Chin. Ann. of Math.
10B (3) 1989

DEGREE OF COPOSITIVE POLYNOMIAL APPROXIMATION

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Abstract

Denoteby $\overline{E}_n(f)$ the degree of copositive approximation to $f(x)$ by polynomials of degree $\leqslant n$. For function $f(x)\in C^k[-1, 1]$ which alternates in sign finitely many timesin $[-1, 1]$, the author obtains the following Jackson type estimates

$$\overline{E}_n(f)\leqslant Cn^{-k}\omega(f^{(k)}, 1/n)$$

foa any positive integer k.

Let $f(x)\in C[-1, 1]$. $f(x)$ changes sign at $y\in(-1, 1)$ if $f(y)=0$ and if i some $\varepsilon>0$, $f(x_1)\ f(x_2)\leqslant 0$ for all $y-\varepsilon<x_1\leqslant\eta\leqslant x_2<y+\varepsilon$. Such a y is called alternation point of $f(x)$. We say that function $g(x)$ is copositive with $f(x)$ if $f($ $g(x)\geqslant 0$ for all $x\in[-1, 1]$. In this paper, we always suppose that $f(x)$ alterna in sign finitely many times, that is, the number of the alternation points of $f($ in $[-1, 1]$ is finite.

The purpose of this paper is to discuss the degree of approximation to suc function $f(x)$ by polynomials that are copositive with $f(x)$. In the past years ma authors paid their attention to this topic and achieved some results. Denote by the class of all polynomials of degree not exceeding n, and write

$$\overline{E}_n(f)=\inf\{\|f(x)-p_n(x)\|p_n(x)\in\Pi_n \text{ and } p_n(x) \text{ copositive with } f(x)\}$$

E. Passow and L. Raymon [3] proved that if $f(x)\in C[-1, 1]$ is proper piecew monotone with nonvanishing peaks, then there is a constant d depending on $f($ but not on n, such that for n sufficiently large

$$\overline{E}_n(f)\leqslant d\omega(f, 1/n),$$

where $\omega(f, t)$ is the modulus of continuity of $f(x)$. Later J. A. Rou-lier repla the condition of proper piecewise monotonicity by a condition that $f(x)$ is prope alternating (see [4] for detail). Obviously both conditions on $f(x)$ made these res not satisfactory. Recently D. Leviatan [2] studied this problem and obtained Jack type estimates for the degree of copositive polynomial approximation with restrictions of the above type. He proved that if $f(x)\in C^k[-1,1]$, $0\leqslant k\leqslant 2$, altern in sign r times in $[-1, 1]$, and $-1<y_1<y_2<\cdots<y_r<1$ are the alternation poi

Manuscript received June 23, 1987.

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then for all n sufficiently large

$$\bar{E}_n(f)\leqslant Cn^{-k}\omega(f^{(k)}, 1/n), \tag{1}$$

where C only depends on $\{y_i\}_{i=1}^r$. How large n should be will depend on $f(x)$. This has greatly improved the preceding results, but, the remainder question is if (1) still hold for $k\geqslant 3$. Here we introduce a new idea and by it give an affirmative answer to the above question

Theorem. 1 *Let k be any positive integer. Suppose that function $f(x)\in C^k[-1, 1]$ alternates in sign r times in $[-1, 1]$, and*

$$Y=\{y_r-1<y_1<y_2<\cdots<y_r<1\}$$

is the set of all alternation points of $f(x)$. Then for all n sufficiently large,

$$\bar{E}_n(f)\leqslant Cn^{-k}\omega(f^{(k)}, 1/n),$$

where C only depends on Y.

The idea is that we first consider the simultaneous polynomial approximation with Hermite interpolatory side conditions, and then use the relative result to the positive polynomial approximation.

(I) Some authors have studied the degree of polynomial approximation with Hermite interpolatory side conditions (e. g. see [1]). Let $f(x)\in C^k[-1, 1]$, and

$$Y=\{y_i\mid -1<y_1<y_2<\cdots<y_r<1\}.$$

denote

$$E_n(f,Y)=\inf\{\|f(x)-p_n(x)\|\mid p_n(x)\in\Pi_n,\ p^{(j)}(y_i)=f^{(j)}(y_i),$$
$$i=1, \cdots, r;\ j=0, 1, \cdots, k\}.$$

The following result is known: if $f(x)\in C^k[-1, 1]$, then for all n sufficiently large,

$$E_n(f, Y)\leqslant Cn^{-k}\omega(f^{(k)}, 1/n),$$

where C only depends on Y.

We now consider the simultaneous polynomial approximation with Hermite interpolatory side conditions, and establish

Theorem 2. *Let k, r be any positive integers, $f(x)\in C^k[-1, 1]$, and*

$$Y=\{y_i\mid -1<y_1\leqslant y_2<\cdots<y_r<1\}.$$

Then there exists a polynomial $Q_n(x)\in\Pi_n$ such that

$$Q_n^{(j)}(y_i)=f^{(j)}(y_i)\ (j=0, 1, \cdots, k;\ i=1, \cdots, r)$$

and for n sufficiently large,

$$|f^{(j)}(x)-Q_n^{(j)}(x)|\leqslant Cn^{-k+j}\omega(f^{(k)}, 1/n),\quad j=0, 1, \cdots, k,$$

where C only depends on Y.

Proof From [5], we know that there exists $p_n(x)\in\Pi_n$ such that

$$|f^{(j)}(x)-p_n^{(j)}(x)|\leqslant Cn^{-k+j}\omega(f^{(k)}, 1/n),\quad j=0, 1, \cdots, k. \tag{2}$$

Setting

$$c_{ij}=\frac{f^{(j)}(y_i)-p_n^{(j)}(y_i)}{n^j}, \tag{3}$$

we have

$$|c_{ij}|\leqslant Cn^{-k}\omega(f^{(k)},\ 1/n),\quad j=0,\ 1,\ \cdots,\ k;\ i=1,\ \cdots,\ r. \tag{4}$$

Now we are going to modify $p_n(x)$ to satisfy Hermite interpolatory side conditions.

For any fixed integer j, $0\leqslant j\leqslant k$, let

$$n'=\left[\frac{n}{j+1}\right]-(k+1)(r-1),$$

$$T_n(x)=\cos\left(n'\ \mathrm{arc}\cos\frac{x}{2}\right),$$

$$p_\lambda=2\cos\frac{2\lambda-1}{2n'}\pi,\ \lambda=1,\ \cdots,\ n'.$$

For any fixed integer i, $1\leqslant i\leqslant r$, and $y_i\in Y$, we can find integer λ_i such that x_i $y<x_{\lambda_i-1}$ and

$$\left[\frac{n'}{3}\right]\leqslant\lambda_i\leqslant\left[\frac{2n'}{3}\right]$$

provided n is sufficiently large.

Define

$$h_{ij}(x)_i=\left(T_n\left(-1+\frac{1+x_{\lambda_i}}{1+y_i}(1+x)\right)-T_n(x_{\lambda_i})\right)^j\left(\frac{\prod\limits_{\substack{\mu=1\\ \mu\neq i}}^{r}(x-y_\mu)^{k+1}}{\prod\limits_{\substack{\mu=1\\ \mu\neq i}}^{r}(y_i-y_\mu)^{k+1}}\right).$$

Then $h_{ij}(x)$ is an algebraic polynomial of degree not exceeding n, and $h_{ij}(x)$ satis the following conditions:

$$h_{ij}^{(\nu)}(y_i)=0\ (\nu<j),$$

$$h_{ij}^{(j)}(y_i)=j!\left(\frac{1+x_{\lambda_i}}{1+y_i}\right)^j\cdot(T_n'(x_{\lambda_i}))^j\sim n^j,$$

$$h_{ij}^{(\nu)}(y_\mu)=0(\mu\neq i,\ \mu=1,\ \cdots,\ r;\ \nu=0,\ 1,\ \cdots,\ k),$$

$$|h_{ij}^{(\nu)}(x)|\leqslant C\ n^\nu,\ x\in[-1,\ 1],\ \mu=0,\ 1,\ \cdots.$$

The estimates (5) and (6) come from the properties of $T_n(x)$ and the choice of x_{λ_i}

Define

$$H_{ij}(x)=\sum_{\nu=0}^{k}b_{i\nu}^{(j)}h_{i\nu}(x),$$

where the $b_{i\gamma}^{(j)}$ are the solution of the equation

$$\begin{pmatrix} h_{i0}(y_i) & 0 & \cdots & & 0\\ h_{i0}'(y_i) & h_{i1}'(y_i) & 0 & & 0\\ h_{i0}^{(j)}(y_i) & h_{i1}^{(j)}(y_i)\cdots h_{ij}^{(j)}(y_i) & & & \\ h_{i0}^{(k)}(y_i) & h_{i1}^{(k)}(y_i)\cdots\cdots\cdots & & & h_{ik}^{(k)}(y_i)\end{pmatrix}\begin{pmatrix} b_{i0}^{(j)}\\ b_{i1}^{(j)}\\ b_{ij}^{(j)}\\ b_{ik}^{(j)}\end{pmatrix}=\begin{pmatrix}0\\0\\n^j\\0\end{pmatrix}.$$

Since the above first matrix is lower triangular with non-zero elements on diagonal, the equation has unique solution. Hence $H_{ij}(x)\in\Pi_n$ and

$$H_{ij}^{(\nu)}(y_\mu)=\delta_{i\mu}\delta_{j\nu}n^j,\ \nu,\ j=0,\ 1,\ \cdots,\ k;\ i,\ \mu=1,\ \cdots r. \tag{7}$$

Furthermore, from (5) and (6), by mathematical induction it is easy to verify

$$|b_{iv}^{(j)}|\leqslant C,$$

where C only depends on k and Y. Thus, by (6),

$$|H_{ij}^{(v)}(x)|\leqslant Cn^{v},\ x\in[-1,1],\ i=1,\cdots,r;\ v,\ j=0,1,\cdots,k. \tag{8}$$

Define

$$Q_n(x)=p_n(x)+\sum_{v=0}^{k}\sum_{\mu=1}^{r}c_{\mu v}H_{\mu v}(x).$$

We have $Q_n(x)\in\Pi_n$ and, by (7) and (3),

$$Q_n^{(j)}(y_i)=p_n^{(j)}(y_i)+c_{ij}H_{ij}^{(j)}(y_i)$$
$$=p_n^{(j)}(y_i)+\frac{f^{(j)}(y_i)-p_n^{(j)}(y_i)}{n^j}n^j=f^{(j)}(y_i),\ j=0,1,\cdots,k;\ i=1,\cdots,r.$$

From (2), (4) and (8), we have

$$|f^{(j)}(x)-Q_n^{(j)}(x)|\leqslant|f^{(j)}(x)-p_n^{(j)}(x)|+\sum_{v=0}^{k}\sum_{\mu=1}^{r}|c_{\omega}|H_{uv}^{(j)}(x)|$$
$$\leqslant Cn^{-k+j}\omega\left(f^{(k)},\frac{1}{n}\right)+\sum_{v=0}^{k}\sum_{\mu=1}^{r}Cn^{-k}\omega\left(f^{(k)},\frac{1}{n}\right)n^j$$
$$\leqslant Cn^{-k+j}\omega\left(f^{(k)},\frac{1}{n}\right).$$

This completes the proof.

(II) Now we can solve the problem of copositive polynomial approximation with the help of Theorem 2. Before proving Theorem 1, we establish the following

Theorem 3. *Let k be any positive integer. Suppose that function $f(x)\in C^k[-1,1]$ alternates in sign r times in $[-1,1]$ and $Y=\{y_i|-1<y_1<y_2<\cdots y_r<1\}$ is the set of all alternation points of $f(x)$. If for each fixed i, $1\leqslant i\leqslant r$, there exist corresponding integer j_i, $1\leqslant j_i\leqslant k$, such uhat $f^{(j)}(y_i)=0(j=0,1,\cdots,j_i-1)$ and $f^{(j_i)}(y_i)\neq 0$, then, for n sufficiently large,*

$$\tilde{E}_n(f)\leqslant Cn^{-k}\omega(f^{(k)},1/n),$$

where C only depends on Y.

Proof Noticing that $f(x)$ changes sign at y_i, we have $f(y_i)=0$, and on the assumption that $f^{(j)}(y_i)=0(j=0,1,\cdots,j_i-1)$ and $f^{(j_i)}(y_i)\neq 0$, we know that j_i must be odd numbers. Suppose $\varepsilon_i f^{(j_i)}(y_i)>0$, $\varepsilon_i=\pm 1$. Then $\varepsilon_i=-1$, if $f(x)$ changes sign at y_i from positive to negative; and $\varepsilon_i=+1$ if $f(x)$ changes sign at y_i from negative to positive. Since $f^{(k)}(x)\in C[-1,1]$, there are positive numbers ε and δ such that $\varepsilon_i f^{(j_i)}(x)\geqslant\delta>0$ for $x\in(y_i-\varepsilon,y_i+\varepsilon)$ and $i=1,\cdots,r$.

By Theorem 2, for $f(x)\in C^k[-1,1]$ and $Y=\{y_i\}$ we get $Q_n(x)\in\Pi_n$ satisfying

$$Q_n^{(j)}(y_i)=f^{(j)}(y_i)$$

and

$$|f^{(j)}(x)-Q_n^{(j)}(x)|\leqslant Cn^{-k+j}\omega(f^{(k)},1/n)\quad(j=0,1,\cdots,k;\ i=1,\cdots,r). \tag{9}$$

Therefore, for $j=0,1,\cdots,j_i-1$; $i=1,\cdots,r$, we have

$$Q_n^{(j)}(y_i)=0 \tag{10}$$

and

$$|f^{(j_i)}(x)-Q_n^{(j_i)}(x)|Cn^{-k+j_i}\omega(f^{(k)},1/n).$$

For $1\leqslant j_i\leqslant k$, the term in the right hand side of the above inequality converges to zero uniformly about $x\in[-1,1]$ and j_i as $n\to\infty$. Thus for n sufficiently large we have

$$s_iQ_n^{(j_i)}(x)\geqslant\delta/2>0,\quad x\in(y_i-s,\ y_i+s),\ i=1,\ \cdots,\ r. \tag{11}$$

According to the above statement about the relation between s_i and the alternation of $f(x)$ in sign at y_i, from (10) and (11), we can affirm that

$$Q_n(x)f(x)\geqslant0,\ x\in(y_i-s,\ y_i-s),\ i=1,\ \cdots,\ r.$$

Define

$$\rho=\min\left\{\left|\prod_{i=1}^{r}(x-y_i)\right|,\ x\in[-1,1]\Big\backslash\bigcup_{i=1}^{r}(y_i-s,\ y_i+s)\right\}$$

and

$$P_n(x)=Q_n(x)+2\rho^{-1}\|f(x)-Q_n(x)\|\cdot s_r\prod_{i=1}^{r}(x-y_i).$$

The second term on the right hand side of (13) is copositive with $f(x)$ in $[-1,$ Moreover, for

$$x\in[-1,1]\Big\backslash\bigcup_{i=1}^{r}(y_i-s,\ y_i+s),$$

the absolute value of the second term is larger than $\|f(x)-Q_n(x)\|$. Hence, f (12) and (13), for n sufficiently large we have

$$P_n(x)f(x)\geqslant0,\quad x\in[-1,1].$$

On the other hand, from (9), we obtain

$$\|f(x)-P_n(x)\|<\|f(x)-Q_n(x)\|+2\|f(x)-Q_n(x)\|\cdot\rho^{-1}\left\|\prod_{i=1}^{r}(x-y_i)\right\|$$
$$\leqslant Cn^{-k}\omega(f^{(k)},1/n).$$

This completes our proof.

Lemma, *Let k be any positive integer. Then there exists $q_n(x)\in\Pi_n$ which is increasing and satisfies*

$$|\operatorname{sgn}x-q_n(x)|\leqslant C\ (1+|nx|)^{-k-1},\ x\in[-1,1].$$

Proof Let

$$N=\left[\frac{n}{4}\right],$$

$P^*_{2N}(x)$ be the Legendae polynomial of degree $2N$, t_i $(i=1,\ \cdots,\ N)$ be positive ze of $P^*_{2N}(x)$ in increasing order

Define

$$\lambda_n(t)=c_n\frac{P^*_{2N}(t)}{(t^2-t_1^2)\cdots(t^2-t_{k+1}^2)^2},$$

where c_n is a constant such that

$$\int_{-1}^{1}\lambda_n(t)\ dt=1.$$

By the simlar method of proving Lemma 1 in [6], we can prove that

$$q_n(x)=\int_{-1}^{1}\frac{\operatorname{sgn}(x+t)+\operatorname{sgn}(x-t)}{2}\lambda_n(t)\,dt$$

an algebraic polynomial of degree $\leqslant n$ in $[-1, 1]$, and

$$|\operatorname{sgn} x-q_n(x)|\leqslant C(1+|nx|)^{-k-1},\quad x\in[-1, 1].$$

ince sgn x is odd and increasing in $[-1, 1]$, obviously $q_n(x)$ is odd and increasing)o.

Proof of Theorem 1 Write

$$Y_1(f)=\{y_i|y_i\in Y;\ f^{(j)}(y_i)=0,\ j=0, 1, \cdots, j_i-1;\ f^{(j_i)}(y_i)\neq 0,\ 1\leqslant j_i\leqslant k\}$$

nd

$$Y_2(f)=\{y_i|y_i\in Y,\ y_i\overline{\in}Y_1(f)\}.$$

[ence $Y=Y_1(f)\cup Y_2(f)$, and

$$Y_2(f)=\{y_i|\ f^{(j)}(y_i)=0,\ j=0, 1. \cdots, k\}.$$

Ve shall prove this theorem by mathematical induction. If $Y_2(f)$ is an empty set, hen Theorem 3 gives the desirsed conclusion. Now suppose we have established 'heorem 1 for the case that $Y_2(f)$ has s points, we shall show that Theorem 1 is still alid if $Y_2(f)$ has $s+1$ points.

Without loss of generality, we assume that $x=0\in Y_2(f)$. Define

$$\tilde{f}(x)=\begin{cases}-f(x), & -1\leqslant x\leqslant 0,\\ f(x), & 0<x\leqslant 1.\end{cases}$$

lecause $f(x)\in C^k[-1, 1]$ and $f^{(j)}(0)=0$, $j=0, 1, \cdots, k$, we have $\tilde{f}(x)\in C^k[-1, 1]$ nd

$$\omega(\tilde{f}^{(k)}, \delta)\leqslant 2\omega(f^{(k)}, \delta). \tag{14}$$

.lso $\tilde{f}(x)$ alternates in sign finitely many times, and

$$Y_1(\tilde{f})=Y_1(f),\ Y_2(\tilde{f})=Y_2(f)\backslash\{0\}.$$

'hat means $Y_2(\tilde{f})$ has s points. By the induction assumption, there exists

$$\tilde{P}_n(x)\in\Pi_{[n/2]}$$

uch that $\tilde{f}(x)\tilde{P}_n(x)\geqslant 0$ and

$$|\tilde{f}(x)-\tilde{P}_n(x)|\leqslant Cn^{-k}\omega(\tilde{f}^{(k)}, 1/n)\leqslant Cn^{-k}\omega(f^{(k)}, 1/n). \tag{15}$$

On the other hand, from Lemma, we have $q_n(x)\in\Pi_{[n/2]}$ which is odd and ncreasing, and satisfies

$$|\operatorname{sgn} x-q_n)|\leqslant C(1+|nx|^{-k-1},\quad x\in[-1, 1]. \tag{16}$$

Define

$$P_n(x)=\tilde{P}_n(x)q_n(x).$$

'hen $P_n(x)\in\Pi_n$, $f(x)P_n(x)=(\tilde{f}(x)\operatorname{sgn} x)\cdot(\tilde{P}_n(x)q_n(x))=(\tilde{f}(x)\tilde{P}_n(x))(\operatorname{sgn} xq_n(x))$ $\geqslant 0$. We only need to prove

$$|f(x)-P_n(x)|\leqslant Cn^{-k}\omega\left(f^{(k)}, \frac{1}{n}\right).$$

Since

$$f(x)-P_n(x)=\tilde{f}(x)\ \mathrm{sgn}\ x-\tilde{P}_n(x)q_n(x)$$
$$=[\tilde{f}(x)-\tilde{P}_n(x)]\ \mathrm{sgn}\ x+\tilde{P}_n(x)\,[\mathrm{sgn}\ x-q_n(x)]=I_1+I_2,$$

and from (15),

$$|I_1|\leqslant Cn^{-k}\omega(f^{(k)},\ 1/n),$$

it remains to show the similar estimate for I_2.

Noticing that $\tilde{f}(x)\in C^k[-1,1]$ with $\tilde{f}^{(j)}(0)=0$, $j=0, 1, \cdots, k$, and (14), we have

$$|\tilde{f}(x)|=|\tilde{f}(x)-\tilde{f}(0)|=|x|\cdot|\tilde{f}'(\theta x)|=|\tilde{f}'(\theta x)-\tilde{f}'(0)|$$
$$\leqslant\cdots\leqslant|x|^k\cdot|\tilde{f}^{(k)}(\theta_1 x)-\tilde{f}^{(k)}(0)|\leqslant|x|^k\omega(\tilde{f}^{(k)},\ |x|)$$
$$\leqslant 2|x|^k\omega(f^{(k)},\ |x|).$$

If $|x|<1/n$, it follows by (15) and (17) that

$$|\tilde{P}_n(x)|\leqslant|\tilde{f}(x)|+C^{-k}\omega(f^{(k)},\ 1/n)\leqslant Cn^{-k}\omega(f^{(k)},\ 1/n),$$

which implies

$$|I_2|\leqslant Cn^{-k}\omega(f^{(k)},\ 1/n).$$

If $|x|\geqslant 1/n$, from (15), (16) and (17), we have

$$|I_2|\leqslant|\mathrm{sgn}\,x-q_n(x)|\cdot|\tilde{f}(x)-\tilde{f}_n(x)|+\tilde{P}|\tilde{f}(x)|\cdot|\mathrm{sgn}\ x-q_n(x)|$$
$$\geqslant C_n^{-k}\omega(f^{(k)},\ 1/n)+C|x|^k\omega(f^{(k)},\ |x|)\,|nx|^{-k-1}$$
$$\leqslant C_n^{-k}\omega(f^{(k)},\ 1/n)+C|x|^k(1+n|x|)\omega(f^{(k)},\ 1/n)\,|nx|^{-k-1}$$
$$\leqslant C_n^{-k}\omega(f^{(k)},\ 1/n).$$

Combining the above inequality with (18) completes the proof.

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