta_t^1)-f_{xy}(\eta_t^2))\alpha(x)\alpha(y)\}.\end{aligned}$$

From (2.2) it follows that

$$\begin{aligned}&E\sum_x(\eta_t^1(x)-\eta_t^2(x))\alpha(x)\\&=\sum_x[(\sum_u\eta^1(u)p(t,u,x)-\sum_u\eta^2(u)p(t,u,x)]\alpha(x)e^{(\lambda_1-\lambda_2)t}\\&\leqslant e^{(\lambda_1-\lambda_2+M-1)t}\|\eta^1-\eta^2\|.\end{aligned}$$

Again from (2.5) it follows that

$$\begin{aligned}&E\sum_{x,y}(f_{xy}(\eta_t^1)-f_{xy}(\eta_t^2))\alpha(x)\alpha(y)\\&\leqslant e^{2(\lambda_1-\lambda_2+M-1)t}\sum_{u,v}|f_{uv}(\eta^1)-f_{uv}(\eta^2)|\alpha(u)\alpha(v)\\&\quad+\lambda_3 2(\sum_x\alpha(x))e^{2(\lambda_1-\lambda_2+M-1)t}\|\eta^1-\eta^2\|\\&\quad+2\lambda_1e^{2(\lambda_1-\lambda_2+M-1)t}\|\eta^1-\eta^2\|\\&\leqslant C_1(t)\{\|\eta^1-\eta^2\|+\sum_{u,v}|f_{uv}(\eta^1)-f_{uv}(\eta^2)|\alpha(u)\alpha(v)\},\end{aligned}$$

here $C_1(t)$ is a positive and locally bounded function on $[0,\infty)$. Thus

$$\begin{aligned}&|P(t)f(\eta^1)-P(t)f(\eta^2)|\\&\leqslant L_2(f)M(t)\{\|\eta^1-\eta^2\|+\sum_{x,y}|f_{xy}(\eta^1)-f_{xy}(\eta^2)|\alpha(x)\alpha(y)\},\end{aligned}$$

here $M(\cdot)$ is a positive and locally bounded function on $[0,\infty)$.

Using the same argument for constructing the semigroup $P(t)$, we can prov that

$$\lim_{n\to\infty}P_n(t)f(\eta)=P(t)f(\eta), \tag{2.9}$$

for all $f\in\mathscr{L}_a$, $\eta\in\mathscr{E}_a$.

Proposition 2.10. *Let S be a countable set. Then for $\eta\in\mathscr{E}_a$,*

$$\begin{aligned}P(t)f_{xy}(\eta)=&e^{2(\lambda_1-\lambda_2)t}\sum_{u,v}p(t,u,x)p(t,v,y)f_{uv}(\eta)\\&+\lambda_3\sum_{u,v}\int_0^t[P(s)f_u(\eta)+P(s)f_v(\eta)]e^{2(\lambda_1-\lambda_2)(t-s)}\\&\times p(t-s,u,x)p(t-s,v,y)ds\\&+2\lambda_1\sum_u P(s)f_u(\eta)e^{2(\lambda_1-\lambda_2)(t-s)}p(t-s,u,x)p(t-s,u,y)ds.\end{aligned}$$

Proof Because $f_{xy}\in\mathscr{L}_2$, the conclusion follows from (2.7) and (2.9).

We had proved the existence uniqueness theorem for the process with state space $\mathscr{E}_M$ in which the function class used to determine the semigroup of the process is

$$\mathscr{L}_M=\{f:\mathscr{E}_M\to R^1,|f(\eta)-f(\zeta)|\leqslant L_M(f)\|\eta-\zeta\|_M$$
$$\text{for some constant and all }\eta,\zeta\in\mathscr{E}_M\}.$$

The corresponding semigroup is denoted by $P_M(t)$. Now we restrict the process to the minimal state space $\mathscr{E}$, the set of functions in $\mathscr{L}_M$ restricted to $\mathscr{E}$ is denoted by $\mathscr{L}_M(\mathscr{E})$. Clearly $\mathscr{L}_M(\mathscr{E})\subset\mathscr{L}_N(\mathscr{E})$ if $M\geqslant N>K$. Let

$$\mathscr{L}=\bigcap_{M>K}\mathscr{L}_M(\mathscr{E}).$$

$\mathscr{L}$ includes all bounded cylinder functions and all f_x $x\in S$. Using the same technique as used in [16], we can prove that the semigroup of the process $\{\eta_t\}$ restricted to $\mathscr{E}$ satisfies: for $f\in\mathscr{L}$ and $\eta\in\mathscr{E}$

$$P(t)f(\eta)=P_M(t)f(\eta),\text{ for all }M>K.\tag{2.11}$$

§3. The Proofs of the Main Theorems.

In this section, the proofs of the main theorems mentioned in the introduction are given.

Proof of Theorem 1.5. When S is a finite set, the state space $\mathscr{E}=Z_+^S$ is countable. Ω determines a Q-matrix$(q(\eta,\zeta))_{\eta,\zeta\in\mathscr{E}}$:

$$q(\eta,\zeta)=\begin{cases}\lambda_1\eta(x), & \zeta=\eta+e_x,\\ \lambda_2\eta(x), & \zeta=\eta-e_x,\\ \eta(x)p(x,y), & \zeta=\eta-e_x+e_y,\ \eta(x)\geqslant1,\\ 0, & \text{otherwise for }\zeta\neq\eta,\end{cases}\tag{3.1}$$
$$-q(\eta,\eta)=q(\eta)=\sum_{\zeta\neq\eta}q(\eta,\zeta).$$

$p(t,\eta,\zeta)$ denotes the transition function for the process $\{\eta_t\}$. By [16], $p(t,\eta,\zeta)$ is the minimal Q-process as well.

Note the following facts at first:

(i) θ is the unique absorbing state of the process, i. e. $p(t,\theta,\theta)=1$ for all $t\geqslant0$. Therefore δ_θ is an invariant measure of the process;

(ii) All other states except θ are transient, therefore

$$\lim_{t\to\infty} p(t, \eta, \zeta) = 0$$

for arbitrbry $\eta \in \mathscr{E}$ and $\zeta \neq \theta$;

(iii) By the forward differential equation, $p'(t, \eta, \theta) \geqslant 0$, therefore for any $\eta \in \mathscr{E}$, the limit

$$p(t, \eta, \theta) \uparrow p_\eta \text{ as } t \to \infty \tag{3.2}$$

exists. Thus the Q-process is ergodic if and only if $p_\eta \equiv 1$. Using the theory of th minimal nonnegative solution[7], we can show that $\{p_\eta: \eta \in \mathscr{E}\}$ is the minima nonnegative solution of the following system of equations:

$$\begin{cases} Y_\eta = \sum_{\zeta \neq \eta} q(\eta, \zeta)/q(\eta) Y_\zeta, \ \eta \neq \theta, \\ Y_\theta = 1. \end{cases} \tag{3.3}$$

Thus, it suffices to show that the system of equations (3.3) has the uniqu nonnegative solution $p_\eta \equiv 1$ if $0 < \lambda_1 \leqslant \lambda_2$, and has other solution except $p_\eta \equiv 1$ if $\lambda_1 > \lambda$ The technique to check this fact can be found in [14]. So we omit the details.

Remark. When $\lambda_1 > \lambda_2$, the process $\{\eta_t\}$ is nonergodic, but the set of invariar measures is yet singleton.

Proof of Theorem 1.13 We complete the proof of Theorem 1.13 throug proving some lemmas.

Lemma3. 4. *Suppose that* μ_n, $n \geqslant 1$ *is a sequence of probability measures on* $(R_+, \mathscr{B}_+)$, *here* $R_+ = [0, \infty)$, *and* $\mathscr{B}_+$ *is Borel field on* $[0, \infty)$. *If*

$$\sup_n \int_0^\infty \mu_n(dr) r = B < \infty, \tag{3.5}$$

then there exists a subsequence $\mu_{n'}$, $n' \geqslant 1$ weakly converges to some probabilit measure μ, and

$$\int_0^\infty \mu(dr) r \leqslant B.$$

Moreover assume

$$\sup_n \int_0^\infty \mu_n(dr) r^2 = A < \infty, \tag{3.6}$$

then

$$\lim_{n'\to\infty} \int_0^\infty \mu_{n'}(dr) r = \int_0^\infty \mu(dr) r.$$

We omit the proof, because the main idea for the proof can be found in [6].

The process with initial configuration $\mathbf{1}$ (here $\mathbf{1} \in \mathscr{E}$, $\mathbf{1}(x) = 1$ for all $x \in S$) i denoted by $\{\eta_t^1\}$. When (1.14) holds for some $x \in S$, then $E\eta_t^1(x) > 0$ by (2.2). Let μ denote the distribution of $\eta_t^1(x)/E\eta_t^1(x)$.

Lemma 3.7. *Suppose that* (1.14) *holds and* $\lambda_1 > \lambda_2$. *Then there exists a sequence* $t_n \uparrow \infty$ *and a probability measure* μ *on* $(R_+, \mathscr{B}_+)$ *such that* μ_{t_n} *weakly converges to* μ *and*

$$\int_0^\infty \mu(dr)r=\lim_{n\to\infty}\int_0^\infty \mu_{t_n}(dr)r.$$

Proof From Lemma 3.4, it suffices to prove

$$\sup_{t>0}\int_0^\infty \mu_t(dr)r=B<\infty$$

and

$$\sup_{t>0}\int_0^\infty \mu_t(dr)r^2=A<\infty.$$

In fact

$$\int_0^\infty \mu_t(dr)r=E(\eta_t'(x)/E\eta_t'(x))=1;$$

and

$$\int_0^\infty \mu_t(dr)r^2=E[\eta_t'(x)]^2/E^2[\eta_t'(x)]$$
$$=P(t)f_{xx}(1)/[P(t)f_x(1)]^2+1/[P(t)f_x(1)]^2.$$

By (2.2) and Proposition 2.10,

$$P(t)f_x(1)=e^{(\lambda_1-\lambda_2)t}\sum_u p(t,u,x)\quad\text{and}$$
$$P(t)f_{xx}(1)\leqslant e^{2(\lambda_1-\lambda_2)t}[\sum_v p(t,u,x)]^2+2\lambda_1(\lambda_1-\lambda_2)^{-1}e^{2(\lambda_1-\lambda_2)t}\sum_u p(t,u,x).$$

So under the assumptions of the lemma

$$\sup_t\int_0^\infty \mu_t(dr)r^2=A<\infty.$$

Now we arein a position to prove Theorem 1.13.

(i) From assumption $\lambda_1-\lambda_2+K-1<0$ we can take $M>K$ such that

$$\lambda_1-\lambda_2+M-1<0.$$

Thus

$$P(t)\|\cdot\|_M(\eta)=\sum_x e^{(\lambda_1-\lambda_2)t}\sum_n \eta(u)p(t,u,x)\alpha_M(x)e^{(\lambda_1-\lambda_2+M-1)t}\|\eta\|_M.$$

Noting that $\mathscr{F}(\mathscr{E})\subset\mathscr{L}$, then for arbitrary $\nu\in P(\mathscr{E})$ and $f\in\mathscr{F}(\mathscr{E})$,

$$C=2\sup_\eta|f(\eta)|,$$

$$\left|\int P(t)f(\eta)\nu(d\eta)-\int f(\eta)\delta_\theta(d\eta)\right|$$
$$=\left|\int P(t)f(\eta)\nu(d\eta)-f(\theta)\right|$$
$$\leqslant\int P(t)|f(\cdot)-f(\theta)|(\eta)\nu(d)$$
$$\leqslant L_M(f)\int P(t)(\|\cdot\|_M\wedge C)\nu(d\eta).$$

Because $P(t)(\|\eta\|_M\wedge C)\leqslant P(t)\|\eta\|_M\wedge C$ and $P(t)\|\eta\|_M\leqslant\|\eta\|_M e^{(\lambda_1-\lambda_2+M-1)t}\to 0$ as $t\to\infty$ by the dominated convergence theorem we have

$$\left|\int P(t)f(\eta)\nu(d\eta)-\int f(\eta)\delta_\theta(d\eta)\right|\to 0\text{ as }t\to\infty.$$

Thus the process $\{\eta_t\}$ is ergodic and the unique invariant measure is δ_θ.

(ii) Suppose that (1.14) holds and $\lambda_1-\lambda_2>0$. If the process is ergodic, then

$\lim \delta_1 P(t) = \delta_\theta$. On the other hand, from Lemma (3.7) there exists a subsequence μ_{t_n}, $n \geq 1$ of distributions μ_t, $t \geq 0$ of $\eta'_t(x)/E\eta'_t(x)$ which weakly converges to some probability measure μ on $(R_+, \mathscr{B}_+)$ and

$$\int_0^\infty \mu(dr)r = 1.$$

It follows that there is some constant $c > 0$ such that $\mu((c, \infty)) > 0$. Because μ_{t_n} weakly converges to μ,

$$\mu_{t_n}((c, \infty)) \geq 1/2 \cdot \mu((c, \infty)) > 0$$

for all sufficient large n. Write $A(t) = E\eta'_t(x)$, (2.2) and the assumption impl $A(t) \to \infty$ as $t \to \infty$. Particularly $cA(t_n) \geq 1$ for all sufficient large n. Thus

$$\begin{aligned} P(t_n)(f_x \wedge 1)(1) &\geq P(t_n)(f_x \wedge 1) \cdot I_{\{\eta: \eta(x) > cA(t_n)\}}(1) \\ &= P(t_n) I_{\{\eta: \eta(x) > cA(t_n)\}}(1) \\ &= P(\eta'_{t_n}(x)/E\eta'_{t_n}(x) > c) \\ &= \mu_{t_n}((c, \infty)) \geq 1/2\mu((c, \infty)) > 0. \end{aligned}$$

Note $f_x \wedge 1 \in \mathscr{F}(\mathscr{E})$, the above fact contradicts that $\delta_1 P(t)$ converges to δ_θ. Th implies that $\{\eta_t\}$ is nonergodic.

The Proof of Theorem 1.19 is similar to the above proof of Theorem 1.13. W only point out that for the case having particle source, i. e. $\lambda_3 > 0$, δ_θ is no longe invariant measure of the process, and the invariant measure which plays the role of δ_θ is that one given in the following lemma. Here assume $\lambda_3 > 0$.

Lemma 3.8. *If* $\lambda_1 - \lambda_2 + K - 1 < 0$, *then*

$$\mu_0 = \lim_{t \to \infty} \delta_\theta P(t)$$

exists and invariant.

Proof Choose $M > K$ such that $\lambda_1 - \lambda_2 + M - 1 < 0$. Using the coupling argumer can prove that if $f \in \mathscr{F}(\mathscr{E})$ is nondecreasing, then so is $P(t)f$. Therefore

$$\begin{aligned} \int P^\theta(\eta_{t+s} \in d\eta) f(\eta) &= \int P^\theta(\eta_t \in d\zeta) P^\zeta(\eta_s \in d\eta) f(\eta) \\ &\geq \int P^\theta(\eta_t \in d\zeta) P^\theta(\eta_s \in d\eta) f(\eta) \\ &= \int P^\theta(\eta_s \in d\eta) f(\eta). \end{aligned}$$

On the other hand, by (2.2)

$$\begin{aligned} &\sum_x \alpha_M(x) \int P^\theta(\eta_t \in d\eta) f_x(\eta) \\ &\leq \lambda_3 \sum_x \alpha_M(x) \int_0^\infty ds\, e^{(\lambda_1 - \lambda_2 + M - 1)s} < \infty. \end{aligned}$$

By Prohorov theorem and Markov inquality, $\{P^\theta(\eta_s \in \cdot), s \geq 0\}$ is relatively compa in the sense of convergence for all finite dimensional distributions. Using th monotone argument, we can prove that $\mu_0 = \lim \delta_\theta P(t)$ exists and is invariant measure for the process.

By the way, we point out tthat (1.20) and (1.21) hold if P is doubly stochastio transition matrix. In this case the line $\lambda_1-\lambda_2=0$ is also the critical line between ergodic domain and nonergodic domain, and the process is nonergodic on this line.

Proof of Theorem 1.16 From now on, we return to the translation case, i. e. suppose $S=Z^d$, $p(x, y)=p(0, y-x)$ for any x, $y\in S$, and suppose $\lambda_3=0$.

The proof of Theorem1.16 is similar to the proof of Theorem 1.9 in [8]. Two parts of the proof of Theorem 1.9 in [8] are coupling result based on a monotonicity property of the process and the first and second moments of the process. Because the first and second moment have been given in section 2, the part remaining to be done is to give the generator of the somigroup for the coupled process required here, and to point out the necessary modifications in carrying out the proof. Particularly, we get Lemma 3.10 instead of Lemma 4.6 in [8]. Readers interested in the details of the proof can refor to the sections 4 and 5 of [8].

Let

$$\mathscr{L}^{(2)}=\{f: |f(\eta^1, \eta^2)-f(\zeta^1, \zeta^2)| \leqslant C_M[\|\eta^1-\zeta^1\|_M+\|\eta^2-\zeta^2\|_M] \text{ for all } (\eta^1, \eta^2), (\zeta^1; \zeta^2)\in\mathscr{E}\times\mathscr{E} \text{ and all } M>K\}.$$

For $f\in\mathscr{L}^{(2)}$, $(\eta^1, \eta^2)\in\mathscr{E}\times\mathscr{E}$, define

$$\begin{aligned}
&\Omega^{(2)}f(\eta^1, \eta^2)\\
&=\Sigma_x \lambda_1\eta^1(x)\wedge\eta^2(x)[f(\eta^1+e_x, \eta^2+e_x)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)>\eta^2(x)}\lambda_1(\eta^1(x)-\eta^2(x))[f(\eta^1+e_x, \eta^2)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)<\eta^2(x)}\lambda_1(\eta^2(x)-\eta^1(x))[f(\eta^1, \eta^2+e_x)-f(\eta^1, \eta^2)]\\
&\quad+\sum_x\lambda_2\eta^1(x)\wedge\eta^2(x)[f(\eta^1-e_x, \eta^2-e_x)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)>\eta^2(x)}\lambda_2(\eta^1(x)-\eta^2(x))[f(\eta^1-e_x, \eta^2)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)<\eta^2(x)}\lambda_2(\eta^2(x)-\eta^2(x))[f(\eta^1, \eta^2-e_x)-f(\eta^1, \eta^2)]\\
&\quad+\sum_x\eta^1(x)\wedge\eta^2(x)\sum_y p(x, y)[f(\eta^1-e_x+e_y, \eta^2-e_x+e_y)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)>\eta^2(x)}(\eta^1(x)-\eta^2(x))\sum_y p(x, y)[f(\eta^1-e_x+e_y, \eta^2)-f(\eta^1, \eta^2)]\\
&\quad+\sum_{x:\eta^1(x)<\eta^2(x)}(\eta^2(x)-\eta^1(x))\sum_y p(x, y)[f(\eta^1, \eta^2-e_x+e_y)-f(\eta^1, \eta^2)].
\end{aligned}$$

We can prove there exists a coupled process (η_t, ζ_t), $t\geqslant0$, with generator $\Omega^{(2)}$ ([16] or [3]). Each of the marginals of the coupled process determined dy $\Omega^{(2)}$ is a version of the basic process. The coupled process possesses the property as follows:

If $\eta, \zeta\in\mathscr{E}$, $\eta\geqslant\zeta$, then for all $t\geqslant0$, $P^{(\eta,\zeta)}(\eta_t\geqslant\zeta_t)=1$.

As Lemma 4.2 in [8], for arbitrary fixed $u\in S$, let

$$f(\eta^1, \eta^2)=|\eta^1(u)-\eta^2(u)|.$$

Then $f\in\mathscr{L}^{(2)}$ and

$$\begin{aligned}\Omega^{(2)} f(\eta^1, \eta^2) &= \sum_x |\eta^1(x)-\eta^2(x)|p(x, u) \\ &+(\lambda_1-\lambda_2-1)|\eta^1(u)-\eta^2(u)| \\ &-\sum_x H(\eta^1(x)-\eta^2(x), p(x, u), \eta^1(u)-\eta^2(u)),\end{aligned} \tag{3.9}$$

where

$$H(\eta^1(x)-\eta^2(x), p(x, u), \eta^1(u)-\eta^2(u)) = \begin{cases} 0, & \text{if } (\eta^1(x)-\eta^2(x))(\eta^1(u)-\eta^2(u)) \geqslant 0, \\ 2|\eta^1(x)-\eta^2(x)|p(x,u), & \text{if } (\eta^1(x)-\eta^2(x))(\eta^1(u)-\eta^2(u)) < 0. \end{cases}$$

Here H does not belong to $\mathscr{L}^{(2)}$ in contrast with [8]. Thus we have to modify tl Lemma 4.6 in [8] as follows .

Lemma 3.10. *Let $G_x(\eta^1(x)-\eta^2(x), p(x, u), (\eta^1(u)-\eta^2(u)) = H(\eta^1(x)-\eta^2(x$ $p(x, u), \eta^1(u)-\eta^2(u)) \wedge 2$. Suppose $\mu \in \mathscr{T}$ (the set of all translation invariant measur on $\mathscr{E} \times \mathscr{E}$) and fo r all $M > K$,*

$$\int \mu(d\eta^1 \times \eta^2)(\|\eta^1\|_M + \|\eta^2\|_M) < \infty.$$

Let

$$g(x, t) = \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) G_x(\eta^1(x)-\eta^2(x), p(x, u), \eta^1(u)-\eta^2(u)),$$

where $P^{(2)}(t)$ is the semigroup determined by $\Omega^{(2)}$. If $\lambda_1 = \lambda_2$, then

$$\lim_{t\to\infty} g(x, t) = 0, \tag{3.11}$$

for each $x \in S$.

Proof Set

$$h(x, t) = \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) H(\eta^1(x)-\eta^2(x), p(x, u), \eta^1(u)-\eta^2(u)).$$

By (3.9) and noting $f \in \mathscr{L}^{(2)}$,

$$\begin{aligned} 0 &\leqslant \int_0^t \sum_x h(x, s) ds \\ &= -\int_0^t \int \mu(d\eta^1 \times \eta^2) P^{(2)}(s) \Omega^{(2)} f(\eta^1, \eta^2) ds \\ &= -\int P^{(2)}(t) f(\eta^1, \eta^2) \mu(d\eta^1 \times \eta^2) + \int \mu(d\eta^1 \times \eta^2)|\eta^1(u)-\eta^2(u)| \\ &\leqslant \int \mu(d\eta^1 \times \eta^2)|\eta^1(u)-\eta^2(u)|. \end{aligned}$$

Let $t \to \infty$, we get

$$\int_0^\infty \sum_x h(x, s) ds \leqslant \int \mu(d\eta^1 \times \eta^2)|\eta^1(u)-\eta^2(u)| < \infty.$$

Because $0 \leqslant G_x \leqslant H$, we have

$$\int_0^\infty \sum_x g(x, s) ds < \infty. \tag{3.12}$$

By the fact that G_x as a bounded cylinde function belongs to $\mathscr{L}^{(2)}$,

$$g((x, t) = \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) G_x(\eta^1(x) - \eta^2(x), p(x, u), \eta^1(u) - \eta^2(u))$$

$$= \int \mu(d\eta^1 \times \eta^2) G_x(\eta^1(x) - \eta^2(x), p(x, u), \eta^1(u) - \eta^2(u))$$

$$+ \int_0^t ds \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) \Omega^{(2)} G_x(\eta^1(x) - \eta^2(x),$$

$$p(x, u), \eta^1(u) - \eta^2(u)),$$

o that $g(x, t)$ is absolutely continuous in t and

$$\frac{d}{dt} g(x, t) = \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) G_x(\eta^1(x) - \eta^2(x), p(x, u), \eta^1(u) - \eta^2(u)).$$

t followt that

$$\left| \frac{d}{dt} g(x, t) \right| \leqslant \int \mu P^{(2)}(t)(d\eta^1 \times \eta^2) C(\|\eta^1\|_M + \|\eta^2\|_M), \ M > K,$$

where the constant C depends on x and M but not t. On the other hand, it follows from $\mu \in \overline{\mathscr{S}}$ that

$$\int \mu P^{(2)}(t)(d\eta^1 \times \eta^2)(\|\eta^1\|_M + \|\eta^2\|_M)$$

$$= \int \mu(d\eta^1 \times \eta^2) e^{(\lambda_1 - \lambda_2)t} \sum_u (\eta^1(u) + \eta^2(u)) \sum_y p(t, u, y) \alpha_M(y)$$

$$\leqslant 2\rho_1 \sum_y \alpha_M(y),$$

here

$$\rho_1 = \bigvee_{i=1}^{2} \int \mu(d\eta^1 \times \eta^2) f_a(\eta^i).$$

Thus

$$\sup_{t > 0} \left| \frac{d}{dt} g(x, t) \right| < \infty.$$

(3.11) follows from above and (3.12).

Using Lemma 3.10 instead of Lemma 4.6 in [8], we can complete the proof of Theorem 1.16 along the way of the sections 4 and 5 in [8].

§ 4. An Example

Theorem 1.13 and Theorem 1.16 mean, generally speaking, when P is a doubly stochastic transition matrix, the ergodicity for the process restricted to the minimal configuration space is almost the same as the ergodicity for the linear birth and death Q-process [4]. In this section an example is givon which shows if there is not any restriction imposed on the configuration space, then diffusion may influence heavily the ergodicity for the process even though P is doubly stochastic. In fact, there may exist some seriously unbounded configurations on the configuration space.

Throughout this section we assum that $S = Z^d$, P is the simple random walk on Z^d, i. e.

$$p(x, y)=\begin{cases} r, & y-x=0, \\ p_i, & y-x=e_i, \\ q_i, & y-x=-e_i, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

where $r\geqslant 0$, p_i, $q_i>0$ and

$$r+\sum_{i=1}^{d}(p_i+q_i)=1;\text{ and } e_i\in Z^d,\ i=1, 2\cdots, d,\ e_i(j)=1 \text{ if } j=i, =0 \text{ if } j\neq i.$$

We choose a fixed $\alpha(\cdot)$ as follows: for each $x=(x_1, \cdots, x_d)\in Z^d$, let

$$\alpha(x)=\prod_{i=1}^{d} s_i^{|x_i|},$$

here $s_i=p_i$ if $x_i<0$, $=q_i$ if $x_i\geqslant 0$.

Straight computation shows that

$$M=d+r+\sum_{i=1}^{d}p_iq_i. \tag{4.}$$

As in the introduction

$$\mathscr{E}_\alpha=\{\eta\in E: \sum_x \eta(x)\alpha(x)<\infty\}.$$

We have

Theorem 4.3. *Assume that $S=Z^d$, P is defined by (4.1), and $\lambda_3=0$. If $\lambda_1-\lambda_2$ $M-1<0$, then the process is ergodic; if $\lambda_1-\lambda_2+M-1>0$, then the process is no ergodic.*

Proof The proof of the first part is the same as the proof of Theorem 1.13 (i The outline of the proof of the second part is similar to that of Theorem 1.13 (ii).

For arbitrary $1\leqslant c_i<1/p_i$ and $1\leqslant c_i'<1/q_i$, let

$$\pi_c(x)=\prod_{i=1}^{d}h_i(x, c)^{|x_i|},$$

where $h_i(x, c)=c_i$ if $x_i\geqslant 0$, $=c_i'$ if $x_i<0$; let

$$\rho_c=\sum_{i=1}^{d}(c_i'q_i+p_i/c_i')(c_ip_i+q_i/c_i)+r.$$

Then

$$\sup\{\rho_c: 1\leqslant c_i<1/p_i,\ 1\leqslant c_i'<1/q_i,\ i=1, 2, \cdots, d\}$$
$$=d+r+\sum_{i=1}^{d}p_iq_i=M,$$

and

$$\sum_x \pi_c(x)\alpha(x)<\infty, \tag{4.}$$

$$\sum_x \pi_c(x)p(x, y)\geqslant\rho_c\pi_c(y). \tag{4.}$$

It follows from (4.5) that

$$\sum_x \pi_c(x)p(t, x, y)\geqslant e^{(\rho_c-1)t}\pi_c(y). \tag{4.}$$

Taking $\eta_c(x)=[\pi_c(x)]+1$, $x\in S$, (4.4) implies that $\eta_c\in\mathscr{E}_\alpha$. From (4.6) and (2._, it follows that

$$P(t)f_x(\eta_c)\geqslant e^{(\lambda_1-\lambda_2+\rho_c-1)t}\pi_c(x). \tag{4.7}$$

Using (4.7), to complete the proof of the second part of Theorem 4.3 is almost to repeat the proof of Theorem 1.13 (ii) for the process with initial configuration η_c.

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