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ON THE TOPOLOGICALLY CONJUGATE CLASSES OF ANOSOV ENDOMORPHISMS ON TORI

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Abstract

This paper considers the following question: Given an Anosov endomorphism f on T^m, whether f is topologically conjugate to some hyperbolic toral endomorphism? It is well known that the answer for Anosov diffeomorphisms and expanding endomorphisms is affirmative. Hwever for the remainder Anosov endomorphisms, a quite different answer is obtained in this paper, i. e., for generic Anosov endomorphisms, they are not topologically conjugate to any hyperbolic toral endomorphism.

§ 0. Introduction

Many authors have worked extensively on the topologically conjugate classes of ıosov diffeomorphisms and expanding maps on compact manifolds (see [1—4]. ıeir results imply the following assertion.

Theorem A[3,4]. *If $f:T^m\to T^m$ is an Anosov diffeomorphism (expanding map), ₙn f is topologically conjugate to some hyperbolic toral automorphism (expanding toral ıomorphism, respectively).*

Because Anosov diffeomorphisms and expanding maps are included in Anosov domorphisms, we have a similar problem: is every Anosov endomorphism on T^m ɔologically conjugate to some hyperbolic toral endomorphism?

Compared with Theorem A, our solution for this problem is quite different.

Let $A(T^m)=\{f:\ T^m\to T^m:f$ is a C^1 Anosov endomorphism$\}$ and $A^*(T^m)=$ $\in A(T^m):f$ is neither an Anosov diffeomorphism nor an expanding map$\}$. In is paper, we will prove the following result.

Main Theorem. *Suppose $m\geqslant 2$. Then for generic $f\in A^*(T^m)$, f is not ɔlogically conjugate to any hyperbolic toral endomorphism.*

In [5], Przytyoki made some discussions on the topologically conjugate classes of (T^m).

§ 1 is some preliminaries. In § 2 we prove a topologically conjugate theorem to ɔe used in § 3. The complete proof of our main theorem is contained in § 3.

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§ 1. Preliminaries

In this paper, we always suppose M is a compact connected smooth Riemannian manifold. Let $\pi : \overline{M} \to M$ be a universal covering map. Let $\alpha = \langle , \rangle$ be a Riemannian metric tensor of M. α induces a metric d on M. Let $\pi * (\alpha)$ be the pull back of α by π. Then $\pi * (\alpha)$ is aRiemannian metric tensor of $\overline{M}$ and induces a metric $\bar{d}$ on $\overline{M}$.

Lemma 1. 1. [7] *Let $f \in C^0(M, M)$ and let $F: \overline{M} \to M$ be a lifting of f to $\overline{M}$ e., $\pi F = f\pi$). Then F is uniformly continuous (with respect to $\bar{d}$). If f is a l homeomorphism of M, then F is a homeomorphis m of $\overline{M}$, and F^{-1} is also unifor continuous.*

To consider the perturbations of a dynamical system, we need a lemma to used in this paper.

Lemma 1. 2. *Let f, g be two local diffeomorphisms and let F, G be the lift of f, g to $\overline{M}$. If g is C^1 near f, and $\bar{d}\ (F(x_0), G(x_0))$ is small for some $x_0 \in \overline{M}$, the is C^1 near F and G^{-1} is also C^1 near F^{-1}.*

Proof It is tedious, we will omit it. When $M = T^m$, one can refer to § 1, Cha 5 in [8].

Definition 1. 3. *An orbit of $f \in C^0(M, M)$ is a bi-infinite sequence (x_n) in such that $f(x_n) = x_{n+1}$, for all $n \in Z$. Let $Orb\ (f) = \{(x_n) : (x_n)$ is an orbit of $f\}$. I easy to prove that $Orb(f)$ is compact in $\prod_{-\infty}^{+\infty} M$.*

The following is the definition of Anosov endomorphisms.

Definition 1. 4. [5] *We call a local diffeomorphism $f \in C^1\ (M, M)$ an Ar endomorphism if f possesses the following properties:*

There exist constants $C > 0$, $0 < \mu < 1$ such that for every $(x_n) \in Orb(f)$ there splitting of $\bigcup_{n=-\infty}^{\infty} T_{x_n}M = E^s \oplus E^u = \bigcup_{n=-\infty}^{\infty} E^s_{x_n} \oplus E^u_{x_n}$ which is preserved by the deriv Tf and satisfies the following conditions:

$$\|Tf^n v\| \leqslant C\mu^n \|v\|, \text{ for } v \in E^s,\ n = 0, 1, 2, \cdots,$$

$$\|Tf^n v\| \geqslant C^{-1}\mu^{-n} \|v\|, \text{ for } v \in E^u,\ n = 0, 1, 2, \cdots$$

Remarks. (1) If f is also a diffeomorphism, then f is called an An diffeomorphism.

(2) If $E^s = \{0\}$, then f is called an expanding map.

In [6], Mañé & Pugh gave a definition of weakly Anosov endomorphisms. not difficult to prove that an Anosov endomorphism (as Def. 1.4) is a weakly An endomorphism in [6].

The following are some simple examples of Anosov endomorphisms.

Example 1. 5. Hyperbolic Toral Endomorphism: Let A be a hyperbolic tomorphism of R^m which preserves the lattice Z^m, i. e., A is an integral matrix d for every eigenvalue λ of A, $|\lambda|\neq 0, 1$. Let $T^m=R^m/Z^m$ be the m-dimensional rus. Then A induces an Anosov endomorphism a (we call it a Hyperbolic Toral ndomorphism), namely, $a([x])=[Ax]$, where $[x]$ denotes points on T^m.

Let $A(M)=\{f: M\to M: f$ is an Anosov endomorphism$\}$ and $A^*(M)=A(M)\setminus$ Anosov diffeomorphisms$\}\cup\{$expanding maps$\})$.

For the properties of Anosov endomorphisms, see [5, 6]. The following are me of these to be used in this paper.

Proposition 1. 6. [5,6] (1) *Let $f\in A(M)$ and let F be the lifting of f to $\overline{M}$. hen F is an Anosov diffeomorphism of $\overline{M}$.*

(2) *$A(M)$ is an open subset in $C^1(M, M)$ (endowed with C^1 topology).*

(3) *If $f\in A^*(M)$, then f is not structurally stable.*

(4) *$A^*(T^m)$ is also open in $C^1(M, M)$.*

The following are some elementary facts in algebraic topology. Let $\pi: R^m\to T^m$ the covering map, $\pi(x)=[x]$. Then the group of the deck transformations $\Gamma=$ (R^m, π) consists of the following maps in R^m:

$$a_k(x)=x+k, \text{ for } x\in R^m, \text{ where } k\in Z^m.$$

A group endomorphism H of Γ can be represented by an integral matrix B $M_m(Z)$, namely, $H(a_k)(x)=x+Bk=a_{B\cdot k}(x)$ for $x\in R^m$.

Proposition 1. 7. (1) *Let $f\in C^0(T^m, T^m)$ and let F be its lifting to R^m. Then re exists a unique matrix $B\in M_m(Z)$ such that $F(x+k)=F(x)+Bk$ for $x\in R^m$, Z^m, i. e., $F-B\in P(R^m)=\{\psi: R^m\to R^m: \psi$ is continuous and satisfies $\psi(x+k)=\psi(x)$ r all $x\in R^m$ and $k\in Z^m\}$. Conversely, if a map $F\in C^0(R^m, R^m)$ satisfies $F-B\in P(R^m)$ r some $B\in M_m(Z)$, then F is a lifting of some $f\in C^0(T^m, T^m)$.*

(2) *Let $F_j(j=1, 2)$ be two liftings of a map $f\in C^0(T^m, T^m)$ and $B_j\in M_m(Z)$ such at $F_j-B_j\in P(R^m)(j=1, 2)$. Then $B_1=B_2$.*

Proof For (1), see Shub ([9], p. 179) or Spanier [10]. From [10], we have $(x)=F_1(x)+k_0$ for all $x\in R^m$ and some $k_0\in Z^m$, which concludes (2).

Now we introduce some fnnctional spaces. Suppose $\|\cdot\|$ is a norm in R^m. Let $B(R^m)=\{\psi: R^m\to R^m: \psi$ is continuous and bounded$\}$. For $\psi\in CB(R^m)$, $\|\psi\|_0=$ $p\{\|\psi(x)\|: x\in R^m\}$. Then $CB(R^m)$ is a Banach space. Let $UCB(R^m)=\{\psi\in CB(R^m)$: is uniformly continuous$\}$. It is easy to prove the following lemma.

Lemma 1. 8. *$UCB(R^m)$ is closed in $CB(R^m)$, therefore a Banach space. Also $R^m)$ is closed in $UCB(R^m)$.*

§ 2. A Topologically Conjugate Theorem

Let $f\in A(M)$ and let F be its lifting to $\overline{M}$. Then F is an Anosov diffeomorphism of $\bar{M}$. Recall that f is not structurally stable unless f is a diffeomorphism or an expanding map. However, the universality of $\bar{M}$ and the fact that F is a lifting system can conclude the following theorem.

Theorem 2. 1. *If $g\in C^1(M, M)$ is sufficiently C^1 near f, and G is a lifting of g such that G is C^1 near F, then there exists a unique homeomorphism $H:\bar{M}\to\bar{M}$ such that H is near $id_{\bar{M}}$ and satisfies $HF=GH$. In fact, H and H^{-1} are uniformly continuous.*

Proof In [6, p. 180] Mañé & Pugh asserted the first half of this theorem. idea of the proof coincides the proof of the structural stability of an An diffeomorphism on a compact manifold. For the details, we can see Chapters 13 in [8]. Similarly, we need only solve the following equation

$$\gamma(x)=\exp_x^{-1}(G(\exp_{F^{-1}(x)}(\gamma(F^{-1}(x))))),\ x\in M. \quad (\text{}$$

In order to prove that γ is uniformly continuous, we solve (2.1) in the Banach s $USB(TM)=\{\sigma:\bar{M}\to T\bar{M}\mid \sigma$ is a uniformly continuous, bounded section$\}$. existence of solution $\gamma\in USB(T\bar{M})$ of (2.1) follows from the local stabilit hyperbolic fixed point. For the verification of hyperbolity, one needs the follov generalized Palais Lemma in nonlinear analysis.

The followiing is the statement of the Lemma.

Let (E, π, X, R^m) be a vector bundle endowed with a Finsler structure. Ass $u: X\to X$ is a homeomorphism such that u and u^{-1} are uniformly continu Assume $U: E\to E$ is a fibre preserving map which covers u, i. e., $\pi U=u\pi$. Ass U is also uniformly continuous and maps a bounded section to a bounded sec Then U induces a map $\tilde{U}: USB(E)\to USB(E)$ in the following way:

$$\tilde{U}(\sigma)=U\sigma u^{-1},\ \sigma\in USB(E). \quad (\text{}$$

Here $USB(E)$ is the space of the uniformly continuous, bounded sections of $(E$ $X, R^m)$.

Generalized Palais Lemma 2. 2. *Let u, U, $\tilde{U}$ be as above. Assume the derivative $\tilde{D}U: E\to L(E, u*(E))$ of U is uniformly continuous. Then differentiable. Yet for $\sigma\in USB(E)$, the derivative $D\tilde{U}(\sigma)$ of $\tilde{U}$ at σ can be repres as follows:*

$$(D\tilde{U}(\sigma)\tau)(x)=\tilde{D}U(\sigma(u^{-1}(x)))\tau(u^{-1}(x)) \text{ for } x\in X,\ \tau\in USB(E). \quad (\text{}$$

Proof The proof of the Palais Lemma [8, p. 212] is also valid under assump-tions.

As an application of Theorem 2.1, weprove a property of Anosov endomorph which is called the Entropy Stability Property. In [11], Yang proved this prop for strongly Anosov endomorphisms. For the definition of the topological entrop a uniformly continuous map, we can see Chapter 7 in [12].

Theorem 2. 3. *Let $f: M\to M$ be an Anosov endomorphism. Then there is a* *eighborhood* $\mathscr{U}\subset C^1(M, M)$ *of* f *such that for every* $g\in\mathscr{U}$, *we have* $\mathrm{ent}(g)=\mathrm{ent}(f)$. *Iere* $\mathrm{ent}(f)=\mathrm{ent}_d(f)$ *denotes the topological entropy of* f *(relative to the metric* d*)*.

Proof Here we only sketch the proof.

(1) Let $\pi: \overline{M}\to M$ be the universal covering map. Then there is a constant $\delta>0$ ıch that

$$\pi|_{B(x,\delta)}: B(x, \delta)\to B(\pi(x), \delta)$$

an isometric homeomorphism for all $x\in\overline{M}$. Here

$$B(x, \delta)=\{y\in\overline{M}\mid d(y, x)\leqslant\delta\}.$$

(2) Let $f\in C^0(M, M)$. Suppose $F: \overline{M}\to\overline{M}$ is a lifting of f. By Theorem 8.12 in [2], we have

$$\mathrm{ent}(f)=\mathrm{ent}_d(f)=\mathrm{ent}_{\bar d}(F).$$

(3) The topological entropy is an invariant of uniformly topological conjugacy. ore precisely, let f, g be two uniformly continuous maps. Suppose there is a meomorphism h such that h and h^{-1} are uniformly continuous and h satisfies $hf=$. Then $\mathrm{ent}(f)=\mathrm{ent}(g)$.

From Theorem 2.1, there is a neighborhood $\mathscr{U}\subset C^1(M, M)$ of f such that for very $g\in\mathscr{U}$, we have

$$\mathrm{ent}(g)\overset{(2)}{=\!=}\mathrm{ent}_{\bar d}(G)\overset{(3)}{=\!=}\mathrm{ent}_{\bar d}(F)\overset{(2)}{=\!=}\mathrm{ent}(f).$$

§3. The Proof of our Main Theorem

Let $A(T^m)$ and $A^*(T^m)$ be as in §0, §1. Let $A^{**}(T^m)=\{f\in A^*(T^m)$: f is no tt- ›logically conjugate to any hyperbolic toral endomorphism$\}$. We need to prove ıt $A^{**}(T^m)$ is a residual subset in $A^*(T^m)$.

Let $\mathrm{HTE}(T^m)=\{a: T^m\to T^m$: a is a hyperbolic toral endomorpism$\}$ and

$$\mathrm{HOM}(T^m)=\{h: T^m\to T^m: h \text{ is a homeomorphism}\}.$$

Proposition 3.1. *If* $m\geqslant 2$, *then* $A^{**}(T^m)$ *is dense in* $A^*(T^m)$.

Proof Aussme the conclusion is not true. Then there exists a nonempty open set $\mathscr{U}\subset A^*(T^m)$ such that $\mathscr{U}\cap A^{**}(T^m)=\phi$. We will prove that every $f\in\mathscr{U}$ is ıcturally stable, which contradicts Proposition 1.6(3).

Fix $f_0\in\mathscr{U}$, then there exist $a_0\in\mathrm{HTE}(T^m)$ and $h_0\in\mathrm{HOM}(T^m)$ such that

$$h_0a_0=f_0h_0. \tag{3.1}$$

›n $h_0([0])=[x_0]$ is a fixed point of f_0, and therefore a hyperbolic fixed point. any $f\in\mathscr{U}$ near f_0, there is a fixed point $[x_0']$ of f near $[x_0]$. Let h' be the eomorphism of T^m: $h'([x])=[x+x_0'-x_0]$ and

$$f_1=(h')^{-1}fh'. \tag{3.2}$$

f_1 is also C^1 near f_0. We may assume $f_1\in\mathscr{U}$. Thus there exist $a_1\in\mathrm{HTE}(T^m)$ and $h_1\in$ $\mathrm{HOM}(T^m)$ such that

$$h_1 a_1 = f_1 h_1. \tag{3.3}$$

Thus $[x_1] = h_1([0])$ is a fixed point of f_1.

Let F_j, A_j and H_j be liftigns of f_j, a_j and h_j such that $F_j(x_j) = x_j$, $A_j(0) = 0$ and $H_j(0) = x_j (j=0, 1)$. By Proposition 1. 7 (1), there exist integral matrixes $C_j \in \mathrm{SL}(m, Z) = \{A \in M_m(Z) : \det(A) = \pm 1\}$ and $D_j \in M_m(Z)$ such that $H_j - C_j$, $F_j - D_j \in P(R^m)$ $(j=0, 1)$. From (3.1) and (3.3), we have

$$H_j A_j = F_j H_j, \ j=0, 1. \tag{3.4}$$

This together with Proposition 1.7 (2) implies

$$C_j A_j = D_j C_j, \ j=0, 1. \quad (\text{[illegible]}$$

Let F'_1 be another lifting of f_1, such that $F'_1(x_0) = x_0$ (because $f_1([x_0]) = [x$ Then $F'_1 - C_1 \in P(R^m)$, by Proposition 1.7 (2). As f_1 is homotopic to f_0,

$$f_{1*[x_0]} = f_{0*[x_0]} : \pi_1(T^m, [x_0]) \to \pi_1(T^m, [x_0]).$$

Thus

$$D_1 = D_0. \quad (\text{[illegible]}$$

(3.5) and (3.6) imply

$$A_1 = C^{-1} A_0 C, \quad (\text{[illegible]}$$

where $C = C_0^{-1} C_1 \in SL(m, Z)$.

From (3.4) and (3.7), we obtain

$$H'' F_0 = F_1 H'', \quad (\text{[illegible]}$$

where $H'' = H_1 C^{-1} H_0^{-1}: R^m \to R^m$. It is easy to check that H'' satisfies $H''(x+$ $H''(x) + C_1 C^{-1} C_0^{-1} k = H''(x) + k$ for $x \in R^m$, $k \in Z^m$. Moreover, $(H'')^{-1}(x+$ $(H'')^{-1}(x) + k$. From Proposition 1.7 (1), H'' induces a homeomorphism $\in \mathrm{HOM}(T^m)$.

Projecting (3.8) to T^m, we have

$$h'' f_0 = f_1 h''. \quad (\text{[illegible]}$$

From (3.2) and (3.9), we have

$$(h'h'') f_0 = f (h'h'').$$

This means $h' h''$ is a topological conjugacy between f_0 and f.

In the same way, we can prove the following result.

Proposition 3. 2. *$A^*(T^m)$ contains infinitely many topologically conj classes.*

Proof Let $a_0 \in \mathrm{HTE}(T^m)$ which is neither a hyperbolic toral automorp nor an expanding toral endomorphism. Suppose $\mathscr{U}$ is any neighborhood of $C^1(T^m, T^m)$. We can prove that there exists $f \in \mathscr{U}$ such that f is not topologi conjugate to any $a \in \mathrm{HTE}(T^m)$.

To prove $A^{**}(T^m)$ is a residual subset, we need a decomposition $A^{**}(T^m) =$ where every Q_i is open, dense in $A^*(T^m)$. Now we begin working out Q_i.

Step 1: As $\mathrm{HTE}(T^m)$ is countable, we assume $\mathrm{HTE}(T^m) = \{a_1, a_2, \cdots\}$. Let $\{f \in A^*(T^m) : f$ is not topologically conjugate to $a_i\}$. Then $A^{**}(T^m) = \bigcap_i R_i$.

Step 2: Let $R'_i = A^*(T^m) \backslash R_i = \{f \in A^*(T^m)$: there exists $h \in \mathrm{HOM}(T^m)$ such that

$hf=a_t h\}$. We construct a Procedure (*) about the liftings of maps in the following ay:

(*) Let $f\in A^*(T^m)$. Suppose there exist $a\in \mathrm{HTE}(T^m)$ and $h\in \mathrm{HOM}(T^m)$ s. t. $=ah$. Then $[x_f]=h^{-1}([0])$ is a fixed point of f. Let F, H, A be the liftings of f, a, satisfying $F(x_f)=x_f$, $A(0)=0$, $H(x_f)=0$. Then we have

$$HF=AH. \tag{3.10}$$

$h\in \mathrm{HOM}(T^m)$, so there is a matrix $C\in \mathrm{SL}(m, Z)$ such that $H-C\in P(R^m)$.

Let $\mathrm{SL}(m, Z)=\{B_1, B_2, \cdots\}$. $R_{ij}=\{f\in R_i'$: the matrix C in Procedure (*) is ual to $B_j\}$. Then we have

$$A^{**}(T^m)=\bigcap_{ij}(A^*(T^m)\setminus R_{ij}). \tag{3.11}$$

Proposition 3.3 *Every R_{ij} is closed in $A^*(T^m)$.*

Proof Fixing $a\in \mathrm{HTE}(T^m)$ and $B\in \mathrm{SL}(m, Z)$, let $R=\{f\in A^*(T^m)$: there ists $h\in \mathrm{HOM}(T^m)$ such that the matrix in (*) is just $B\}$.

Let $f_n\in R(n=1, 2, \cdots)$ and $f_n\to f_0\in A^*(T^m)$ for some f_0 as $n\to\infty$. We need to ove $f_0\in R$.

By the definition of R, there exist $h_n\in \mathrm{HOM}(T^m)$ such that

$$h_n f_n=a h_n,\ n=1, 2, \cdots. \tag{3.12}$$

For $n=1, 2, \cdots$, let $[x_n]=h_n^{-1}([0])$. Then the liftings H_n, F_n in (*) satisfy $F_n(x_n)$ x_n, $H_n-B\in P(R^m)$ and

$$H_nF_n=AH_n,\ n=1, 2, \cdots. \tag{3.13}$$

the compactness of T^m, we may assume that $\{[x_n]\}$ converges to some $[x_0]\in T^m$. e also may assume $\{x_n\}$ converges to x_0. It is easy to check that $f_0([x_0])=[x_0]$. As (*), let F_0 be the lifting of f_0 to R^m, satisfying $F_0(x_0)=x_0$. By Lemma 1.2, $F_n\to$ as $n\to\infty$.

Consider the following functional equations

$$(B+\psi)F_n=A(B+\psi),\ n=0, 1, 2, \cdots. \tag{3.14}$$

om (3.13), for $n=1, 2, \cdots$, $\psi=\xi_n=H_n-B\in P(R^m)$ is a solution of (3.14).

Because A is a hyperbolic automorphism of R^m, there is a norm $\|.\|$ in R^m and a perbolic splitting $R^m=E^s\oplus E^u$ and a constant $\tau\in(0, 1)$ s. t. $AE^s=E^s$, $AE^u=E^u$ d

$$\|Av\|\leqslant\tau\|v\|,\ \text{for } v\in E^s;\ \|Av\|\geqslant\tau^{-1}\|v\|,\ \text{for } v\in E^u.$$

Let P_s, P_u be the projections of R^m to E^s, E^u resp., and $A_s=A|E^s$, $A_u|E^u$.

By Lemma 1.8, $\mathrm{UCB}(R^m)$ with $\|.\|_0$ is a Banach space. We have a splitting of $\mathrm{CB}(R^m)=UC^s\oplus UC^u$, here $UC^{s(u)}=\{\psi\in \mathrm{UCB}(R^m)$: $\psi(x)\in E^{s(u)}$ for $x\in R^m\}$. For $\psi\in$ $\mathrm{CB}(R^m)$, we have the decomposition $\psi=\psi^s+\psi^u=P_s\psi+P_u\psi$. Define an equivalent rm $\|.\|$ on $\mathrm{UCB}(R^m)$ as follows

$$\|\psi\|=\|\psi^s+\psi^u\|=\max(\|\psi^s\|_0,\ \|\psi^u\|_0).$$

Let $\phi_n(x)=A^{-1}BF_n(x)-Bx$. From (3.13), it is easy to prove $\phi_n\in P(R^m)$ for $n=1$, 2, $\cdots$. So $\phi_0\in P(R^m)$ because $F_n\to F_0$.

Now (3.14) is equivalent to $\psi=A^{-1}\psi F_n+\phi_n$. Projecting this equation to E^s, E^u resp., we have

$$\begin{cases}\psi^s=A_s^{-1}\psi^s F_n+\phi_n^s,\\ \psi^u=A_u^{-1}\psi^u F_n+\phi_n^u,\end{cases}$$

i. e.,

$$\begin{cases}\psi^s=A_s\psi^s F_n^{-1}-A_s\phi_n^s F_n^{-1}\overset{\text{def.}}{=\!=}T_n^s(\psi),\\ \psi^u=A_u^{-1}\psi^u F_n+\phi_n^u\overset{\text{def.}}{=\!=}T_n^u(\psi).\end{cases}\tag{3.}$$

Because F_n, F_n^{-1}, ϕ_n^s, ϕ_n^u are uniformly continuous, we can define maps $\mathrm{UCB}(R^m)\to\mathrm{UCB}(R^m)$ as follows:

$$T_n(\psi)=T_n(\psi^s+\psi^u)\overset{\text{def}}{=\!=}T_n^s(\psi)+T_n^u(\psi)\ \text{for}\ \psi\in\mathrm{UCB}(R^m).$$

Now (3.15) becomes

$$\psi=T_n(\psi),\ n=0,\ 1,\ 2,\ \cdots.\tag{3.}$$

Lemma 3.4. *For $n=0, 1, 2, \cdots$, T_n is contractive. More precisely, we have*

$$\|T_n(\psi_1)-T_n(\psi_2)\|\leqslant\tau\|\psi_1-\psi_2\|\ \text{for}\ \psi_1,\ \psi_2\in\mathrm{UCB}(R^m).$$

Proof For ψ_1, $\psi_2\in\mathrm{UCB}(R^m)$, we have

$$\begin{aligned}&\|T_n^s(\psi_1)-T_n^s(\psi_2)\|_0\\ &=\sup\{\|A_s(\psi_1^s(F_n^{-1}x)-\psi_2^s(F_n^{-1}x))\|:x\in R^m\}\\ &=\sup\{\|A_s(\psi_1^s(x)-\psi_2^s(x))\|:x\in R^m\}\\ &\leqslant\tau\|\psi_1^s-\psi_2^s\|_0\leqslant\tau\|\psi_1-\psi_2\|.\end{aligned}$$

In the samy way, we have

$$\|T_n^u(\psi_1)-T_n^u(\psi_2)\|_0\leqslant\tau\|\psi_1-\psi_2\|.$$

These two inepualities conclude the lemma.

We continue to prove Proposition 3.3.

From Lemma 3.4, for every $n=0, 1, 2, \cdots$, there exists a unique $\psi_n\in\mathrm{UCB}($ s. t. $\psi_n=T_n(\psi_n)$, i. e., ψ_n is the solution of (3.14). By the uniqueness, $\psi_n=\xi_n\in P($ for $n=1, 2, \cdots$. Now we prove $\psi_n\to\psi_0$ as $n\to\infty$. Thus $\psi_0\in P(R^m)$.

$$\begin{aligned}\|\psi_n-\psi_0\|&=\|T_n(\psi_n)-T_0(\psi_0)\|\\ &<\|T_n(\psi_n)-T_n(\psi_0)\|+\|T_n(\psi_0)-T_0(\psi_0)\|\\ &\leqslant\tau\|\psi_n-\psi_0\|+\|T_n(\psi_0)-T_0(\psi_0)\|.\end{aligned}$$

This implies

$$\|\psi_n-\psi_0\|\leqslant 1/(1-\tau)\|T_n(\psi_0)-T_0(\psi_0)\|.\tag{3.}$$

As $\psi_0\in\mathrm{UCB}(R^m)$, for arbitrary $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ s. t.

$$\|\psi_0^s(x)-\psi_0^s(y)\|<\varepsilon/2\tau\ \text{for}\ x,\ y\in R^m\ \text{and}\ \|x-y\|<\delta.\tag{3.}$$

By Lemma 1.2, $F_n\to F_0$ and $F_n^{-1}\to F_0^{-1}$ as $n\to\infty$. Thus there is

$$N_1=N_1(\varepsilon,\ \delta)=N_1(\varepsilon)\gg 1$$

such that

$$\|F_n(x)-N_0(x)\|<s_1 \text{ and } \|F_n^{-1}(x)-F_0^{-1}(x)\|<s_1 \tag{3.19}$$

r all $x\in R^m$ and $N>N_1$. Here $s_1=\min(s/(2\tau\|B^s\|),\ \delta)$.

(3.18) and (3.19) imply

$$\begin{aligned}&\|T_n^s(\psi_0)(x)-T_0^s(\psi_0)(x)\|\\ &=\|A_s(\psi_0^s(F_n^{-1}(x))-\psi_0^s(F_0^{-1}(x)))-A_sB^s(F_n^{-1}(x)-F_0^{-1}(x))\|\\ &\leqslant s \text{ for } n\leqslant N_1 \text{ and } x\in R^m.\end{aligned}$$

We can prove a similar inequality for T_n^u. Then we have proved $\|\psi_n-\psi_0\|<s$ for $\gg 1$.

Now we prove the following lemma.

Lemma 3. 5. *$H_0=B+\psi_0$: $R^m\to R^m$ is a homeomorphism and the projection h_0 ' H_0 is also a homeomorphism of T^m.*

Proof Because $f_0\in A^*(T^m)$ and $f_n\to f_0$. From Theorem 2.1, for some $N\gg 1$, ıere exists $\bar{H}_N=id_{R^m}+\bar{\psi}_N\in \mathrm{HOM}(R^m)$ such that

$$\bar{H}_N F_0=F_N\bar{H}_N, \tag{3.20}$$

here $\bar{\psi}_N\in \mathrm{UCB}(R^m)$.

(3.14) for $n=N$ and (3.20) imply

$$((B+\psi_N)(id_{R^m}+\bar{\psi}_N))F_0=A((B+\psi_N)(id_{R^m}+\bar{\psi}_N)).$$

s $(B+\psi_N)(id_{R^m}+\bar{\psi}_N)-B=\psi_N(id_{R^m}+\bar{\psi}_N)+B\bar{\psi}_N\in \mathrm{UCB}(R^m)$, the uniqueness of the ılution of (3.14) for $n=0$ implies that $B+\psi_0=(B+\psi_N)(id_{R^m}+\bar{\psi}_N)\in \mathrm{HOM}(R^m)$, ecause $B+\psi_N$ and $id_{R^m}+\bar{\psi}_N\in \mathrm{HOM}(R^m)$.

Because $\psi_0\in P(R^m)$, it is easy to prove that $H_0^{-1}=B^{-1}+\bar{\psi}_0$ for some $\bar{\psi}_0\in P(R^m)$. s B, $B^{-1}\in \mathrm{SL}(m, Z)$, from Proposition 1.7(1), the projection h_0 of H_0 to T^m is a omeomorphism.

Now projecting (3.14) for $n=0$ to T^m, we have $h_0 f_0=ah_0$. By the definition of , $f_0\in R$. This completes the proof of Proposition 3.3.

Proof of our Main Theorem Proposition 3.1 implies $A^*(T^m)\setminus R_{ij}$ is dense in $^*(T^m)$ and Proposition 3.3. asserts it is open in $A^*(T^m)$. The formula (3.11) ›mpletes the proof of the Main Theorem.

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