Chin. Ann. of Math.
10B (4) 1989

GENERALIZED DILATIONS OF OPERATOR SEQUENCES TO KREIN SPACE

WU JINGBO (伍镜波)*

Abstract

In this paper, the author considers the generalized dilations of operator sequences in a Hilbert space to a Krein space. In order to obtain the unitary and self-adjoint dilations, only some boundedness and symmetry assumptions are needed.

In the dilation theory it is used to consider that the dilation space is a Hilbert ace[5,6]. Davis[2,3] has considered the unitary dilations of a single operator and a niformly continuous semigroup to a Krein space. In this paper we consider the neralized dilations of operator sequences in a Hilbert space to a Krein space. For $T_k\}$, an operator sequence in a Hilbert space H, we try to find a Krein space K, an erator B from H into K and an operator A on K such that

$$T_k=B^+A^kB,$$

here B^+ is the dual of B with respect to the inner product of H and the indefinite ner product of K. Theorems 2 and 4 deal with unitary and self-adjoint dilations, spectively.

Throughout this paper, by an operator we mean a linear bounded transformation nless it is otherwise stated. The Banach algebra of all operators on Hilbert space H denoted by $L(H)$. The inner product of Hilbert space is denoted by $(.,\ .)$ and definite inner product of Krein space is denoted by $\langle .,\ .\rangle$. For fundamental mmetry J, the J-inner product is denoted by $[.,\ .]_J$ or simply by $[.,\ .]$. As for her notions and symbols of Krein space, we refer to [1].

Theorem 1. *Let H be a Hilbert space and $\{T_k\}_{-\infty}^{\infty}$ a doubly infinite sequence of erators in H. Assume that there is an $M>0$ and an a, $0<a<1$, such that $T_k=T^*_{-k}$ d $\|T_k\|\leqslant Ma^{|k|}$ for all $k=0, \pm1, \pm2, \cdots$. Then*

(a) *There is a fundamentally reducible unitary operator U in a Krein space K ch that*

$$T_k=B^+U^kB \quad (k=0,\ \pm1,\ \pm2,\ \cdots), \tag{1}$$

here B is an operator on H into K and B^+, defined by

$$(B^+f,\ x)=\langle f,\ Bx\rangle \quad (\forall f\in K,\ \forall x\in H), \tag{2}$$

Manuscript received March 24, 1987.

* Department of Computer and System Sciences, Nankai University, Tianjin, China.

is an operator on K into H.

(b) *K is spanned by the elements of the form $U^k Bx$ where $x \in H$ and $k=0, \pm 1, \pm 2, \cdots$.*

Proof Let $\bar{H} = \oplus_{-\infty}^{\infty} H$. Define $\bar{T}$ on $\bar{H}$ by

$$(\bar{T}\bar{x})_j = \sum_{i=-\infty}^{\infty} T_{i-j} x_i \quad (j=0, \pm 1, \pm 2, \cdots)$$

where $\bar{x} = \{x_i\}_{-\infty}^{\infty} \in \bar{H}$. By the assumption, we have

$$\begin{aligned}
\|\bar{T}\bar{x}\|^2 &= \sum_{j=-\infty}^{\infty} \left\| \sum_{i=-\infty}^{\infty} T_{i-j} x_i \right\|^2 \\
&\leqslant M^2 \sum_{j=-\infty}^{\infty} \left(\sum_{i=-\infty}^{\infty} a^{|i-j|} \|x_i\| \right)^2 \\
&= M^2 \sum_{j=-\infty}^{\infty} \left(\sum_{i=-\infty}^{\infty} a^{|i-j|} \|x_i\| \right)\left(\sum_{l=-\infty}^{\infty} a^{|l-j|} \|x_l\| \right) \\
&= M^2 \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \|x_i\| \cdot \|x_l\| \sum_{j=-\infty}^{\infty} a^{|i-j|+|l-j|} \\
&= M^2 \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \|x_i\| \cdot \|x_l\| a^{|i-l|} [|i-l| + (1+a^2)(1-a^2)^{-1}] \\
&\leqslant M^2 \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \|x_i\|^2 a^{|i-l|} [|i-l| + (1+a^2)(1-a^2)^{-1}] \\
&= M^2 (1+a)^2 (1-a)^{-2} \|\bar{x}\|^2.
\end{aligned}$$

This means that $\bar{T}$ is bounded and hence $\bar{T}$ is a symmetric operator in $\bar{H}$.

Denote by K_0 the set of all doubly infinite sequences $f = \{f_k\}_{-\infty}^{\infty}$ such that

$$f_k = \sum_{i=-\infty}^{\infty} T_{i-k} x_i$$

and $\bar{x} = \{x_i\}_{-\infty}^{\infty} \in \bar{H}$. Define

$$\langle f, g \rangle = (\bar{T}\bar{x}, \bar{y}), \ [f, g] = (|\bar{T}|\bar{x}, \bar{y}),$$

where f and $\bar{x}$ are as above and $g = \{g_k\}_{-\infty}^{\infty}$,

$$g_k = \sum_{i=-\infty}^{\infty} T_{i-k} y_i,$$

$\bar{y} = \{y_i\}_{-\infty}^{\infty} \in \bar{H}$. Thus

$$\langle f, g \rangle = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} (T_{i-j} x_i, y_j) = \sum_{j=-\infty}^{\infty} (f_j, y_j).$$

If $f=0$, then $|\bar{T}|\bar{x} = \bar{T}\bar{x} = 0$. It is easy to see that $\langle f, g \rangle$ and $[f, g]$ are well defir If $[f, f]=0$, then $\bar{T}\bar{x} = |\bar{T}|\bar{x} = 0$, so $f=0$. Therefore K_0 is a pre-Hilbert space v inner product $[\cdot, \cdot]$. Denote by K the completion of K_0. Write

$$\bar{H}_- = \ker \bar{T}^+, \ \bar{H}_+ = \bar{H} \ominus \bar{H}_-.$$

Let K_-^0 be the set of all doubly infinite sequences $f^- = \{f_k^-\}_{-\infty}^{\infty}$ such that

$$f_k^- = \sum_{i=-\infty}^{\infty} T_{i-k} x_i^-,$$

$\bar{x}^- = \{x_i^-\}_{-\infty}^{\infty} \in \bar{H}_-$ and let K_+^0 be the set of all doubly infinite sequences $f^+ = \{f_k^+\}_{-\infty}^{\infty}$ such that

$$f_k^+=\sum_{i=-\infty}^{\infty}T_{i-k}x_i^+,$$

$^+=\{x_i^+\}_{-\infty}^{\infty}\in\overline{H}_+$. It is easy to check that K_+^0 and the anti-space of K_-^0 are both pre-Iilbert spaces with respect to $\langle.,\ .\rangle$. Let K_- and K_+ be the completions of K_-^0 and K_+^0, respectively. For $f^-\in K_-^0$ and $f^+\in K_+^0$, it follows from (3) that

$$[f^-,\ f^+]=(|\overline{T}|\bar{x}^-,\ \bar{x}^+)=(\overline{T}^-\bar{x}^-,\ \bar{x}^+)=(\bar{x}^-,\ \overline{T}^-\bar{x}^+)=0.$$

hen $K_+\perp K_-$ and $K=K_+\oplus K_-$ with respect to $[.,\ .]$. Let P_- and P_+ be the rthogonal projections of K onto K_- and K_+ with respect to $[.,\ .]$, respectively. efine $J=P_+-P_-$. It follows from (3) that

$$\langle f,\ g\rangle=[Jf,\ g]\quad(\forall f,\ g\in K).$$

Ience K is a Krein space with indefinite inner product $\langle.,\ .\rangle$ and $K=K_+\oplus K_-$ is fundamental decomposition.

For $f=\{f_k\}_{-\infty}^{\infty}\in K_0$, define $Uf=\{f_{k-1}\}_{-\infty}^{\infty}$. Since

$$f_{k-1}=\sum_{i=-\infty}^{\infty}T_{i-k+1}x_i=\sum_{i=-\infty}^{\infty}T_{i-k}x_{i-1},$$

re have

$$\langle Uf,\ Ug\rangle=\sum_i\sum_k(T_{i-k}x_{i-1},\ y_{k-1})=\sum_i\sum_k(T_{i-k}x_i,\ y_k)=\langle f,\ g\rangle.$$

or $\bar{x}=\{x_i\}_{-\infty}^{\infty}\in\overline{H}$, define $\overline{S}\bar{x}=\{x_{i-1}\}_{-\infty}^{\infty}$. Then

$$\|\overline{S}{*}|\overline{T}|\overline{S}\bar{x}\|^2=\||\overline{T}|\overline{S}\bar{x}\|^2=\|\overline{T}\overline{S}\bar{x}\|^2=\sum_j\|\sum_i T_{i-j}x_{i-1}\|^2=\sum_j\|\sum_i T_{i-j}x_i\|^2=\|\overline{T}\bar{x}\|^2=\|\overline{T}|\bar{x}\|^2.$$

his implies that $\overline{S}{*}|\overline{T}|\overline{S}=|\overline{T}|$. Thus

$$[Uf,\ Ug]=(\overline{S}{*}|\overline{T}|\overline{S}\bar{x},\ \bar{y})=(|\overline{T}|\bar{x},\ \bar{y})=[f,\ g].$$

t follows that U may be considered to be both a unitary operator with respect to $.,.\rangle$ and a J-unitary operator with respect to $[.\ ,\ .]$. Clearly $JU=UJ$ and K_+ nd K_- are reducing subspaces of U.

For $x\in H$, define $\bar{x}^0=\{x_i^0\}_{-\infty}^{\infty}$, where $x_0^0=x$, $x_i^0=0$ $(i\neq 0)$. Let $Bx=\{T_{-k}x\}_{-\infty}^{\infty}$. ince

$$[Bx,\ Bx]=(|\overline{T}|\bar{x}^0,\ \bar{x}^0)\leqslant\|\overline{T}\|\cdot\|\bar{x}^0\|^2=\|\overline{T}\|\cdot\|x\|^2,\tag{5}$$

is an operator on H into K. Clearly $U^lBx=\{T_{l-k}x\}_{k=-\infty}^{\infty}$ $(l=0,\ \pm1,\ \pm2,\ \cdots)$. For $,\ y\in H$, we have

$$\langle U^lBx,\ By\rangle=(T_lx,\ y)\quad(l=0,\ \pm1,\ \pm2,\ \cdots).\tag{6}$$

hus (1) holds. The statement (b) is clear. This completes the proof of the theorem.

Theorem 2. *Let H be a Hilbert space and $\{T_k\}_{-\infty}^{\infty}$ a doubly infinite sequence of perators in H. As ume that there are positive M and a such that $\|T_k\|\leqslant Ma^{|k|}$ and $T_k=T_{-k}^*$ for all $k=0,\ \pm1,\ \pm2,\ \cdots$. Then there is a unitary operator U in a Krein space K and an opera or B on H into K such that* (1) *holds and the statement* (b) *of Theorem is valid.*

Proof Making use of Theorem 1 for $\{(a+\varepsilon)^{-|k|}T_k\}_{-\infty}^{\infty}$ where $\varepsilon>0$, we see that there is a fundamentally reducible unitary operator U in a Krein space K and an operator B on H into K such that

$$T_k=(a+\varepsilon)^{|k|}B^+U^kB \quad (k=0, \pm1, \pm2, \cdots), \tag{7}$$

where B^+ is defined by (2). Let $K=H_+\oplus H_-$ be a fundamental decomposition where H_+ and the anti-space of H_- are Hilbert spaces and reduce U. By the Davis Theorem ([2] or [1, Theorem VI. 8.6]), there exist Krein spaces $K_+\supset H_+$ and $K_-\supset H_-$ such that $(a+\varepsilon)U|H_+$ and $(a+\varepsilon)U|H_-$ have minimal unitary dilations U_+ and U_- to K_+ and K_-, respectively. Write $K_1=K_+\oplus K_-$ and $U_1=U_+\oplus U_-$. Then K_1 is a Krein space and U_1 is a unitary operator in K_1. Clearly $K_1\supset K$. Let P be the orthogonal projection of K_1 onto K and E the embedding operator of K into K_1. Then

$$PU_1^kE=(a+\varepsilon)^{|k|}U^k \quad (k=0, \pm1, \pm2, \cdots).$$

Write $B_1=EB$. Let B_1^+ be defined by (2). For $f\in K_1$, $x\in H$, we have

$$(B_1^+f, x)=\langle f, B_1x\rangle=\langle f, EBx\rangle=\langle Pf, Bx\rangle=(B^+Pf, x).$$

It follows from (7) and (8) that

$$T_k=B^+PU_1^kEB=B_1^+U_1^kB_1 \quad (k=0, \pm1, \pm2, \cdots).$$

Denote by K_2 the closure of the linear set of all elements of the form U_1^k where $x\in H$, $k=0, \pm1, \pm2, \cdots$. Let K_0 be the isotropic part of K_2. Clearly K_0 closed subspace and reduces U_1 with respect to the indefinite inner product. Writ

$$K_3=\{f\in K_2\colon [f, f_0]=0 \text{ for all } f_0\in K_0\}.$$

Then K_3 is a closed non-degenerate subspace [1, Lemma I. 5.1]. Let $\tilde K=K_2/$ For $\tilde f, \tilde g\in\tilde K$, $f\in\tilde f$, $g\in\tilde g$, define $\langle\tilde f, \tilde g\rangle=\langle f, g\rangle$. It is easy to see that $\langle\tilde f, \tilde g\rangle$ is defined. Therefore $\tilde K$ is a non-degenerate indefinite inner product space. For $\tilde f$, $\tilde K$, we define $[\tilde f, \tilde g]=[f, g]$ where $f\in\tilde f\cap K_3$ and $g\in\tilde g\cap K_3$. Here f and g uniquely determited by $\tilde f$ and $\tilde g$, respectively. Then $[., .]$ is a Hilbert majoran $\tilde K$. For $\tilde f\in\tilde K$, $f\in\tilde f$, define $\tilde U\tilde f=\widetilde{U_1f}$. It is easy to check that $\tilde U$ is well defined. $\tilde f, \tilde g\in\tilde K$, $f\in\tilde f$, $g\in\tilde g$, we have

$$\langle\tilde U\tilde f, \tilde U\tilde g\rangle=\langle U_1f, U_1g\rangle=\langle f, g\rangle=\langle\tilde f, \tilde g\rangle.$$

For $\tilde f\in\tilde K$, $f\in\tilde f$, we have $\tilde U^{-1}\tilde f=\widetilde{U_1^{-1}f}$. Using Theorem IV. 5.2 of [1], there fundamental decomposition $\tilde K=\tilde H_+\oplus\tilde H_-$ such that the positive definite subs $\tilde H_+$ and the negative definite subspace $\tilde H_-$ are closed with respect to $[., .]$. De by $\|.\|_{J_1}$ the decomposition majorant. For the fundamental decomposition $\tilde K=(\tilde U$ $\oplus(\tilde U\tilde H_-)$, denote by $\|.\|_{J_2}$ the decomposition majorant. For $\tilde f\in\tilde K$, decompose $\tilde f$ $+\tilde f_-$, where $\tilde f_+\in\tilde H_+$ and $\tilde f_-\in\tilde H_-$, By Theorem IV. 6.4 of [1], there is an $\alpha>0$ that

$$\alpha\|\tilde U\tilde f\|^2_{J_1}\leqslant\|\tilde U\tilde f\|^2_{J_2}=\langle\tilde U\tilde f_+, \tilde U\tilde f_+\rangle-\langle\tilde U\tilde f_-, \tilde U\tilde f_-\rangle=\langle\tilde f_+, \tilde f_+\rangle-\langle\tilde f_-, \tilde f_-\rangle=\|\tilde f\|^2_{J_1}$$

for all $\tilde f$. Let $\tilde K_+$ and $\tilde K_-$ be the completions of $\tilde K_+$ and $\tilde K_-$ with respect to $\langle$ and $-\langle ., .\rangle$, respectively. Define the Krein space $\tilde K'=\tilde K_+\oplus\tilde K_-$. By (10) and (11), $\tilde U$ may be considered to be a unitary operator in $\tilde K'$. For $x\in H$, define $\tilde Bx=\widetilde{B_1x}$. It is easy to check that $\widetilde{B}$ is continuous with respect to the Hilbert majorant of $\widetilde{K}$, so it continuous with respect to $\|.\|_{J_1}$. Therefore $\widetilde{B}$ may be considered to bean operator H into $\widetilde{K}'$. For $x, y\in H$, it follows from (9) that

$$\langle\widetilde{U}^k\widetilde{B}x, \widetilde{B}y\rangle=\langle U_1^kB_1x, B_1y\rangle=(T_kx, y) \quad (k=0, \pm1, \pm2, \cdots).$$

ıus (1) holds. For $\tilde{f}\in\widetilde{K}$, let $f\in\tilde{f}$. Given $\varepsilon>0$, there exist $x_1, \cdots, x_n\in H$ such that

$$\left[f-\sum_{k=1}^{n}U_1^kB_1x_k, f-\sum_{k=1}^{n}U_1^kB_1x_k\right]<\varepsilon.$$

Lemma IV. 5.4 of [1], there exists a constant $M>0$ such that

$$\left\|\tilde{f}-\sum_{k=1}^{n}\widetilde{U}^k\widetilde{B}x_k\right\|_{J_1}^2\leqslant M\left[f-\sum_{k=1}^{n}U_1^kB_1x_k, f-\sum_{k=1}^{n}U_1^kB_1x_k\right]<M\varepsilon.$$

ıerefore, the statement (b) is valid for $\widetilde{U}$ and $\widetilde{B}$. This completes the proof of ıeorem 2.

Corollary 3. *Let the assumption of Theorem 2 be satisfied. If T_0 is an invertible itive operator, then ran B i a uniformly po itive closed subspace of K. If $T_0=I$, m $K\supset H$, $\langle.,.\rangle$ on H coincide with $(.,.)$, B is the natural embedding of H into and B^+ is the orthogonal projection of K onto H with respect to $\langle.,.\rangle$.*

Proof Assume that T_0 has a lower bound $m>0$. By Theorem 2, there exists an >0 such that for $x\in H$

$$[Bx, Bx]\leqslant M\|x\|^2\leqslant m^{-1}M(T_0x, x)=m^{-1}M\langle Bx, Bx\rangle\leqslant m^{-1}M[Bx, Bx].$$

is implies that ran B is a uniformly positive closed subspace of K.

Assume further that $T_0=I$. By Theorem 2, we obtain

$$\langle Bx, By\rangle=(x, y) \quad (\forall x, y\in H).$$

ɘ original space H can be embedded as a subspace in K. The embedding operator 3. Let Q be the orthogonal projection of K onto ran B with respect to $\langle.,.\rangle$. For H, $f\in K$, it follows from (2) that

$$\langle Qf, By\rangle=\langle f, By\rangle=(B^+f, y).$$

s completes the proof of the corollary.

Theorem 4. *Let H be a Hilbert space and $\{T_k\}_0^\infty$ a infinite sequence of self-oint operators in H. Assume that there are positive M and a such that $\|T_k\|\leqslant Ma^k$ all $k=0, 1, 2, \cdots$. Then*

(a) *There is a self-adjoint operator A in a Krein space K such that*

$$T_k=B^+A^kB \quad (k=0, 1, 2, \cdots), \tag{12}$$

re B is an operator on H into K and B^+ is defined by (2).

(b) *K is spanned by the elements of the form A^kBx where $x\in H$ and $k=0, 1, 2,$*

Proof Without loss of generality, we may assume that $0<a<1$. Let $\overline{H}=\bigoplus_0^\infty H$. ne $\overline{T}$ on $\overline{H}$ by

$$(\overline{T}\bar{x})_j=\sum_{i=0}^{\infty}T_{i+j}x_i \quad (j=0, 1, 2, \cdots),$$

where $\bar{x}=\{x_i\}_0^\infty\in\bar{H}$. By the assumption, we have

$$\|\bar{T}\bar{x}\|^2=\sum_j\|\sum_i T_{i+j}x_i\|^2\leqslant M^2\sum_j(\sum_i a^{i+j}\|x_i\|)^2\leqslant M^2(1-a^2)^{-2}\|\bar{x}\|^2.$$

This means that $\bar{T}$ is bounded and hence $\bar{T}$ is a symmetric operator in $\bar{H}$.

Denote by K_0 the set of all infinite sequences $f=\{f_k\}_0^\infty$ such that

$$f_k=\sum_{i=0}^\infty T_{i+k}x_i$$

and $\bar{x}=\{x_i\}_0^\infty\in\bar{H}$. Define

$$\langle f,\ g\rangle=(\bar{T}\bar{x},\ \tilde{y}),\ [f,\ g]=(|\bar{T}|\bar{x},\ \tilde{y}),$$

where f and $\bar{x}$ are as above and $g=\{g_k\}_0^\infty$,

$$g_k=\sum_{i=0}^\infty T_{i+k}y_i,$$

$\tilde{y}=\{y_i\}_0^\infty\in\bar{H}$. Thus

$$\langle f,\ g\rangle=\sum_i\sum_j(T_{i+j}x_i,\ y_j)=\sum_j(f_j,\ y_j).$$

As in the proof of Theorem 1, we can prove that $[.,\ .]$ and $\langle .,\ .\rangle$ are well defin and that K, the completion of K_0 with respect to $[.,\ .]$, is a Krein space w indefinite inner product $\langle .,\ .\rangle$.

For $f=\{f_k\}_0^\infty\in K_0$, define $Af=\{f_{k+1}\}_0^\infty$. We have

$$f_{k+1}=\sum_i T_{i+k+1}x_i=\sum_i T_{i+k}x'_i,$$

where $x'_0=0$, $x'_i=x_{i-1}(i>0)$. Write $\bar{x}'=\{x'_i\}_0^\infty$. For $f,\ g\in K_0$, it is easy to check th

$$\langle Af,\ g\rangle=\langle f,\ Ag\rangle. \qquad (\text{:}$$

For $\bar{x}=\{x_i\}_0^\infty\in\bar{H}$, define $\bar{S}\bar{x}=\bar{x}'$. Then

$$\|\bar{S}^*|\bar{T}|\bar{S}\bar{x}\|^2\leqslant\||\bar{T}|\bar{S}\bar{x}\|^2=\|\bar{T}\bar{S}\bar{x}\|^2=\sum_j\|\sum_i T_{i+j+1}x_i\|^2\leqslant\sum_j\|\sum_i T_{i+j}x_i\|^2=\|\bar{T}\bar{x}\|^2$$
$$=\||\bar{T}|\bar{x}\|^2.$$

By the Heinz inequality[4], we have

$$(\bar{S}^*|\bar{T}|\bar{S}\bar{x},\ \bar{x})\leqslant(|\bar{T}|\bar{x},\ \bar{x}).$$

Thus

$$[Af,\ Af]=(|\bar{T}|\bar{S}\bar{x},\ \bar{S}\bar{x})\leqslant(|\bar{T}|\bar{x},\ \bar{x})=[f,\ f].$$

It follows from (13) that A may be considered to be a self-adjoint operator in K.

For $x\in H$, define $\bar{x}^0=\{x_i^0\}_0^\infty$ where $x_0^0=x$, $x_i^0=0$ $(i>0)$. Define $Bx=\{T_kx\}_0^\infty$. Si

$$[Bx,\ Bx]=(|\bar{T}|\bar{x}^0,\ \bar{x}^0)\leqslant\|\bar{T}\|\cdot\|\bar{x}^0\|^2=\|\bar{T}\|\cdot\|x\|^2,$$

B is an operator on H into K. Clearly $A^lBx=\{T_{l+k}x\}_{k=0}^\infty$ $(l=0,\ 1,\ 2,\ \cdots)$. For x, H, we have

$$\langle A^lBx,\ By\rangle=(T_lx,\ y)\quad(l=0,\ 1,\ 2,\ \cdots).$$

Thus (1) holds. The statement (b) is clear. This completes the proof of the theor

As in the proof of Corollary 3, we obtain the following

Corollary 5. *Le; the assump ions of Theorem 4 be sa isfied. If T_0 is an invert positive operator, then ran B is a uniformly positive closed subspace of K. If $T_0=I$, then $K\supset H$, $\langle .,\ .\rangle$ on H coincides with $(.,\ .)$, B is the na ural embedding of H into*

K and B^+ is the orthogonal projection of K onto H with respect to $\langle .,\ .\rangle$.

References

[1] Bognar, J., Indefinite inner product spaces, Springer-Verlag, 1974.

[2] Davis, C., J-unitary dilation of a general operator, *Acta Sci. Math.* (*Szeged*), **31** (1970), 75—86.

[3] Davis, C., Dilation of uniformly continuous semi-groups, *Rev. Roumaine Math. Pures Appl.*, **15** (1970), 975—983.

[4] Kato, T., Perturbation theory for linear operators, Springer-Verlag, 1966.

[5] Mlak, W., Unitary dilations of contraction operators, *Rozprawy Mat.*, **46** (1965).

[6] Sz. Nagy, B., Extensions of linear transformations in Hilbert space which extend beyond this space, Appendix to "Functional analysis" by F. Riesz and B. Sz. -Nagy, Ungar, 1960.