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ON THE INVARIANCE PRINCIPLE FOR ρ-MIXING SEQUENCES OF RANDOM VARIABLES

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Abstract

In this note the author establishes the invariance principle for ρ-mixing sequences under combinations of moment assumptions and ρ-mixing rates. The result answers a problem from a recent survey paper of Peligrad.

§1. Introduction

First some notation: log denotes the logarithm with base 2 and $\log^+ x := \max\{$ $\log x\}$. The indicator function of a set A is denoted by $I_{(A)}$. The notation $a \ll b$ mea $a = O(b)$. The greatest integer $\leqslant x$ is denoted by $[x]$. The norm in L_p is denoted $\|\cdot\|_p (p \geqslant 1)$. $N(0, 1)$ denotes the standard normal distribution. $\{W(t), 0 \leqslant t \leqslant$ denotes the standard Wiener process.

Throughout the paper we suppose that $\{X_k, k \in Z\}$ is a strictly stationa sequence of real-valued random variables on a probability space (Ω, F, P). F $-\infty \leqslant m \leqslant n \leqslant \infty$ let F_m^n denote the σ-field of events generated by the rando variables $(X_k, m \leqslant k \leqslant n)$. For each natural $n \geqslant 1$ define the dependence coefficient

$$\rho(n) := \sup_{f \in L_2(F_{-\infty}^0), g \in L_2(F_n^\infty)} |\operatorname{corr}(f, g)|.$$

The stationary sequence $\{X_k\}$ is said to be ρ-mixing if $\rho(n) \to 0$ as $n \to \infty$.

For each $n \geqslant 1$ define the partial sum $S_0 := 0$, $S_n := \sum_{k=1}^{n} X_k$ and denote by $S(t)$: $S_{[t]}$ for each $t \geqslant 0$, $\sigma_t^2 := \operatorname{var} S(t)$. Peligrad[2] proved the following weak invarian principle:

Theorem A. *Suppose $\{X_k\}$ is a strictly stationary sequence of random variabl satisfying*

$$EX_k = 0,\ EX_k^2 < \infty,\ \sigma_n \to \infty \text{ as } n \to \infty, \tag{1.1}$$

$$\sum_{k=1}^{\infty} \rho^{1/2}(2^k) < \infty. \tag{1.2}$$

For each $t \in [0, 1]$, put $W_n(t) := \sum_{k=1}^{[nt]} X_k / \sigma_n$. Then

$$W_n(t) \Rightarrow W(t) \text{ as } n \to \infty. \tag{1.3}$$

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Shao[5] showed that the condition (1.2) can be replaced by

$$\sum_{k=1}^{\infty}\rho(2^k)<\infty. \tag{1.2}'$$

Recently Peligrad in her survey paper proposed the study of the following general problem: Suppose $\{X_k\}$ satisfies (1.1) and $EX_0^2g(|X_0|)<\infty$, where g: $[0, \infty)\to[0, \infty)$ is such that

$$g(x) \text{ and } x^\delta/g(x) \text{ are increasing functions, for some } 0<\delta<1. \tag{1.4}$$

Then, under these conditions, what is the slowest mixing rate for $\rho(n)$ that will still imply that W_n is weakly convergent to W. She conjectured that: if $\{X_k\}$ is strictly stationary and satisfies

$$EX_0^2g(|X_0|)<\infty \text{ and}$$

$$g(n^{1/2})\gg\exp\Big(d\sum_{k=1}^{n}k^{-1}\rho(k)\Big) \tag{1.5}$$

as $n\to\infty$ for every $d>0$, then $W_n\Rightarrow W$.

Fortunately, Peligrad[4] has proved

Theorem B. *Let $g(x)$ satisfy* (1.4). *Suppose that $\{X_k\}$ is a strictly stationary sequence satisfying* (1.1) *and*

$$EX_0^2g(|X_0|)<\infty \tag{1.6$_a$}$$

and

$$g(n^{1/2})\gg\exp\Big((2+\varepsilon^*)\sum_{k=1}^{[\log n]}\rho(2^k)\Big) \tag{1.6$_b$}$$

for some $0<\varepsilon^*<1$. Then $S_n/\sigma_n\to N(0, 1)$ in distribution as $n\to\infty$.

We now can establish the following

Theorem. *Let $g(x)$ satisfy* (1.4). *Suppose $\{X_k\}$ is a strictly stationary sequence satisfying* (1.1), (1.6)$_a$ *and* (1.6)$_b$, *then W_n is weakly convergent to W.*

This theorem contains Theorem A. By taking $g(x)=$constant for every $x\geq0$, we get the conclusion of Theorem A under (1.2)′. By simple computation we get the following corollaries:

Corollary 1. *Assume $\{X_k\}$ is strictly stationary satisfying* (1.1) *and for some $\varepsilon>0$, and $c>0$*

$$EX_0^2(\log^+|X_0|)^{2c/(1-\varepsilon)}<\infty$$

and

$$\rho(n)\leq c\log^{-1}n \text{ for every } n \text{ sufficiently large.} \tag{1.7}$$

Then the invariance principle holds.

Corollary 2. *Assume $\{X_k\}$ is strictly stationary satisfying* (1.1) *and for some $0<\beta<1$, $\varepsilon>0$ and $c>0$*

$$EX_0^2\exp\Big(\frac{2c(1+\varepsilon)}{1-\beta}(2\log^+|X_1|)^{1-\beta}\Big)<\infty \tag{1.8}$$

and

$$\rho(n)\leq c\log^{-\beta}n \text{ for every } n \text{ suffiietly large.} \tag{1.9}$$

Then the invariance principle hilds.

Corollary 3. *Assume $\{X_k\}$ is strictly stationary satisfying* (1.1) *and for some*

$r>0$, $\varepsilon>0$ *and* $c>0$

$$EX_0^2 \exp\left(\frac{4c \log^+|X_0|}{(1-\varepsilon)(\log^+\log^+|X_0|)^r}\right)<\infty \tag{1.10}$$

and

$$\rho(n)\leqslant c\log^{-r}\log n \text{ for every } n \text{ sufficiently large.} \tag{1.11}$$

Then the invariance principle holds.

§ 2. Proof of Theorem

We shall give first two preliminary lemmas followed by the proof of Theorem

Lemma 1. *Suppose* $\{X_k\}$ *satisfies* (1.1). *We can find two positive constan* $c_1(\varepsilon^*)$ *and* $c_2(\varepsilon^*)$ *such that for every* $n\geqslant 1$

$$\sigma_n^2\leqslant c_1 nEX_0^2 \exp\left(\sum_{k=1}^{[\log n]}\rho(2^k)\left(1+\frac{1}{4}\varepsilon^*\right)\right) \tag{2.:}$$

and

$$\sigma_n^2\geqslant c_2 n \exp\left(-\left(1+\frac{1}{2}\varepsilon^*\right)\sum_{k=1}^{[(1-\varepsilon^*)\log n]}\rho(2^k)\right). \tag{2.:}$$

For the proof of this lemma see Lemma 1 in [4]. The following lemma is a mo precise form of Lemma 1 of [5].

Lemma 2. *Suppose* $\{X_k\}$ *satisfies* (1.1) *and* $E|X_0|^{2+\delta}<\infty$ *for some* $0<\delta<$: *Then there is a positive constant* c_3 *such that for every* $n\geqslant 1$

$$E|S_n|^{2+\delta}\leqslant c_3\left(\sigma_n^{2+\delta}+E|X_0|^{2+\delta} n \exp\left(30\sum_{k=1}^{[\log n]}\rho^{2/(2+\delta)}(2^k)\right)\right). \tag{2.3}$$

Proof of Theorem Shao[5] has established the invariance principle under th assumption $\sum_k \rho(2^k)<\infty$. We shall treat here the case when $\sum_k \rho(2^k)=\infty$, when w shall consider that $g(x)\to\infty$ as $x\to\infty$. Without loss of generality, we can assum that

$$\rho(n)\geqslant(\log n)^{-1}\cdot(\log^{-2}\log n) \tag{2.4}$$

for every n sufficiently large.

In order to establish the theorem, by Theorem 1.4 of [3] and Theorem 1 of [4] it suffices to show that for each positive ε there exists $\lambda>1$ such that

$$P(\max_{i\leqslant n}|S_i|\geqslant 6\lambda\sigma_n)\leqslant 6\varepsilon/\lambda^2. \tag{2.5}$$

The proof of (2.5) is somewhat similar to the proof of Lemma 2 in [5]. W shall truncate at level $J:=n^{1/2}/T$, where

$$T:=\exp\left(\frac{40}{\delta}\sum_{k=1}^{[\log n]}\rho^{2/(2+\delta)}(2^k)\right). \tag{2.6}$$

Put

$$X_{i1}=X_iI_{\{|X_i|<J\}}-EX_iI_{\{|X_i|<J\}},$$
$$X_{i2}=X_iI_{\{|X_i|>J\}}-EX_iI_{\{|X_i|>J\}},$$

$$S_{n1}(k)=\sum_{i=1}^{k}X_{i1},\ S_{n2}(k)=\sum_{i=1}^{k}X_{i2},$$
$$\sigma_{n1}^2(k)=ES_{n1}^2(k),\ \sigma_{n2}^2(k=)ES_{n2}^2(k).$$

bviously, $S_i=S_{n1}(i)+S_{n2}(i)$ and

$$P(\max_{i<n}|S_i|\geqslant 6\lambda\sigma_n)\leqslant P(\max_{i<n}|S_{n1}(i)|\geqslant\lambda\sigma_n)+P(\max_{i<n}|S_{n2}(i)|\geqslant 5\lambda\sigma_n).$$

We first note that

$$\text{og } T=\frac{40}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\leqslant\frac{40}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=1}^{[\log n/T^2]}\rho(2^i)+\frac{40}{\delta}\rho^{2/(2+\delta)}\left(\frac{u}{T^2}\right)\sum_{i=1+[\log(n/T^2)]}^{[\log n]}$$
$$\leqslant\frac{40}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=1}^{[\log(n/T^2)]}\rho(2^i)+\frac{90}{\delta}\rho^{2/(2+\delta)}\left(\frac{n}{T^2}\right)\log T.$$

lence we have for every n sufficiently large

$$\log T\leqslant\frac{50}{\delta}\rho^{-\delta/(2+\delta)}\left(\frac{n}{T^2}\right)\sum_{i=1}^{[\log(n/T^2)]}\rho(2^i)\tag{2.7}$$

ud

$$\sum_{i=1}^{[\log n]}\rho(2^i)\leqslant\left(1+\frac{\varepsilon^*}{12}\right)\sum_{i=1}^{[\log(n/T^2)]}\rho(2^i).\tag{2.8}$$

'rom this and by $(1.6)_b$ and the fact that $g(x)$ is increasing we have

$$g(J)\geqslant\exp\left(\frac{2+\varepsilon^*}{1+\varepsilon^*/12}\sum_{i=1}^{[\log n]}\rho(2^i)\right)\tag{2.9}$$

nd by (2.1), (2.2) and (2.9) for every $k\leqslant n$ and n sufficiently large

$$\sigma_{n2}^2(k)\leqslant c_1kEX_0^2I_{\{|X_0|>J\}}\exp\left(\sum_{i=1}^{[\log n]}\left(1+\frac{1}{4}\varepsilon^*\right)\rho(2^i)\right)$$
$$\leqslant\frac{c_1c_2^{-1}\sigma_k^2EX_0^2g(|X_0|)}{g(J)}\exp\left(\left(2+\frac{3}{4}\varepsilon^*\right)\sum_{i=1}^{[\log n]}\rho(2^i)\right)$$
$$\leqslant c_1c_2^{-1}\sigma_k^2EX_0^2g(|X_0|)\exp\left(\frac{\varepsilon^*}{52}\sum_{i=1}^{[\log n]}\rho(2^i)\right).$$

'rom this and because $\sum\rho(2^i)=\infty$, we deduce that

$$\max_{1<k\leqslant n}\frac{\sigma_{n2}(k)}{\sigma_k}=o(1)\ \text{as}\ n\to\infty.$$

Whence it is easy to see that for $k=1, 2, \cdots, n$ and n sufficiently large

$$\sigma_{n1}^2(k)\leqslant 2\sigma_k^2.\tag{2.10}$$

By Lemma 2 and (2.10)

$$E|S_{n1}(k)|^{2+\delta}\leqslant 4c_3\left(\sigma_k^{2+\delta}+kE|X_0|^{2+\delta}I_{\{|X_0|<J\}}\exp\left(30\sum_{i=1}^{[\log n]}\rho(2^i)\right)\right).$$

From this and by (1.1), (2.1), $(1.6)_a$ $(1.6)_b$, (2.6), (2.9) and Corollary 3 of Morio we see that there exists a constant c_4 such that

$$E\max_{1<k\leqslant n}|S_{n1}(k)|^{2+\delta}$$
$$\leqslant c_4\left(\sigma_n^{2+\delta}+n\log^{2+\delta}nE|X_0|^{2+\delta}I_{\{|X_0|<J\}}\exp\left(30\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right)\right)$$
$$\leqslant c_4\left(\sigma_n^{2+\delta}+\frac{n^{2+\delta}\log^{2+\delta}nEX_0^2g(|X_0|)}{g(J)T^\delta}c_1\exp\left(35\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\right)\right)$$
$$\leqslant c_4\sigma_n^{2+\delta}(1+EX_0^2g(|X_0|))$$

whence there exists a constant $\lambda>1$ such that for every n sufficiently large

$$p(\max_{k\leqslant n}|S_{n1}(k)|\geqslant\lambda\sigma_k)\leqslant\varepsilon/\lambda^2. \tag{2.11}$$

We now estimats $P(\max_{k\leqslant n}|S_{n2}(k)|\geqslant 5\lambda\sigma_n)$. Let

$$p=\exp\Big(\frac{50}{\delta}\sum_{i=1}^{[\log n]}\rho^{2/(2+\delta)}(2^i)\Big),$$

$$r=\Big[\frac{n}{p}\Big],\ p_1=\Big[\frac{p}{2}\Big],\ p_2=\Big[\frac{p-1}{2}\Big].$$

Put

$$y_i=\sum_{j=1+(2i-1)r}^{2ir}X_{j2},\ i=1,\ 2,\ \cdots,\ p_1;$$

$$z_i=\sum_{j=1+2ir}^{(2i+1)r}X_{j2},\ i=0,\ 1,\ \cdots,\ p_2;$$

$$T_1(i)=\sum_{j=1}^{i}y_j,\ T_2(i)=\sum_{j=0}^{i}z_j.$$

Noting that $\{X_{j2}\}_{j=1}^{n}$ is stationary we have

$$\begin{aligned}
&P(\max_{k\leqslant n}|S_{n2}(k)|\geqslant 5\lambda\sigma_n)\\
&\leqslant P(\max_{k\leqslant p_1}|T_1(k)|\geqslant 2\lambda\sigma_n)+P(\max_{k\leqslant p_2}|T_2(k)|\geqslant 2\lambda\sigma_n)\\
&\quad+(p+1)P(\max_{k\leqslant r}|S_{n2}(k)|\geqslant\lambda\sigma_n)\\
&:=I_1+I_2+I_3.
\end{aligned}$$

In terms of (2.1) and (2.2) we have for every n sufficiently large

$$\begin{aligned}
&P(\max_{k\leqslant r}|S_{n2}(k)|\geqslant\lambda\sigma_n)\\
&\leqslant P\Big(\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n-2\sum_{i=1}^{r}E|X_{n2}(i)|\Big)\\
&\leqslant P\Big(\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n-2r\,\frac{EX_0^2g(|X_0|)}{g(J)J}\Big)\\
&\leqslant P\Big(\sum_{i=1}^{r}(|X_{n2}(i)|-E|X_{n2}(i)|)\geqslant\lambda\sigma_n/2\Big)\\
&\leqslant 4c_1r\sigma_n^{-2}\exp\Big(\Big(1+\frac{1}{4}\varepsilon^*\Big)\sum_{i=1}^{[\log n]}\rho(2^i)EX_0^2I_{\{|X_0|>J\}}\cdot\lambda^{-2}
\end{aligned}$$

whence by (2.9)

$$\begin{aligned}
I_3&\leqslant 4c_1n\sigma_n^{-2}\exp\Big(\Big(1+\frac{1}{4}\varepsilon^*\Big)\sum_{i=1}^{[\log n]}\rho(2^i)\Big)EX_0^2I_{\{|X_0|>J\}}\cdot\lambda^{-2}\\
&\leqslant\frac{4c_1}{c_2\lambda^2g(J)}\exp\Big(\Big(2+\frac{3}{4}\varepsilon^*\Big)\sum_{i=1}^{[\log n]}\rho(2^i)\Big)EX_0^2g(|X_0|)\\
&\leqslant\frac{4c_1}{c_2\lambda^2}\exp\Big(-\frac{\varepsilon^*}{52}\sum_{i=1}^{[\log n]}\rho(2^i)\Big)EX_0^2g(|X_0|),
\end{aligned}$$

therefore

$$I_3\leqslant\varepsilon/\lambda^2. \tag{2.1}$$

In order to establish the estimation of I_1, let

$$G_0=(\Omega,\ \phi),\ G_k=\sigma(X_i;\ 1\leqslant i\leqslant 2rk);$$

$$u_k = E(y_k | G_{k-1}),\ k=1,\ 2,\ \cdots,\ p_1;$$

$$U_i(k) = \sum_{j=1+i}^{i+k} u_j,\ T^*(k) = T_1(k) - U_0(k).$$

ɔviously

$$I_1 \leqslant P(\max_{i=p_1} |T^*(i)| \geqslant \lambda\sigma_n) + P(\max_{i\leqslant p_2} |U_0(i)| \geqslant \lambda\sigma_n)$$

$$:= I_1^{(1)} + I_2^{(2)}.$$

ɔcause $\{T^*(i),\ i=1,\ \cdots,\ p_1\}$ is a martingale sequence, we have

$$I_1^{(1)} \leqslant \frac{16}{\lambda^2\sigma_n^2} \sum_{i=1}^{p_1} Ey_i^2.$$

ı a way somewhat similar to the estimation of I_3 we also have for every $\lambda>1$ and r every n sufficiently large

$$I_1^{(1)} \leqslant \varepsilon/\lambda^2. \tag{2.13}$$

Finally, we shall prove that for every i, k, n, by induction on k

$$EU_i^2(k) \leqslant c_1 k\rho^2(r) \log^2 2k EX_0^2 I_{\{|X_0|>J\}} \cdot r \cdot \exp\left(\left(1+\frac{1}{4}\varepsilon^*\right)\sum_{i=1}^{[\log n]} \rho(2^i)\right). \tag{2.14}$$

Vhen $k=1$, by the definition of ρ-mixing

$$EU_i^2(1) = EE^2(y_{i+1}|G_i) = E(y_{i+1}E(y_{i+1}|G_i)) \leqslant \rho(r) \|y_{i+1}\|_2 \cdot \|E(y_{i+1}|G_i)\|_2,$$

hus (2.14) is true for $k=1$ and for every $i+1\leqslant p_1$ by (2.1). When $k\geqslant 2$, assume 2.14) holds for every integer less than k. Put $k_1=[k/2]$, $k_2=k-k_1$, then

$$EU_i^2(k) = EU_i^2(k_1) + EU_{i+k_1}^2(k_2) + 2EU_i(k_1)U_{i+k_1}(k_2)$$

$$= EU_i^2(k_1) + E_{i+k_1}^2(k_2) + 2EU_i(k_1) \sum_{j=1+i+k_1}^{k+i} y_j$$

$$\leqslant EU_1^2(k_1) + EU_{i+k_1}^2(k_2) + 2\|U_1(k_1)\|_2 \cdot \|\sum_j y_j\|_2 \rho(r).$$

3y induction hypothesis and (2.1)

$$EU_i^2(k) \leqslant c_1(k_1 \log^2 2k_1 + k_2 \log^2 2k_2 + 2(k_1k_2)^{1/2} \log 2k_1).$$

$$\cdot \rho^2(r) \cdot r \exp\left(\left(1+\frac{1}{4}\varepsilon^*\right)\sum_{j=1}^{[\log n]} \rho(2^j)\right) EX_0^2 I_{\{|X_0|>J\}}$$

$$\leqslant c_1 k (\log^2 2k) r \cdot \exp\left(\left(1+\frac{1}{4}\varepsilon^*\right)\sum_{j=1}^{[\log n]} \rho(2^j)\right) EX_0^2 I_{\{|X_0|>J\}} \cdot \rho^2(r),$$

which proves that (2.14) holds.

From (2.14) we obtain by Corollary 4 of Moricz

$$E \max_{i\leqslant p_1} U_0^2(i)$$

$$\leqslant 3c_1 r p_1 \rho^2(r) \log^4(2p_1) \cdot \exp\left(\left(1+\frac{1}{4}\varepsilon^*\right)\sum_{j=1}^{[\log n]} \rho(2^j)\right) EX_0^2 I_{\{|X_0|>J\}}$$

$$\leqslant \frac{3c_1\sigma_n^2 \rho^2\left(\frac{n}{p_1}\right) \log^4(2p_1)}{c_2 g(J)} \exp\left(\left(2+\frac{3}{4}\varepsilon^*\right)\sum_{j=1}^{[\log n]} \rho(2^j)\right) EX_0^2 g(|X_0|)$$

$$\leqslant \frac{3c_1\sigma_n^2 \sigma'^2\left(\frac{n}{p_1}\right) \log^4(2p_1)}{c_2} \exp\left(-\frac{\varepsilon^*}{52}\sum_{j=1}^{[\log n]} \rho(2^j)\right) EX_0^2 g(|X_0|)$$

By (2.7)

$$\rho^2\left(\frac{n}{p_1}\right)\log^4 2p_1\leqslant\left(\frac{50}{\delta}\right)^4\rho^{2/3}\left(\frac{n}{T^2}\right)\sum_{j=1}^{[\log n]}\rho(2^j),$$

hence we finally get that for every $\lambda>1$ and for every n sufficiently large

$$I_1^{(2)}\leqslant\varepsilon/\lambda^2,$$

therefore

$$I_1\leqslant 2\varepsilon/\lambda^2. \tag{2.15}$$

Similiarly, we have

$$I_2\leqslant 2\varepsilon/\lambda^2. \quad (2.$$

(2.5) now follows from (2.11)-(2.12) and (2.15)-(2.16), which proves theorem.

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