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# YANG-MILLS CONNECTION OVER KÄHLER SURFACE

SHEN CHUNLI (沈纯理)*

**Abstract**

Let $M$ be a compact Kähler surface with positive scalar curvature, $P(M, SU(N))$ a principal fibre bundle. Then an irreducible Yang-Mills connection $\nabla$ on $P$ must be an anti-self-dual connection, provided

$$\int_M |\Phi|^2 < C.$$

Here $\Phi$ is the projection of the curvature $F$ of the connection $\nabla$ on the Kähler form of $M$, and $C$ is an a-priori postive constant independent of connections on $P$.

## § 1. Introduction

Let $M$ be a compact 4-dimensional Riemannian manifold and $P$ a principa $G$-bundle over $M$, where $G$ is a compact Lie group. We consider the Yang-Mil functional

$$YM(\nabla) = \int_M |F|^2, \tag{1}$$

where $F$ is the curvature of the connection $\nabla$ on $P$ and the norm is defined in term of the Riemannian metric on $M$ and a fixed $Ad_G$-invariant scalar product on the Lie algebra $\mathfrak{G}$ of $G$.

Let $X_a$ be an orthonormal base of $\mathfrak{G}$, i. e.,

$$\langle X_a, X_b \rangle = \delta_{ab}, \quad a, b, =1, \cdots, \dim G$$

and the commutator

$$[X_a, X_b] = C^c_{ab}, X_c,$$

where $C^c_{ab}$ are the structure constants of Lie group $G$.

The critical points of the Yang-Mills functional are called Yang-Mill connections. Yang-Mills connections should satisfy

$$d^{\nabla *} F = 0, \tag{2}$$

where $d^{\nabla *}$ is the adjoint of the exterior covariant differential $d^\nabla$ of the connection $\nabla$. Meanwhile, the curvature $F$ automatically satisfies the Bianchi identities

$$d^\nabla F = 0. \tag{3}$$

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* Department of Mathematics, East China Normal University, Shanghai, China.

Using Hodge * operator, we have

$$F=F^{+}+F^{-},$$

where $F^{+}$, $F^{-}$ are the self-dual and anti-self-dual parts of $F$ respectively. Hence

$$YM(\nabla)=2\int_M |F^{-}|^2+8\pi^2 k$$

or

$$YM(\nabla)=2\int_M |F^{+}|^2-8\pi^2 k,$$

here $k$ is the Pontrjagin index of $P$. Thus, the connections satisfying

$$F^{+}=0 \quad \text{or} \quad F^{-}=0$$

are called self-dual and anti-self-dual connections respectively. Such connections minimize the Yang-Mills functoinal $YM(\nabla)$.

Therefore, it is a very interesting problem for us to study when Yang-Mills connection is self-dual (or anti-self-dual).

In [1], Bourguignon-Lawson proved that the Yang-Millsconnetion $\nabla$ on $P$ ($M$, $SU$ ($N$)), where $M$ is a four-dimensional self-dual compact Riemannian manifold with positive scalar curvature $R$, must be self-dual, provided

$$|F^{-}|<\frac{R}{12}.$$

Further, Min-Oo [4] proved that the Yang-Mills connection $\nabla$ on $P$ ($S^4$, $SU$ (2)) must be self-dual, whenever

$$\int_{S^4} |F^{-}|^2<C,$$

here $C$ is a suitable positive constant.

In [5], Shen discussed the relationship between the Yang-Mills connections and self-dual connections in the non-compact case. In [2], Itoh investigated tho moduli space of anti-solf-dual connections on the principal fibre bundle P whoso base manifold is a Kähler surface.

## §2. Main Theorem and Notations

In this paper, we assume $M$ is a Kähler surface and want to study the relationship between the Yang-Mills connections and anti-self-dual connections. The main theorem is as follows.

**Theorem.** *Let $M$ be a compact Kähler surface with positive scalar curvature, $P$ ($M$, $SU(N)$) a principal fibre bundle with base manifold $M$ and structure group $SU$ ($N$). Let $\nabla$ be an irreducible Yang-Mills connection on $P$ and $F$ the curvature of $\nabla$. There exists a positive constant $C$ independent of connections on $P$ such that if*

$$\int_M |\Phi|^2<C,$$

*then $\nabla$ must be anti-self-dual. Here $\Phi$ is the projection of the curvature $F$ on the Kähler form $\Omega$ of $M$.*

we use the following notations in [2].

$A^k$: the $k$-form bundle;

$A^{p,q}$: the bundle consisting of complex forms of type $(p, q)$;

$$\mathfrak{G}_p = P X_{Ad} \mathfrak{G};$$

$\mathfrak{G}_p^C$: the complexification of $\mathfrak{G}_p$;

$$A^k(\mathfrak{G}_p) = \Gamma(M, A^k \otimes \mathfrak{G}_p);$$

$$A^{p,q}(\mathfrak{G}_p) = \Gamma(M, A^{p,q} \otimes \mathfrak{G}_p);$$

$$A^{p,q}(\mathfrak{G}_p^C) = \Gamma(M, A^{p,q} \otimes \mathfrak{G}_p^C);$$

$[\cdot A \cdot]$: $A^p(\mathfrak{G}) \times A^q(\mathfrak{G}) \to A^{p+q}(\mathfrak{G})$ is the Lie bracket for exterior forms;

$A$: connection 1-form of $\nabla$;

$d^\nabla$: the exterior covariant differential of $\nabla$, $d^\nabla$ splits into $d^\nabla = \partial^\nabla + \bar\partial^\nabla$ on $A^{p,q}(\mathfrak{G}_p^C)$ as follows.

$\partial^\nabla$: $A^{p,q}(\mathfrak{G}_p^C) \to A^{p+1,q}(\mathfrak{G}_p^C)$;

$\bar\partial^\nabla$: $A^{p,q}(\mathfrak{G}_p^C) - A^{p,q+1}(\mathfrak{G}_p^C)$;

$\partial^{\nabla *}$: the adjoint of $\partial^\nabla$;

$\bar\partial^{\nabla *}$: the adjoint of $\bar\partial^\nabla$;

$d^{\nabla *}$: the adjoint of $d^\nabla$;

$\langle \Phi, \Psi \rangle = -Tr(\Phi \Lambda \overline{*\Psi})$ for $\Phi, \Psi \in A^p(\mathfrak{G}_p^C)$;

$[\cdot, \cdot, \cdot]$: the mixed product on $\mathfrak{G}$;

$\Phi^+$ the $(1, 0)$ part of $\Phi \in A^1(\mathfrak{G}_p^C)$;

$\Phi^-$: the $(0, 1)$ part of $\Phi \in A^1(\mathfrak{G}_p^C)$;

$A^+$: the $(1, 0)$ part of the connection $A$;

$A^-$: the $(0, 1)$ part of the connection $A$;

$\nabla^+$: exterior covariant differential of connection $A^+$;

$\nabla^-$: exterior covariant differential of connection $A^-$;

$\Omega$: the Kähler form of $M$;

$\Phi \in A^1(\mathfrak{G}_p)$ splits into $\Phi = \Phi^+ + \overline{\Phi^+}$ for $\Phi^+ \in A^{1,0}(\mathfrak{G}_p^C)$, we have

$$\langle \Phi, \Psi \rangle = 2R \langle \Phi, \Psi^+ \rangle \quad \text{for } \Phi, \Psi \in A'(\mathfrak{G}_p).$$

Every $\mathfrak{G}_p$-valued self-dual 2-form $\Psi$ can bo expressed as

$$\Psi = \Psi^{2,0} + \Psi^0 \otimes \Omega + \overline{\Psi^{2,0}},$$

where $\Psi^{2,0} \in A^{2,0}(\mathfrak{G}_p^C)$, $\Psi^0 \in A^0(\mathfrak{G}_p)$, and $\Omega$ is the Kähler form of $M$. Hence for $F^+$ we have

$$F^+ = \alpha + \Phi \otimes \Omega + \bar\alpha,$$

where $\alpha \in A^{2,0}(\mathfrak{G}_p^C)$, $\Phi \in A^0(\mathfrak{G})$. We call $\Phi$ the projection of $F^+$ on $\Omega$.

## §3. The Proof

Let $F^+$, $F^-$ be the self-dual and anti-self-dual parts of $F$ respectively. Then

$$F^+=\alpha+\Phi\otimes\Omega+\bar{\alpha}, \tag{4}$$

where $\alpha\in A^{2,0}(\mathfrak{G}^c)$, $\Phi\in A^0(\mathfrak{G})$. In local coordinates,

$$\alpha=\alpha_{12}\,dz^1\Lambda\, dz^2,$$
$$\Phi=\Phi^a X_a,$$

where

$$\alpha_{12}=A+\sqrt{-1}\,B=A^a X_a+\sqrt{-1}\,B^a X_a.$$

Since $\nabla$ is a Yang-Mills connection, from (2) and Bianchi identities (3), we have

$$\begin{aligned} d^{\nabla^*}F^+=0, \quad d^\nabla F^+=0,\\ d^{\nabla^*}F^-=0, \quad d^\nabla F^-=5. \end{aligned} \tag{5}$$

From the definitions in § 2 we have

$$\begin{aligned} d^\nabla=\partial^\nabla+\bar{\partial}^\nabla,\\ d^{\nabla^*}=\partial^{\nabla^*}+\bar{\partial}^{\nabla^*}. \end{aligned} \tag{6}$$

Let

$$\triangle_+^\nabla=d_+^\nabla\cdot d_+^{\nabla^*},$$

where $d_+^\nabla$ is the projection of $d^\nabla$ on the self-dual part. Because $F^+$ is self-dual,

$$d_+^{\nabla^*}F^+=d^{\nabla^*}F^+,$$

then

$$\triangle_+^\nabla F^+=d_+^\nabla\cdot d_+^{\nabla^*}F^+=d_+^\nabla d^{\nabla^*}F^+=0. \tag{7}$$

Through a straightforward but complicated computation and using the Kähler condition, we obtain

$$\begin{aligned} 0=\langle\triangle_+^\nabla F^+,\ \alpha+\bar{\alpha}\rangle=4\mathrm{Re}(\langle(\partial^\nabla\partial^{\nabla^*}+\partial_+^{\nabla^*}\partial^\nabla)\alpha,\ \alpha\rangle)-8\mathrm{Im}(\langle[\Phi\Lambda\alpha],\alpha\rangle),\\ 0=\langle\triangle_+^\nabla F^+,\ \Omega\otimes\Phi\rangle=-8\sqrt{-1}\,\mathrm{Tr}(\langle\partial^\nabla\bar{\partial}^\nabla\Phi,\ \Omega\rangle\Phi)+\mathrm{Tr}(\langle\overline{\partial^\nabla\partial^{\nabla^*}_+\alpha}+\bar{\partial}^\nabla\partial^{\nabla^*}\alpha,\Omega\rangle\Phi). \end{aligned} \tag{8}$$

Intgrating the above equalities over the whole $M$, we get

$$\int\langle\nabla^+\alpha,\ \nabla^+\alpha\rangle+\int\rho\langle\alpha,\ \alpha\rangle-\int[\Phi,\ A,\ B]=0, \tag{9}$$

$$\int\langle\nabla^+\Phi,\nabla^+\Phi\rangle-\int[\Phi,\ A,\ B]=0, \tag{10}$$

where $\rho$ is the scalar curvature of $M$.

By Kato's inequality

$$\|\nabla^+\alpha\|_2^2\geqslant\|d|\alpha|\,\|_2^2$$

from (9) one can see

$$\begin{aligned} 0&\geqslant\int|d|\alpha\|^2+\int\rho|\alpha|^2-\int|\Phi|\cdot|\alpha|^2\\ &\geqslant\int|d|\alpha\|^2+\int\rho|\alpha|^2-\sqrt{\int|\Phi|^2}\cdot\sqrt{\int|\alpha|^4}. \end{aligned} \tag{11}$$

According to the Sobolev inequality of P. Li[3] and from (11), we have

$$0 \geqslant k_1 \sqrt{\int |\alpha|^4} - k_2 \int |\alpha|^2 + \int \rho |\alpha|^2 - \sqrt{\int |\Phi|^2} \cdot \sqrt{\int |\alpha|^4}, \tag{12}$$

where

$$k_1 = \frac{1}{9} \cdot \frac{\sqrt{c_1}}{2}, \quad k_2 = \frac{1}{9} \cdot \frac{\sqrt{c_1}}{\sqrt{\mathrm{vol}(M)}} \tag{13}$$

and

$$\mathrm{vol}(M) = \text{the volume of } M,$$

$$C_1 = \inf_s \frac{(\mathrm{vol}(s))^4}{(\min(\mathrm{vol}(M_1), \mathrm{vol}(M_2)))^3},$$

here $S$ runs over the set of all hypersurfaces in $M$, and the whole manifold $M$ divided by $S$ into two parts $M_1$, $M_2$.

Let $\rho_0 = \min \rho$, then from (12) we obtain

$$0 \geqslant \left(k_1 - \sqrt{\int |\Phi|^2}\right) \cdot \sqrt{\int |\alpha|^4} + (\rho_0 - k_2) \int |\alpha|^2. \qquad (1$$

Assuming

$$\|\Phi\|_{L^2} < k_1, \qquad (1$$

we have two cases.

(i) If $\rho_0 > k_2$, then from (14) we get $\alpha = 0$.

(ii) If $\rho_0 \leqslant k_2$, then from (11) we have

$$0 \geqslant \rho_0 \int |\alpha|^2 - \sqrt{\int |\Phi|^2} \cdot \sqrt{\int |\alpha|^4}. \qquad ($$

Moreover, we assume

$$\|\Phi\|_{L^2} < \frac{1}{2} \rho_0 \cdot \sqrt{\mathrm{vol}(M)}. \qquad ($$

If $\alpha \neq 0$, then from (16), (17), we have

$$0 > \rho_0 \left( \int |\alpha|^2 - \frac{1}{2} \sqrt{\mathrm{vol}(M)} \cdot \sqrt{\int |\alpha|^4}, \qquad ($$

that is

$$\int |\alpha|^2 < \frac{1}{2} \sqrt{\mathrm{vol}(M)} \cdot \sqrt{\int |\alpha|^4}. \qquad ($$

Hence we obtain from (14), (19)

$$\begin{aligned} 0 &\geqslant \left(k_1 - \frac{1}{2}\rho_0 \cdot \sqrt{\mathrm{vol}(M)}\right) \sqrt{\int |\alpha|^4} + (\rho_0 - k_2) \int |\alpha|^2 \\ &> \left(\frac{2k_1}{\sqrt{\mathrm{vol}(M)}} - \rho_0\right) \cdot \int |\alpha|^2 + (\rho_0 - k_2) \int |\alpha|^2. \qquad ( \end{aligned}$$

Notice

$$k_2 = \frac{2k_1}{\sqrt{\mathrm{vol}(M)}},$$

so the right hand (20) equals to zero and we get a contradiction. Hence $\alpha = 0$.

Thus, we take a positive constant

$$C=\min\left(k_1, \frac{1}{2}\cdot\rho_0\cdot\sqrt{\mathrm{vol}(M)}\right),$$

which is independent of connections on $P$, and then we have $\alpha=0$, provided

$$\|\Phi\|_{L^2}<C. \tag{21}$$

From (10), now we have

$$\int\langle\nabla^+\Phi, \nabla^+\Phi\rangle=0,$$

so $\nabla^+\Phi=0$. Since $\Phi$ is real, we get

$$\nabla\Phi=\nabla^+\Phi+\overline{\nabla^+\Phi}=0.$$

Because $\nabla$ is an irreducible connection, one can see $\Phi=0$. Hence $F^+=\alpha+\Phi\otimes\Omega+\bar{\alpha}=0$, i. e., $\nabla$ is anti-self-dual.

## References

[1] Bourguignon, J. P. & Lawson, H. B., Stability and isolation phenomena for Yang-Mills fields, *Comm. Math. Phys.*, **79** (1981), 189-230.

[2] Itoh, M., On the moduli space of anti-self-dual Yang-Mills connection on Kähler surfaces, *Publ. RIMS, Kyoto Univ.*, **19** (1983), 15-32.

[3] Li, P., On the Sobolev constant and P-spectrum of a compact Riemannian manifold, *Ann. Scient. Ec. Norm. Sup.*, 4e série, t. 13(1980), 451—469.

[4] Min-Oo, An $L_2$-isolation theorem for Yang-Mills field, *Compositio Math.*, **47** (1982), 153—163.

[5] Shen, C. L., The gap phenomena of Yang-Mills fields over the complete manifold, *Math. Z.*, (1982), **180** (1982), 69-77.