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ON NEW SIMPLE LIE ALGEBRAS OF SHEN GUANGYU

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Abstract

In [1] Shen Guangyu constructed two classes of simple Lie algebras Σ^*, $\tilde{\Sigma}$, and proved that Σ^* is new. This paper gives the invariant filtrations and the associated graded algebras of Σ^*, $\tilde{\Sigma}$. Itfollows that $\tilde{\Sigma}$ is new too. Moreover, the author gives some new invariants of Σ^*, $\tilde{\Sigma}$ and refines some results of Shen Guangyu [1, 4].

§ 0. Introdution

In [1] Shen Guangyu constructed three classes of finite-dimensional simple Lie lgebras $\bar{\Sigma}$, Σ^*, $\tilde{\Sigma}$, and proved that they are of generalized Cartan type $H(2n+2)$. n [2] he proved that $\tilde{\Sigma}$ is a class of Lie algebras associated with some nodal oncommutative Jordan algebras. In particular, [1] showed that Σ^* is a class of ew simple Lie algebras from their dimensions. In [1] it was claimed that "the outer erivation algebras of all known (p^N-1)-dimensional simple Lie algebras of eneralized Cartan type H have dimensionality $\geqslant N$", and so $\tilde{\Sigma}$ was new. But the irst claim follows from the results of R. H. Oehmke[5] which are wrong as pointed ut by H. Strade[6]. In fact, it is still an open question whether the claim is right or ot. So [1] did not show if $\tilde{\Sigma}$ is new. Our notations follow [1].

Let F be an algebraically closed field of characteristic $p>3$, n, m be two ositive integers. Let Σ be the truncated polynomial ring $F[x_{00}, x_{01}, \cdots, x_{0s_0}, x_{10}, \cdots, x_{1s_1}, \cdots, x_{n0}, \cdots, x_{ns_n}, y_{10}, \cdots, y_{1t_1}, \cdots, y_{n0}, \cdots, y_{nt_n}, \cdots, z_1, \cdots, z_m]$ where

$$x_{ij}^p=0,\ i=0, 1, \cdots, n,\ j=0, 1, \cdots, s_i,$$
$$y_{ij}^p=0,\ i=1, \cdots, n,\ j=0, 1, \cdots, t_i,$$
$$z_i^p=1,\ i=1, \cdots, m.$$

et

$$D_i=\frac{\partial}{\partial x_{i0}}+\sum_{v=1}^{s_i}\left(\prod_{u=0}^{v-1} x_{iu}^{p-1}\right)\frac{\partial}{\partial x_{iv}},\ i=0, 1, \cdots, n,$$

$$D_i'=\frac{\partial}{\partial y_{i0}}+\sum_{v=1}^{t_i}\left(\prod_{u=0}^{v-1} y_{iu}^{p-1}\right)\frac{\partial}{\partial y_{iv}},\ i=1, \cdots, n.$$

: k_i and l_i are integers such that $0\leqslant k_i<p^{s_i+1}$, $0\leqslant l_i<p^{t_i+1}$, they can be uniquely xpressed in p-adic form

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$$k_i=\sum_{v=0}^{s_i}k_{iv}p^v,\ 0\leqslant k_{iv}<p,\ l_i=\sum_{v=0}^{t_i}l_{iv}p^v,\ 0\leqslant l_{iv}<p. \tag{0.1}$$

We put

$$x_i^{k_i}=x_{i0}^{k_{i0}}\cdots x_{is_i}^{k_{is_i}},\ y_i^{l_i}=y_{i0}^{l_{i0}}\cdots y_{it_i}^{l_{it_i}}. \tag{0.2}$$

So, if $k_i,\ l_i\neq 0$, then

$$D_i x_i^{k_i}=k_i^* x_i^{k_i-1},\ D_i' y_i^{l_i}=l_i^* y_i^{l_i-1}, \tag{0.3}$$

where k_i^* and l_i^* are the first nonzero numbers of $(k_{i0},\ \cdots,\ k_{is_i})$ and $(l_{i0},\ \cdots,\ l_{it_i})$ respectively (conventionally, $x_i^{k_i}=0$ if $k_i<0$ and $y_i^{l_i}=0$ if $l_i<0$). It follows that

$$D_i x_i^{k_i}=0\Leftrightarrow k_i=0,\ D_i' y_i^{l_i}=0\Leftrightarrow l_i=0.$$

Let

$$\bar{k}=(k_0,\ k_1,\ \cdots,\ k_n),\ 0\leqslant k_i<p^{s_i+1},\ i=0,\ 1,\ \cdots,\ n, \quad (0.$$

$$l=(l_1,\ \cdots,\ l_n),\ 0\leqslant l_i<p^{t_i+1},\ i=1,\cdots,n, \quad (0.$$

$$x^{\bar{k}}y^l=x_0^{k_0}\cdots x_n^{k_n}y_1^{l_1}\cdots y_n^{l_n}. \quad (0.$$

We assume that S is the prime field of F and $\gamma_1,\ \cdots,\ \gamma_m$ are linearly independe over S. Let G be the additive subgroup of F generated by $\gamma_1,\ \cdots,\ \gamma_m$. Every elem of G can be uniquely expressed as

$$u=\sum_{i=1}^m u_i\gamma_i,\ 0\leqslant u_i<p.$$

We write

$$z^u=z_1^{u_1}\cdots z_m^{u_m},\ u\in G,$$

and have

$$z^u.\ z^v=z^{u+v},\ u,\ v\in G.$$

Let $\mu_i,\ \nu_i,\ i=1,\ 2,\ \cdots,n$, be $2n$ elements of F such that

$$\mu_i+\nu_i=1,\ i=1,\ \cdots,\ n.$$

Let

$$\partial_0'=I-\sum_{i=1}^m\left(\mu_i x_{i0}\frac{\partial}{\partial x_{i0}}+\nu_i y_{i0}\frac{\partial}{\partial y_{i0}}\right)-\sum_{i=1}^m\gamma_i z_i\frac{\partial}{\partial z_i},$$

Where I is the identity mapping. In Σ, we define an operation

$$[f,\ g]=(D_0 f)(\partial_0' g)-(D_0 g)(\partial_0' f)+\sum_{i=1}^n((Df)(D_i' g)-(D_i g)(D_i' f)). \quad (($$

It can be verified that Σ becomes a Lie algebra. Σ is said to be of type I if 1 and of type II if $1\bar{\in}G$. We have

Proposition 0. 1[1] *If Σ is of type* I, *then $\langle z\rangle$ is the center and commut subalgebra* $\Sigma'=\langle x^{\bar{k}}y^l z^r\,|\,(\bar{k},\ l,\ r)\neq(\pi,\ \pi',\ n+2)\rangle$, *where*

$$\bar{\pi}=(\pi_0,\ \pi_1,\ \cdots,\ \pi_n),\ \pi'=(\pi_1',\ \cdots,\ \pi_n'),$$

$$\pi_i=p^{s_i+1}-1,\ i=0,\ 1,\ \cdots,\ n,\ \pi_i'=p^{t_i+1}-1,\ i=1,\ 2,\ \cdots,\ n.$$

Proposition 0. 2[1] *If Σ is of type* II *then $\Sigma'=\Sigma$ when* $n+2\not\equiv 0$ (mod p) *an* $=\langle x^{\bar{k}}y^l z^u\,|\,(\bar{k},\ l,\ u)\neq(\bar{\pi},\ \pi',\ 0)\rangle$ *when* $n+2\equiv 0$ (mod p).

Definition 0. 1[1] *If* $1\in G$, *set* $\bar{\Sigma}=\Sigma'/\langle z\rangle$; *if* $1\bar{\in}G$ *and* $n+2\not\equiv 0$ (mod p) $\Sigma^*=\Sigma$, *and if* $1\bar{\in}G$ *and* $n+2\equiv 0$ (mod p), *set* $\tilde{\Sigma}=\Sigma'$.

We denote $\tilde{\Sigma}$, Σ^*, or $\bar{\Sigma}$ by X. Set

$$s(z)=\sum_{r\in G} z,\ e=x^{\bar{\pi}}y^{\pi'}s(z),\ E=\mathrm{ad}\ e|_x.$$

n [1] ker E, denoted by $\mathscr{L}_0$, was computed. Det $\mathscr{L}_{-1}=X$ and

$$\mathscr{L}_i=\{x\in\mathscr{L}_{i-1}\,|\,[x,\ \mathscr{L}_{-1}]\subset\mathscr{L}_{i-1}\},\ i>0.$$

1] claimed to have found these $\mathscr{L}_i$, $i>0$, but the result contained some errors. So Ve need some other computations to find $\mathscr{L}_i$, $i>0$.

§1. Filtrations

We define the degree of $x=x^k y^l$ or xz^r:

$$d_x=\deg x=\deg xz^r=\sum_{i=0}^{n} k_i+\sum_{i=1}^{n} l_i,$$

hich is different from the degree defined in [1] and is a key to determine $\mathscr{L}_i$, $i>$ Let

$$\alpha=\{\alpha_r\,|\,r\in G\}\subset F,\ P_t(\alpha)=P_t(\{\alpha_r\})=\sum_{r\in G}(1-r)^t\alpha_r,$$

$$t=0,\ 1,\ \cdots,\ \alpha(z)=\sum_{r\in G}\alpha_r z^r.$$

hen we have

$$\mathscr{L}_{-1}=\langle xz^r\,|\,d_x\geqslant 1\rangle\oplus\langle\alpha(z)\rangle,$$

ıd can reduce Lemma 5.1 of [1] to the following proposition.

Proposition 1. 1. $\mathscr{L}_0=\langle xz^r\,|\,d_x\geqslant 2\rangle\oplus\langle x\alpha(z)\,|\,d_x=1,\ P_0(\alpha)=0\rangle\oplus\langle\alpha(z)\,|\,P_1(z)=$ *. It is an invariant maximal subalgebra of X.*

Now we have

Theorem 1. 1. $\mathscr{L}_i=\langle xz^r\,|\,d_x\geqslant i+2\rangle\oplus\sum_{j=1}^{i+1}\langle x\alpha(z)\,|\,d_x=j,\ P_t(\alpha)=0,$

$$t=0,\ 1,\ \cdots,\ i+1-j\rangle\oplus\langle\alpha(z)\,|\,P_t(\alpha)=0,\ t=1,\ 2,\ \cdots,\ i+1\rangle,\ i=0,\ 1,\ 2,\cdots. \quad (1.1)$$

§2. The Associated Graded Algebras

To give the graded algebras Gr X of X, we need some lemmas.

Lemma 2. 1. $P_t(\alpha)=0,\ t=0,\ 1,\ \cdots,\ i\Leftrightarrow\sum_{r\in G} r^t\alpha_r=0,\ t=0,\ 1,\ \cdots,\ i.$

Proof It is because $(1-r)^t=\sum_{j=0}^{t}\binom{t}{j}(-r)^j$ and $r^t=(1-(1-r))^t$.

Lemma 2. 2. *If $P_t(\alpha)=0,\ t=0,\ 1,\cdots,\ i$, then $P_{i+1}(\alpha)=\sum_{r\in G}(-r)^{i+1}\alpha_r$.*

Proof It is because $(1-r)^{i+1}=\sum_{j=0}^{i+1}\binom{i+1}{j}(-r)^j$ and Lemma 2.1.

Lemma 2. 3. *Assume that $f(z)=\sum_{r\in G} f_r z^r,\ f_r\in F,\ P_t(f)=0,\ t=0,1,\cdots,c-1, g(z)=\sum_{r\in G} g_r z^r,\ g_r\in F,\ P_t(g)=0,\ t=0,\ \cdots,\ q-1$, and $f(z)\ g(z)=h(z)=\Sigma\ h_r z^r$. Then $P_t(h)=0,\ t=0,\ 1,\ \cdots,\ c+q-1$ and $P_{c+q}(h)=\binom{c+q}{c}P_c(f)P_q(g)$.*

Proof $P_i(h)=\sum_{r\in G}(1-r)^i h_r=\sum_{r\in G}(1-r)^i\sum_{u+v=r}f_u g_v=\sum_{r\in G}\sum_{u+v=r}(1-u-v)^i f_u g_v=$ $\sum_{u\in G}\sum_{v\in G}(1-u-v)^i f_u g_v=\sum_{u\in G}\sum_{v\in G}\left(\sum_{j=0}^{i}\binom{i}{j}(1-u)^j(-v)^{i-j}\right)f_u g_v=\sum_{j=0}^{i}\binom{i}{j}(\sum_u(1-u)^j f_u)$ $(\sum_v(-v)^{i-j}g_v)=\sum_{j=0}^{i}\binom{i}{j}P_j(f)(\sum_v(-v)^{i-j}g_v)$.

If $i=0, 1, \cdots, c+q-1$, then $P_j(f)=0$ where $j=0, 1, \cdots, c-1$, and $\sum_v(-v)^{i-j}g_v$ $=P_{i-j}(g)=0$ by lemma 2.1 where $j=c, \cdots, i$. Hence $P_i(h)=0$ if $i=0, 1, \cdots, c-1$.

Similarly, by Lemma 2.2,

$$P_{c+q}(h)=\sum_{j=0}^{c+q}\binom{c+q}{j}P_j(f)\sum_v(-v)^{c+q-j}g_v=\binom{c+q}{c}P_c(f)P_q(g).$$

The proof is completed.

Now let B be the commutative associative algebra $F[z_1, \cdots, z_m]$ and

$$B_i=\langle f(z)=\Sigma f_r z^r\,|\,P_t(f)=0,\ t=0, 1, \cdots, i-1\rangle,\ i=1, 2, \cdots.$$

We have

$$B=B_0\supset B_1\supset B_2\supset\cdots\supset B_i\supset\cdots\supset B_A=0,$$

where $A=p^m$. By Lemma 2.3 $B_uB_v=B_{u+v}$, B becomes a filtered algebra. Hence can construct a commutative associative graded algebra $\bar{B}=\mathrm{Gr}\ B=\prod_{i=0}^{A-1}B_i/B_{i+1}$: dim $B_i/B_{i+1}=1$. Let $G=\{r_1, r_2, \cdots, r_A\}$ and $a_i=((1-r_1)^i, \cdots, (1-r_A)^i)$, $i=0, 1$, $A-1$. Then $a_0, a_1, \cdots, a_{A-1}$ is a basis of vector space $F^A\cong B$. Let $u=(u_0, u_1,$ $u_{A-1})$, $v=(v_0, v_1, \cdots, v_{A-1})\in F^A$ and $u*v=\sum_{i=0}^{A-1}u_iv_i$. Take $b_0, b_1, \cdots, b_{A-1}$ as a basi F^A such that $a_i*b_i=\delta_{ij}$. We identify F^A with B, so B is an algebra with a bilir form $*$, and $b_0, b_1, \cdots, b_{A-1}$ is also a basis of B. It is easy to see that b_i+B_{i+1} forn basis of B_i/B_{i+1} and will be denoted by $b^{(i)}$, $i=0, 1, \cdots, A-1$. Moreover $b^{(0)}$, $b^{(1)}$, $b^{(A-1)}$ is a basis of $\bar{B}$ and $\bar{B}=\prod_{i=0}^{A-1}\langle b^{(i)}\rangle$.

Lemma 2. 4. $b^{(c)}b^{(q)}=\binom{c+q}{c}b^{(c+q)}$ *and hence B becomes a divided power alge*

Proof $(b_cb_q)*a_t=0$, $t=0, 1, \cdots, c+q-1$ and $(b_cb_q)*a_{c+q})=\binom{c+q}{c}(b_c*a_c)(b_q$ $=\binom{c+q}{c}$ by Lamma 2.3. Therefore $b_cb_q=\binom{c+q}{c}b_{c+q}+\sum_{i<c+q}\lambda_ib_i$, $\lambda_i\in F$. Since $\sum_{i<c+q}$ $\in B_{c+q+1}$,

$$b^{(c)}b^{(q)}=(b_c+B_{c+1})(b_q+B_{q+1})=b_cb_q+B_{c+q+1}=\binom{c+q}{c}b^{(c+q)}.$$

In L_i of Gr X, $\overline{xb_i}=xb_i+\mathscr{L}_{i+1}$ may be denoted by $xb^{(i)}$ where $x=x^ky^l$. Let $S=$ $t_j\,|\,i=0,1, \cdots,n,\ j=1, \cdots, n\}$, $\delta=\sum_{\mu\in S}(p^{\mu+1}-1)$, $\Delta=\delta+A-1$. We have

Theorem 2. 1. $\mathrm{Gr}\tilde{\Sigma}=\prod_{i=1}^{\Delta-3}L_i$, *where*

$$L_i=\Pi\langle xb^{(j)}\mid 0\leqslant j\leqslant A-1,\ d_x=i+2-j,\ and\ 0\leqslant d_x\leqslant\delta\rangle,$$
$$i=-1,0,1,\cdots,A-3,A-1,\cdots,\Delta-3.$$
$$L_{A-2}=\Pi\langle xb^{(j)}\mid 0\leqslant j\leqslant A-1,\ d_x=A-j,\ and\ 0\leqslant d_x\leqslant\delta\rangle\oplus\langle b^{(c)}\rangle.$$

: $\Sigma^*=\prod_{i=-1}^{\Delta-2}L_i$, *where* L_i *is the same as* Gr $\widetilde{\Sigma}$, $i=-1, 0, \cdots, \Delta-3$, *and* $L_{\Delta-2}=$ $\langle x^{\pi}y^{\pi'}b^{(A-1)}\rangle$. Gr $\overline{\Sigma}=\prod_{i=-1}^{\Delta-3}L_i$, *where* $L_i=\Pi\langle xb^{(j)}\mid 0\leqslant j\leqslant A-1,\ d_x=i+2-j,\ and\ 0\leqslant d_x\leqslant \delta\rangle$.

Proof By direct computation.

Lemma 2.5. *For Gr X, the* ∂_0 *in* (0.7) *may be reduced to*

$$\partial_0'=I-\sum_{i=1}^{m}\gamma_i z_i\frac{\partial}{\partial z_i}.$$

Proof Let $x=x^{k}y^{l}$, $w=x^{k'}y^{l'}$, $xb^{(c)}\in L_{i-2}$, $wb^{(q)}\in L_{j-2}$, and

$$\partial''=\sum_{i=1}^{n}\left(\mu_i x_{i0}\frac{\partial}{\partial x_{i0}}+\nu_i y_{i0}\frac{\partial}{\partial y_{i0}}\right).$$

We consider that

$$E=D_0(xb_c)\partial''(wb_q)=(D_0x)b_c(\partial'w)b_q=(D_0v)(\partial'w)\binom{c+q}{c}b_{c+q}+b\in\mathscr{L}_{i+j-4},$$

where $b\in\mathscr{L}_{i+j-3}$. However deg $((D_0x)(\partial''w))+(c+q)=(i-c)+(j-q)-1+(c+q)$ $=i+j-1\Rightarrow E\in\mathscr{L}_{i+j-3}\Rightarrow E+\mathscr{L}_{i+j}-3=0$ in Gr X. So $\partial_0'=I-\sum_{i=1}^{m}\gamma_i z_i\frac{\partial}{\partial z_i}$ for Gr X.

New ∂_0' can be viewed as an operator acting on Gr X in the obvious way, so by Lemma 2.5 we have

Lemma 2.6. $\partial_0' b^{(c)}=b^{(A-1)}$ *in* L_{A-2} *and* $\partial_0'(xb^{(c)})=xb^{(c-1)}$.

And we have

Theorem 2.2.

$$\text{Gr }\widetilde{\Sigma}=H(2n+2,\underline{r}),\tag{2.1}$$
$$\text{Gr }\overline{\Sigma}=H(2n+2,\underline{r})+\langle x^{\Delta e_{n+2}}\rangle,\tag{2.2}$$
$$\text{Gr }\Sigma^*=H(2n+r,\underline{r})+\langle x^{\Delta e_{n+2}}\rangle+\langle x^{(\pi)}\rangle,\tag{2.3}$$

where

$$\underline{r}=(s_0+1,s_1+1,\cdots,s_n+1,m,t_1+1,\cdots,t_n+1),$$
$$\pi=(p^{s_0+1}-1,\cdots,p^{s_n+1}-1,p^m-1,p^{t_1+1}-1,\cdots,p^{t_n+1}-1),$$
$$e_{n+2}=(\delta_{1\,n+2},\delta_{2\,n+2},\cdots,\delta_{2n+2\,n+2}).$$

Proof First we show (2.2). Let

$$w^{(\bar{k},l)}=(-1)^{\lambda_{\bar{k}l}}\frac{x^{\bar{k}}y^{l}}{\bar{k}!\,l!},$$

where

$$\bar{k}!=\prod_{i=0}^{n}\prod_{j=0}^{s_i}k_{ij}!,\quad l!=\prod_{i=1}^{u}\prod_{j=0}^{t_i}l_{ij}!,$$
$$\lambda_{\bar{k}l}=\sum_{i=0}^{n}\sum_{j=0}^{s_i}jk_{ij}+\sum_{i=1}^{n}\sum_{j=0}^{t_i}jl_{ij}\ (\text{see}(0.1)(0.4)(0.5)).$$

R. L. Wilson[8] has shown that if $u=\sum_{i=0}^{t}u_ip^i$, $v=\sum_{i=0}^{t}v_ip^i$, $0\leqslant u_i, v_i<p$, then

$$\binom{u+v}{u}=\prod_{i=0}^{t}\binom{u_i+v_i}{u_i}\quad(\mathrm{mod}\ p).$$

From this, putting $h=(\bar{k}, \bar{l})=(k_0, k_1, \cdots, k_n, l_1, \cdots, l_n)$, $h'=(\bar{k}', \bar{l}')=(k'_0, k'_1, \cdots, k'_n, l'_1, \cdots, l'_n)$, and by careful computation, we can verify

$$w^{(h)}w^{(h')}=\binom{h+h'}{h}w^{(h+h')}.$$

Thus $F[x_{00}x_{01}\cdots x_{0s_0}x_{10}\cdots x_{1s_1}\cdots x_{ns_n}y_{10}\cdots y_{1t_1}\cdots y_{n0}\cdots y_{nt_n}]$ becomes a divided power algebra and $\{w^{(h)}\}$ is its standard basis. On the other hand, we have

$$D_i w^{(\bar{k}, l)}=w^{(\bar{k}-\bar{e}_i, l)},\quad D'_j w^{(\bar{k}, l)}=w^{(\bar{k}, l-e'_j)}.$$

Let $x^{p^m e_{n+1}}=b^{(0)}$ in L_{A-2} of Gr $\tilde{\Sigma}$ and $x^{(\alpha)}=w^{(\bar{k}, l)}b^{(j)}$, $=(k_0, k_1, \cdots, k_n, j, l_1, \cdots, l_n)$ the other cases. Because of the above explanations and (0.7), we can obtain (2. Similarly, (2.1) (2.3) can be shown.

Theorem 2. 3. *$\tilde{\Sigma}$ is a class of new Lie algebras of characteristic $p>3$.*

Proof We have dim $\tilde{\Sigma}=p^N-1$, where $N=\sum_{i=0}^{n}(s_i+1)+\sum_{i=1}^{n}(t_i+1)+m$. On other hand, Shen Guangyu[3] proves that is J is any simple Lie algebra v dimension p^N-1 associated with a nodal noncommutative Jordan algebra, then $J=H(2q, \underline{r}')+\langle x^{(\pi)}\rangle$, where the $x^{(\pi)}$ is the "highest term" which is determined the $2q$-tuple r'. Evidently Gr $\tilde{\Sigma}$ and Gr J are not isomorphic by (2.2), and sc $\tilde{\Sigma}$ and J by [10].

§ 3. Invariants

In this section we discuss first intrinsic properties of graded Lie algebra H v). Let $N(2u)$ be the divided power algebra of $2u$ variables over F with basis $\{x$ where

$$\alpha=(\alpha_1, \cdots, \alpha_{2u}),$$

and α_i are non-negative integers. The multiplication table is

$$x^{(\alpha)}x^{(\beta)}=\binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)},$$

where

$$\binom{\alpha+\beta}{\alpha}=\prod_{i=1}^{2r}\binom{\alpha_i+\beta_i}{\alpha_i}.$$

Let

$$e_i=(\delta_{i1}, \cdots, \delta_{i2u}),\ x_i=x^{(e_i)},\ i=1, 2, \cdots, 2u.$$

Define derivations ∂_i of $N(2u)$:

$$\partial_i(x^{(\alpha)})=x^{(\alpha-e_i)},\ i=1, \cdots, 2u.$$

Define a bracket operation in $N(2u)$:

$$[f, g]=\sum_{i=1}^{2u}\theta(i)(\partial_i f)(\partial_{\sigma i} g), \tag{3.1}$$

here $\sigma i=i+u$, $\theta(i)=1$ if $1\leqslant i\leqslant u$, and $\sigma i=i-u$, $\theta(i)=-1$ if $u<i\leqslant 2u$. Then (2u) becomes a Lie algebra. It is easy to see that $\langle 1\rangle$ is the center of Lie algebra (2u). Let $H(2u)=N(2u)/\langle 1\rangle$, and

$$\underline{v}=(v_1, \cdots, v_{2u}),\ \pi=(\pi_1, \cdots, \pi_{2u}),$$

$$\pi_i=p^{v_i}-1,\ i=1, \cdots, 2u,$$

here v_i are positive integers. Write

$$H=H(x_1x_{\sigma 1}\cdots x_ux_{\sigma u})=H(2u,\ v)=\langle x^{(\alpha)}\in H(2u)\,|\,\alpha_i\langle p^{v_i},\ \alpha\neq\pi\rangle.$$

$$|\alpha|=\sum_{i=1}^{2u}\alpha_i,\ H_i=\langle x^{(\alpha)}|\ |\alpha|=i+2\rangle.$$

hen $H=\coprod_{i\geqslant -1}H_i$ is a graded Lie algebra. Because the (invariant) associated graded gebra of H is isomorphic to itself, all H_i are intrinsically determined. Let S be a bspace of H such that $[H_{-1}, S]\subset S$. If $y\in H_{-1}$, $(\mathrm{ad}\ y)^t S=0$ and $(\mathrm{ad}\ y)^{t-1} S\neq 0$, en t is called the nilpotent index of y for S and will be denoted by $NI_S(y)$. By .1), we have

$$NI_H(x_i)=p^{v\sigma_i},\ i=1, 2, \cdots, 2u. \tag{3.2}$$

rite $A_{\sigma i}=p^{v\sigma_i}$. We arrange $x_1, \cdots, x_{2u}$ in the order

$$x_{j_1},\ x_{j_2},\ \cdots,\ x_{j_{2u}}, \tag{3.3}$$

ch that

$$A_{j_1}\leqslant A_{j_2}\leqslant\cdots\leqslant A_{j_{2u}}.$$

Lemma 3. 1. *$\{NI_H(y)\,|\,y\in H_{-1}-\{0\}\}$ consists of A_i, $i=1, 2, \cdots, 2u$.*

Proof If $y\in H_{-1}-\{0\}$, then we may assume that

$$y=\sum_{q=1}^{t}c_qx_{jq},\ c_t\neq 0.$$

nce $A_{\sigma j_t}$ is a power of p and ad x_i ad x_j = ad x_j ad x_i, it is not difficult to see that

$$(\mathrm{ad}\ y)^{A_{\sigma jt}}=0,\ (\mathrm{ad}\ y)^{A_{\sigma j_t}-1}=0, \text{ and } NI_H(y)=A_{\sigma j_t}.$$

e Lemma is proved.

Lemma 3. 2. *For $H(2u, \underline{v})$, $\{v_1, v_2, \cdots, v_{2u}\}$ is an invariant set.*

Proof Since H_{-1} is intrinsically determined, dim H_{-1} is in-trinsic. Set

$$E_t=\langle y\in H_{-1}|NI_H(y)\leqslant A_{\sigma j_t}\rangle,\ t=1, 2, \cdots, 2u.$$

en E_t $(t=1, 2, \cdots, 2u)$ are intrinsically determined. Suppose that $A_{\sigma j_1}\leqslant A_{\sigma j_2}\leqslant\cdots$ $A_{\sigma j_{t-s}}<A_{\sigma j_{t-s+1}}=\cdots=A_{\sigma j_t}<A_{\sigma j_{t+1}}\leqslant\cdots\leqslant A_{\sigma j_{2u}}$. We have

$$E_{t-s+1}=\cdots=E_t=\langle x_{j_1},\ x_{j_2},\ \cdots x_{j_t}\rangle,\ E_{t-s}=\langle x_{j_1},\ x_{j_2}\ \cdots,\ x_{j_{t-s}}\rangle.$$

erefore $s=\dim E_t-\dim E_{t-s}$ and $\{A_{\sigma j_1}, \cdots, A_{\sigma j_{2u}}\}=\{A_1, A_2, \cdots, A_{2u}\}$ are intrinsic. follows that the set $\{x_1, v_2, \cdots, v_{2u}\}$ is an invariant for $H(2u, v)$.

Theorem 3. 1. *Let*

$$\underline{v}=(v_1, v_2, \cdots, v_u, v_{\sigma 1}, \cdots, v_{\sigma u}),\ \underline{v}'=(v_2', v_2', \cdots, v_u', v_{\sigma 1}', \cdots, v_{\sigma u'}').$$

Then $H(2u, \underline{v})$ and $H(2u', \underline{v}')$ are isomorphic if and only if $u=u'$ *and* $\{\{v_1, v_{\sigma 1}\}, \cdots, \{v_u, v_{\sigma u}\}\}=\{\{v_1', v_{\sigma 1}'\}, \cdots, \{v_u', v_{\sigma u}'\}\}$.

Proof The "if" part is obvious. Now we prove that $\{\{v_1, v_{\sigma 1}\}, \cdots, \{v_u, v_{\sigma u}\}\}$ is intrinsic for $H(2u, \underline{v})$.

Suppose that

$$A_{\sigma j_1} < A_{\sigma j_2} \leqslant \cdots \leqslant A_{\sigma j_{2u}}.$$

The subspace $E_1 = \langle y \in H_{-1} \mid NI_H(y) \leqslant A_{j_1} \rangle = \langle x_{j_1} \rangle$ is an invariant of $H(2u, \underline{v})$. Since $NI_H(y) = A_{\sigma j_1}$ for every $y \in N_1 - \{0\}$, the number $v_{\sigma j_1}$ is an invariant of $H(2u, \underline{v})$. Set

$$L = \{z \in H(2u, \underline{v}) \mid \text{ad } E_1.\ z = 0\}.$$

Then

$$L = \langle x^{(\alpha)} \in H \mid \alpha_{\sigma j_1} = 0 \rangle$$

and it is an invariant subalgebra of $H(2u, \underline{v})$. The center of L

$$C = \langle x_{j_1}^{(\beta)} \mid \beta = 1, \cdots, A_{j_1} - 1 \rangle$$

is invariant too. If $y \in H_{-1}$, then $NI_c(y)$ can take and can only take $A_{j_1} - 1$ or Hence the $\{v_{j_1}, v_{\sigma j_1}\}$ is intrinsically determined. Notice that the $H(x_{j_2}, x_{\sigma j_2}, \cdots,$ $x_{\sigma j_u})$ is simple. It is not difficult to compute the commutator subalgebra L' of L a the unique maximal ideal M of L'. We have

$$L'/M = H(x_{j_2}, x_{\sigma j_2}, \cdots, x_{j_u}, x_{\sigma j_u}).$$

By induction on u, the result is obtained. We omit the general proof.

For graded Lie algebras of Cartan type $K(m, \underline{n})$ where $m = 2r+1$, we similarly show (cf. [4])

Theorem 3.2. *Let $p > 3$. If $m+3 \not\equiv 0$ (mod p), Lie algebras $K(m, \underline{n})$ $K(m', \underline{n}')$ are isomorphic if and only if $m = m'$, $n_m = n'_m$ and*

$$\{\{n_1, n_1\}, \cdots, \{n_r, n_r\}\} = \{\{n'_1, n'_1\}, \cdots, \{n'_r, n'_{r'}\}\}.$$

when $m+3 \equiv 0$ (mod p), similar result holds for $K(m, \underline{n})'$.

Note 3.1. Theorem 3.2 refines [4, Theorem 3.2 and Theorem 3.4]. In the conditions of isomorphism of [4, Theorem 3.2, 3.4] are necessary but sufficient.

Theorem 3.3. *For $\Sigma^*, \tilde{\Sigma}$ or $\bar{\Sigma}$, $\{\{m, s_0+1\}, \{s_1+1, t_1+1\}, \cdots, \{s_n+1, t_n+$ is invariant.*

Proof Let $X = \Sigma^*$, $\tilde{\Sigma}$ or $\bar{\Sigma}$. By Theorem 2.2 we have

$$(\text{Gr } X)' = H(2n+2, \underline{r}).$$

By Theorem 3.1 the conclusion is reached.

Note 3.2 Theorem 3.3 refines [1, Theorem 5.3].

Let $N(2u)_k = \langle x^{(\alpha)} \mid |\alpha| = k \rangle$ and $V^* = \langle \partial_1, \cdots, \partial_{2u} \rangle$. For $f \in N(2u)_k$ define $f^\perp =$ $\in V^* \mid \partial f = 0\}$. We have

Lemma 3.3 *If $f \in N(2u)_k$ and $\dim f^\perp = 2u-1$, then $f = (\Sigma a_i x_i)^{(k)}$, $a_i \in F$.*

Theorem 3.4 *For Σ^* or $\tilde{\Sigma}$, the triple (n, m, s_0) is invariant.*

Proof Let $L = \coprod_{i \geqslant -1} L_i$ be Gr Σ^* or Gr $\tilde{\Sigma}$. First, n is invariant by $\dim L_{-1} = 2n + 2$. Second, Let L' be derived algebra of L. Write $L' = \coprod_{i \geqslant -1} L'_i$, L'_i being a subspace

? L_i. For $i \leqslant p^m-2$, we have

$$\dim L_i/L'_i=\begin{cases}1, & \text{if } i=p^m-2,\\ 0, & \text{otherwise,}\end{cases}$$

y (2.2) (2.3) of [1]. Hence m is invariant too. Finally, set $u=n+1$, L can be iewed as a graded subspace of $N(2u)$. On $L_i(i>-1)$ the acting of V^* means the ›ting of ad L_{-1}. Let

$$M_{p^m-2}=\{f\in L_{p^m-2}\,|\dim f^{\perp}=(2n+2)-1\},$$

ıd

$$M=\langle(\text{ad } L_{-1})^{p^m-1}M_{p^m-2}\rangle.$$

Obviously M_{p^m-2} and M are intrinsically determined. By Lemma 3.3 eve_y ement in M_{p^m-2} can be expressed as

$$f=(\Sigma\, a_i x_i)^{(p^m)},$$

here the highest index of $x_i \geqslant p^m$, i. e. $i=n+2$ or with $p^{v_i}>p^m$. It is easy to see that

$$M=\langle x_i\in L_{-1}\,|\, i=n+2 \text{ or } v_i>m\rangle.$$

›t $L'=H$ and $T=\{NI_H(y)\,|\,y\in M\}$. Take out numbers from T each of which pairs ith some p^{v_i} as in Theorem 3.1 with $v_i>m$, the remaining number is p^{s_0+1}. By an gument similar to the proofs of Lemma 3.1, and Theorem 3.1 of [1], we can ove that s_0+1, or s_0, is an invariant.

Combining with [1, Theorem 5.3 and Corollary 5.1] we have the following finement of [1, Corollary 5.1] [2, Theorem 3.2]:

Theorem 3. 5 *$X(n, m, \underline{r}, G)\cong X(n', m' \underline{r}', G')$ implies $n=n'$, $m=m'$, $s_0=s'_0$, $s_1, t_1\}, \cdots, \{s_n, t_n\}\}=\{\{s'_1, t'_1\}, \cdots, \{s'_n, t'_n\}\}$.*

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