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ON THE 2ND POWER OF PRIME MEROMOPRHIC FUNCTIONS

ZHENG JIANHUA (郑建华)*

Abstract

In this paper, the author respectively constructs the prime entire function and the prime non-entire meromorphic function of finite order whose 2nd power and derivative are not pseudo-prime so that the answer to the Problem (A) Song-Huang[6] posed is in the affirmative. And it is exhibited that none but periodic meromorphic functions has the property mentioned in Problem (A).

§1. Introduction and Statement of Results

A meromorphic function $F(z)$ is said to be prime, if any factorization $f(g$ F for meromorphic f and entire g may always imply that either f or g is linear 1971, F. Gross[1] asked whether there exists a periodic, prime entire function or From that time on, such functions of finite and infinite order have been constru See references Ozawa[2], Baker-Yang[3], Gross-Yang[4] and Urabe[5]. Recently, S Huang[6] and Zheng[7] independentely proved that periodic entire functions

$$F(z)=\sin z\cdot\exp(\cos z) \text{ and } G(z)=\cos^n z\cdot\sin z\cdot\exp(\cos^{2m+1} z)$$

are both prime, where both n and m are non-negative integers. Obviously, F a have a special property that F^2 and G^2 are not pseudo-prime. For example, we write

$$F^2(z)=[(1-w^2)\exp(2w)]\circ\cos z,$$

where $\circ$ denotes the composition of function. This shows an elementary important result that the product of prime functions is not always pseudo-p But the above F and G are both of infinite order, hence Song-Huang[6] pose following problem:

(A) Does there exist a primo entiro function of finite order such that F^2 i pseudo-prime?

In this paper, we are to construct the prime entire function and the prime entire meromorphic function of finite order whose 2nd power is not pseudo-prin that the answer to the Problem (A) is in the affirmative. And we shall poin

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* Department of Mathematics, Anhui University, Hefei, Anhui, China.

that none but periodic meromorphic function possesses the same property. By the y, the first derivative of the function which we shall construct is not pseudo-me, either. Hence at the same time, we also show such an elementary and impor-t result that the derivative of prime function is not always pseudo-prime. The ve assertion is contained in the following theorems.

We assume that the reader is familiar with Navenlinna's fundamental theory, particular, with symbolisms $n(r, f=0)$, $\overline{N}(r, f)$ and $\Theta(r, f)$ etc. And we denote order of f by $\rho(f)$ (cf. [8]).

Theorem 1. *Let F be a non-periodic entire function. Then F^2 is still pseudo-me.*

Theorem 2. *Let $\{v_n\}$ be a sequence of prime numbers with $v_n\geqslant 3$ and $v_{n+1}>v_n$ $=0, 1, \cdots$), and $\{b_n\}$ a sequence of finite complex numbers with $b_n\neq b_m$ $(n\neq m)$ and ± 1 for each n. If $L(z)=\prod\limits_{n=1}^{\infty}(1-z/b_n)^{v_n}$ forms an entire function and $\rho(L(\cos z))$ ∞, then $F(z)=\sin^{v_0} z\cdot L(\cos z)$ is prime.*

Theorem 3. *Let $L(z)$ be as in the above. If $\rho(L(\cos z))<+\infty$, then $F(z)=$ $z/L(\cos z)$ is prime.*

Remark. i) By the same method as in the proof of Theorem 2 of Ozawa[9], we easily prove Theorem 1, so we omit its proof.

ii) Let $F(z)$ be as in Theorem 2, then

$$F'(z)=v_0\sin^{v_0-1}z\cos z\cdot L(\cos z)-\sin^{v_0+1}z\cdot L'(\cos z)$$
$$=[v_0w(1-w^2)^{(v_0-1)/2}L(w)-(1-w^2)^{(v_0+1)/2}L'(w)]\circ\cos z$$

ot pseudo-prime.

(iii) The function $L(z)$ in Theorem 2 can be found. For example, we may me $b_n=\exp v_n$. By the method of Hayman[8, p.27], we can conclude that the order of $os z)$ is not greater than 3.

§2. Proof of Theorems

Before proving thoorems, we need four known results.

Lemma 1[10, p.27]. *Assume that $g(z)$ is an entire function with period $2k\pi$, of mential type σ. Then $g(z)$ has the form*

$$g(z)=\sum_{j=-m}^{m}c_j\exp((ji/k)z)\quad(0\leqslant m\leqslant k\pi\sigma),$$

constant c_j's.

Lemma 2[11]. *Let $f(z)$ be an entire function. Assume that there exists an unded sequence $\{a_n\}$ such that all the roots of $f(z)=a_n(n=1, 2, \cdots)$ lie on a straight line. Then $f(z)$ must be a polynomial of degree at most 2.*

Lemma 3 (Borel's Theorem[10, p.165]. *Let $f_1(z), \cdots, f_n(z)$ be n entire functions* *such that $f_i(z)-f_j(z)$ is non-constant for $i\neq j$. Let $g_1(z), \cdots, g_n(z)$ be n meromorphic functions of finite order such that*

$$\rho(g_t)<\min\{\rho(\exp(f_i-f_j));\ i, j=1, 2, \cdots, n.\ i\neq j\}\ (t=1, 2, \cdots, n).$$

If

$$\sum_{i=1}^{n} g_i(z)\exp(f_i(z))=0,$$

then $g_1(z)\equiv g_2(z)\equiv\cdots\equiv g_n(z)\equiv 0$.

Lemma 4[17]. *If $\cos z=P(g(z))$, where P is a polynomial and $g(z)$ an entire function, then $g(z)=A\cos(z/k)+B$, for some positive integer k and constants A and B.*

Proof of Theorem 2 Let $F(z)=\sin^{v_0} z\cdot L(\cos z)=f(g)$.

We treat several cases, separately.

(I) The case when both f and g are transcendental entire functions. By Pol Theorem[12], $\rho(f)=0$. Since $F(z)$ is a periodic entire function with period 2π, also a periodic function with period $2k\pi$ for some positive integer k[10, p.106].

Let $\{w_n\}$ denote the set of all the zeros of f. Since g is entire, there exist at ı two positive integers m_0 and m_1 such that $g-w_{m_i}(i=0, 1)$ only have multiple ze If $g-w_s$ has a simple zero for some natural number s, then since $v_n\neq v_m(n\neq m)$, a $v_{t(s)}$ multiple zero of f, where $t(s)$ is a non-negative integer corresponding 1 further since all $v_n(n=1, 2, \cdots)$ are prime numbers, we may get that $g-w_s$ only simple zeros. Thus we can choose two distinct positive integers n_1 and n_2 such $g-w_{n_i}(i=1, 2)$ only have simple zeros. And we can find out two natural num $t(n_i)(i=1, 2)$ corresponding to $n_i(i=1, 2)$ such that the zeros of $g-w_{n_i}(i=1, 2)$ ones of $(1-\cos z/b_{t(n_1)})(1-\cos z/b_{t(n_2)})\cdot\sin z$. Then

$$\begin{aligned}T(r, g)&<N(r, w_{n_1}, g)+N(r, w_{n_2}, g)+O(\log r)\\&<N(r, b_{t(n_1)}, \cos z)+N(r, b_{t(n_2)}, \cos z)+N(r, ,0, \sin z)+O(\log r)\\&<Ar,\ (A>0).\end{aligned}$$

Hence g is of exponential type. By Lemma 1, g has the form

$$g(z)=\sum_{j=-v}^{m} c_j\exp((ji/k)z)\quad (o\leqslant m,\ v\leqslant k\pi A),$$

where $c_m\cdot c_{-v}\neq 0$. Put $P(w)=\sum_{j=-v}^{m} c_j\cdot w^j$, hence we have $g(z)=P(\exp(iz/k))$. Th follows from $F(z)=f(g)$ that

$$f(P(w))=(1/2i)^{v_0}(w^k-w^{-k})^{v_0}L((w^k+w^{-k})/2)=-f(P(1/w)).$$

Let $\lambda(r, M, f(w))$ denote the number of the zeros of $f(w)$ in $M<|w$ (counting multiplicity). If $m\neq v$, we may assume that $m>v$ and take a sufficie small $\varepsilon>0$ such that $|c_m|-\varepsilon>0$ and $|c_{-v}|-\varepsilon>0$. As $w\to\infty$, we have the forms

$$P(w)=c_m w^m(1+o(1/w)),$$

$$P(1/w)=c_{-v}w^v(1+o(1/w)).$$

Obviously, (2) and (3) imply that for sufficiently large n, $P(w)-w_n$ has and only has m distinct zeros whose modules are between $[|w_n|/(|c_m|+\varepsilon)]^{1/m}$ and $[|w_n|/(|c_m|-\varepsilon)]^{1/m}$ while $P(1/w)-w_n$ has and only has v distinct zeros whose modules are etween $[|w_n|/(|c_{-v}|+\varepsilon)]^{1/v}$ and $[|w_n|/(|c_{-v}|-\varepsilon)]^{1/v}$. Hence there exists a ıfficiently large $M>0$ such that $r\to\infty$,

$$\lambda(r, M, f(P(w)))>mn(r^m(|c_m|-\varepsilon), M^m(|c_m|+\varepsilon), f(w))$$
$$=(m+o(1))n(r^m(|c_m|-\varepsilon), f(w)),$$

ut

$$\lambda(r, M, f(P(1/w)))<vn(r^v(|c_{-v}|+\varepsilon), M^v(|c_{-v}|-\varepsilon), f(w))$$
$$=(v+o(1))n(r^v(|c_{-v}|+\varepsilon), f(w)).$$

learly, from (1) and $m>v$, we have a contradictory inequality. Hence $m=v$. If $'(w)-w_n$ has a simple zero, then $g-w_n$ also has simple zeros, and by the previous iscussion, $g-w_n$ only has simple zeros, which implies that $P(w)-w_n$ only has mple zeros. Since for each n, v_n is prime number with $v_n\geqslant 3$, g is entire and ːp$(iz/k)-a(a\neq 0)$ only has simple zeros, there exists at most a w_q such that every ro of $P(w)-w_q$ has corresponding multiplicity v_t not greater than deg P. Thus we n choose a v_t such that $P(w)-w_q$ has no v_t multiple zeros. And let $\{w_{n_j}\}_{j=1}^p(1\leqslant p$ $+\infty)$ be the set of v_t multiple zeros of $f(w)$. Of course, for each j, $w_{n_j}\neq w_q$. Then follows that v_t multiple zeros of $f(g(z))$ only consist of the zeros of $g(z)-w_{n_j}(j=$ $2, \cdots, p)$, i. e.

$$C\prod_{j=1}^{p}(g(z)-w_{n_j})=\exp(uiz/k)(1-\cos z/b_t)\quad (C\neq 0), \tag{4}$$

here u is an integer. Since the right side of the above equality is periodic, of ponential type and g is periodic, by Lemma 1, p is finite. We rewrite (4) as

$$C\prod_{j=1}^{p}\left(\sum_{d=-m}^{m}c_d\cdot w^d-w_{n_j}\right)=w^u(1-(w^k+w^{-k})/2b_t),$$

that we imply that $pm=u+k$ and $-pm=u-k$, further $u=0$. Therefore from (4), have $\cos z=N(g(z))$, where

$$N(w)=-C\prod_{j=1}^{p}(w-w_{n_j})b_t+1.$$

Lemma 4, we know $g(z)$ is even, so is $f(g)=F(z)$, which is a contradiction.

(II) The case when f is a polynomial and g is a transcendental entire function. e denote all the distinct zeros of f by $\{A_k\}_1^q$. Assume that f has two distinct ıltiple zeros, say A_1 and A_2. Since $v_n(n=1, 2, \cdots)$ are prime numbers and $v_n\neq v_m$ $\neq m)$, there are $b_{n_i}(i=1, 2)$ such that all the zeros of $g-A_i(i=1, 2)$ are contained the set of zeros of $(1-\cos z/b_{n_1})(1-\cos z/b_{n_2})\cdot\sin z$. It follows that

$$T(r, g)<N(r, A_1, g)+N(r, A_2, g)+O(\log r)$$
$$<N(r, b_{n_1}, \cos z)+N(r, b_{n_2}, \cos z)+N(r, 0, \sin z)+O(\log r)$$
$$<Kr\quad (K>0),$$

namely, g is of exponential type, so is $f(g)$. But $f(g)=F$ is not of exponential type.

Thus f has at most one multiple zero. We need only treat two subcases for f, separately:

(II. 1) $f(w)=C(w-A_1)\cdots(w-A_{m-1})(w-A_m)^{v_s}$,

(II. 2) $f(w)=C(w-A_1)\cdots(w-A_m)$.

We may assume $m\geqslant 2$. In fact, when $m=1$, the result is trivial. For subcase (II. 1), since the roots of $g=A_m$ are ones of $\cos z=b_s$ for some s, we have

$$\overline{N}(r, A_m, g)\leqslant\overline{N}(r, b_s, \cos z)\leqslant T(r, \cos z)+O(1)=o(T(r, f(g))=o(T(r, g)),$$

(cf. [13]), so that $\Theta(A_m, g)=1$.

Put $D(z)=L(\cos z)/(1-\cos z/b_s)^{v_s}$. Since A_k $(k=1, 2, \cdots, m-1)$ are mutu distinct, we have

$$\sum_{k=1}^{m-1}N(r, 1/(g-A_k))\geqslant N(r, 0, \sin^{v_0}z)+N(r, 0, D(z))$$
$$>v_0N(u, 0, \sin z)+3\overline{N}(r, 0, D)$$
$$\geqslant 3\sum_{k=1}^{m-1}\overline{N}(r, A_k, g)+(v_0-3)\overline{N}(r, 0, \sin z)-3N(r, 0, 1-\cos z/b_s),$$

namely, $(m-1+o(1))T(r, g)\geqslant 3\sum_{k=1}^{m-1}\overline{N}(r, A_k, g)$. Hence we have $\sum_{k=1}^{m-1}\Theta(A_k, g)\geqslant$ $-1)/3\geqslant 2/3$. Since g is entire, by Navenlinna's Theory, this is impossible.

For subcase (II. 2), by the same method as in the above, we can get

$$\sum_{k=1}^{m}\Theta(A_k, g)\geqslant 2m/3\geqslant 4/3,$$

since $m\geqslant 2$. This is also impossible. Hence $m=1$, i. e. $f(w)$ is linear.

(III) The case when f is a transcendental entire function and g is a polyno Since $f(g)$ is periodic, by the theorem of Renyi[14] the degree of g is at most 2. $g=a(z-b)^2+c$. Then from $F=f(g)$, we have

$$\sin^{v_0}(z+b)\cdot L(\cos(z+b))=-\sin^{v_0}(z-b)\cdot L(\cos(z-b)).$$

Since $v_n(n=0, 1, \cdots)$ are prime numbers and $v_n\neq v_m(n\neq m)$, it follows from that

$$\sin(z+b)=\sin(z-b)\cdot\exp E_0(z),$$
$$1-\cos(z+b)/b_n=(1-\cos(z-b)/b_n)\cdot\exp E_n(z),$$

$(n=1, 2, \cdots)$, where each $E_n(z)$ is a linear function. By Lemma 3, we immed see that E_n's are constants. Differentiating both the sides of (6), we get

$$\cos(z+b)=\cos(z-b)\cdot\exp E_0.$$

Combining (7) and (8), we have

$$1-\cos(z+b)/b_n=1-\cos(z-b)\cdot\exp(E_0)/b_n=\exp E_n-\cos(z-b)\cdot\exp E_n/b_n,$$
$$(1/b_n)(\exp E_n-\exp E_0)\cos(z-b)=\exp E_n-1,$$
$$\exp E_0=\exp E_n=1\quad(n=1, 2, \cdots).$$

This is a contradiction to the equality (5).

(IV) The case when $f(w)$ is a transcendental, non-entire, meromorphic function and $g(z)$ an entire function. Then we can write $f(w)=f^*(w)/(w-w_1)^n$, (n a fixed positive integer), $g(z)=w_1+\exp(M(z))$, where $f^*(w)$ is an entire function ith $f^*(w_1)\neq 0$ and $M(z)$ is entire, too. Therefore

$$f(g)=(f^*(w_1+\exp w)\exp(-nw))\circ M(z).$$

ccording to the above discussion, we know $M(z)=Cz+E$ and $C=ip/q$, where both and q are integers and E is a constant. And we may assume $p\geqslant 0$ and $q>0$ without ss of generality, then

$$f(w_1+e^E\exp(ip/q)z)=\sin^{v_0}z\cdot L(\cos z).$$

t $w=\exp(iz/q)$, then the above equality becomes

$$f(w_1+e^E w^p)=(1/2i)^{v_0}(w^q-w^{-q})^{v_0}L((w^q+w^{-q})/2). \tag{9}$$

he left side of (9) is entire, but the right side of (9) has an essentially singular int at $w=0$, which is an absurd equality.

(V) The case when f is a non-polynomial rational function and g a eromorphic function. We may write

$$f(w)=P(w)/((w-a)^n(w-b)^m),$$

here n ($\geqslant 0$) and $m(>0)$ are both integers and both a and b are complex numbers d $P(w)$ is a polynomial with $P(a)\neq 0$ and $P(b)\neq 0$. Making a linear ansformation $\lambda(w)=(w-a)/(w-b)$, we reduce the factorization $f(g)$ to $\lambda^{-1}(\lambda(g)))$, where $\lambda(g)$ is entire. By the same method as in case (IV) and case I), we can imply that $f(\lambda^{-1})$ is linear so that $f(w)$ is linear.

Thus Theorem 2 follows.

Proof of Theorem 3 Let $F(z)$ be $f(g(z))$. Here we only state the proof of the lowing cases. In fact, for the other cases, the method is similar to that in the oof of Theorem 2.

(I) The case when f is a transcendental meromorphic function and g a nscendental entire function. By Edrei-Fuch's Theorem[15] $\rho(f)=0$. Put $f=f_1/f_2$, ere both f_1 and f_2 are entire functions of zero order without common zeros.

$$f(g)=f_1(g)/f_2(g)=\sin z/L(\cos z).$$

en there exists an entire function $M(z)$ such that

$$f_1(g)=\sin z\cdot\exp(M(z)), \tag{10}$$

$$f_2(g)=L(\cos z)\cdot\exp(M(z)). \tag{11}$$

If f_1 is transcendental, then by $\rho(f_1)=0$, f_1 has infinitely many zeros. By mma 2, g is a polynomial. Hence f_1 ought to be a polynomial. Then it follows m (10) that there exist two distinct integers m_1 and m_2 such that $g(2m_1\pi)=$ $2m_2\pi)$. Put $A(z)=g(z)$ and $B(z)=g(z+2(m_2-m_1)\pi)$. Then $A(2m_1\pi)=B(2m_1\pi)$. ice $2m_1\pi$ is a simple zero of $F(z)$, $f'(A(2m_1\pi))\neq 0$. And $f(A(z))=f(B(z))$. $A(2m_1\pi)$ is an analytic point of f and both A and B are entire functions. Therefore by these conditions, one can conclude that $A(z)=B(z)$, namely g is periodic. Then the remaining proof is similar to the proof of Theorem 2, so we omit its proof.

(II) The case when g is a polynomial. By the same method as in the above, we can get (10), where $M(z)$ is an entire function. f_1 certainly has infinitely many zeros, further by Lemma 2, g is a polynomial of degree at most 2. Then the remaining proof is similar to case (III) of Theorem 2, so we omit its proof here.

Remark. In the factorization theory of meromorphio functions, so far the present auther knows, few prime meromorphic functions have been listed and when dealing with meromorphic functions, one often assumed that there are only finite many poles. In 1981, Urabe[16] found out a sort of special prime meromorpl functions but required that the lower order and order of the functions are betweer and 2. Thus the function of Theorem 3 also has a special significance in this sense.

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