Chin. Ann. of Math.
10B (4) 1989

SOME CONSISTENT RESULTS ON* LINDELÖFNESS AND CALIBRE

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Abstract

This paper gives some topological propositions whioh are equivalent to the continuum hypothesis. The following results are also given: In the class of 1-st countable Hausdoff spaces, the existence of space which has calibre (ω_1, ω) but no calibre ω_1 is equivalent to the existence of space which has calibre (ω_1, ω) but is not point-countablely Lindelög, the existence of space which has calibre ω_1 but is not separable is equivalent to the existence of space which has calibre ω_1 but is not *Lindelöf, too.

A topological space X is called a *Lindelöf space iff for any open cover $\mathscr{G}$ of X, there exists a countable subset $A\subset X$ such that $\bigcup\{st(x, \mathscr{G}): x\in A\}=X$. X is called a pc-lindelöf (point-countable Lindelöf) iff every point countable open cover of X has a countable subcover. X has calibre ω_1 (calibre (ω_1, ω), respectively) iff every point-countable (point-finite, respectively) family of open sets has the cardinality less than ω_1. It is obvious that *Lindelöfness implies pc-lindelöfness, and every space with calibre ω_1 is pc-lindelöf. We refer the reader to [1] for the related results.

In this paper, we give an equivalent characterization for CH (Continuum Hypothesis) and two consistent results concerning calibre ω_1 and calibre (ω_1, ω).

§1. An Equivalent Proposition of CH

First of all, we give a lemma which is easy to prove by pigeon-hole principle.

Lemma. *If X is a first countable space with calibre ω_1 and $\mathscr{U}$ is an uncountable family of open sets, there exists an uncountable sub-family $\mathscr{U}_0$ of $\mathscr{U}$ such that* Int $(\bigcap\mathscr{U}_0)=\varnothing$.

Theorem 1. *The following are equivalent:*

(1) CH,

(2) *Every T_2 space which has calibre ω_1 and cardinality $\leqslant 2^{\omega}$ is separable,*

(3) $(R, d\vee c)$ *does not have calibre ω_1,*

Manuscript received April 27, 1987.

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4) $(R, s\vee c)$ *does not have calibre* ω_1,

5) *Every* T_2 *space which has calicre* ω_1 *and cardinality* $\leqslant 2^\omega$ *is* *Lindelöf.

Here R is the set of all reals, d, s, c denote the open interval topology, afrey topology and co-countable topology, respectively. $d\vee c$, $s\vee c$ are topologies ated by $d\cup c$ or $s\cup c$, respectively.)

roof For (1)→(2) see [4] (Proposition 3.20). Note that $(R, d\vee c)$ is a non- ible Urysohn space, $|R|=2^\omega$ and $s\vee c$ is stronger than $d\vee c$, (2)→(3)→(4) 2)→(5) are obvious.

)→(4). Assuming CH be falfe, let $\mathscr{G}$ be an uncountable subset of $s\vee c$. Every $G\in\mathscr{G}$ denoted by $G=U_G-A_G$, here $U_G\in s$ and $|A_G|\leqslant\omega$. By the lemma, there exists $U_\alpha-A_\alpha$: $\alpha<\omega_1\}\subset\mathscr{G}$ and an open interval (a, b) such that $(a, b)\subset\bigcap_{\alpha<\omega_1}U_\alpha$. we have $\bigcap\mathscr{G}_0=\bigcap_{\alpha<\omega_1}U_\alpha-\bigcap_{\alpha<\omega_1}A_\alpha\supset(a, b)-\bigcup_{\alpha<\omega_1}A_\alpha\neq\varnothing$ because of $|\bigcup_{\alpha<\omega_1}A|=\omega_1<2^\omega$. ays that $(R, s\vee c)$ has calibre ω_1.

o prove (5) →(1), it is sufficient to construct a countorexample under $\neg$CH. $_0=R\times\{0\}$, $L_1=R\times\{1\}$, and $X=L_0\cup L_1$. We define the topology on X as ing:

$\subset X$ is open iff

if point $(x, 1)\in G\cap L_1$, then there exists $n<\omega$ and $A\in[R]^{<\omega}$ such that $((x,)-A)\times\{1\}\subset G$;

if point $(x, 0)\in G$, then there exists $n<\omega$ and $A\in[R]^{<\omega}$ such that $((x-1/n,)\times\{1\}\subset G$.

ith this topology, every pair of points is functinally separated. L_1 is an open subspace and homeomorphic to $(R, s\vee c)$. L_0 is a closed discrete set in X. If holds, then L_1 has calibre ω_1, so does the space X. Now we show that X is not löf. Let $\{A_\alpha$: $\alpha<2^\omega\}$ enumerate $[R]^{<\omega}$ such that for every $A\in[R]^{<\omega}$, $|\{\alpha$: $A_\alpha=$ 2^ω. Let $\{x_\alpha$: $\alpha<2^\omega\}$ enumerate R. For every $\alpha<2^\omega$ let $G_\alpha=\{(x_\alpha, 0)\cup(R-A_\alpha)$. $\mathscr{G}=\{G_\alpha$: $\alpha<2^\omega\}$ is an open cover of X. $(x_\alpha, 0)\in G_\beta$ iff $\alpha=\beta$. For any $M\in$ let $M=A\times\{1\}$. Then $A_\alpha=A$ implies $G_\alpha\cap M=\varnothing$. Hence $(x_\alpha, 0)\overline{\in}\cup\{st(p, \mathscr{G})$: . But $\{\alpha$: $A_\alpha=A\}$ is uncountable. For any $N\in[L_0]^{<\omega}$, there exists α such $_\alpha=A$ and $(x_\alpha, 0)\overline{\in}\cup\{st(p, \mathscr{G})$: $p\in N\}$. This shows that $\{st(p, \mathscr{G})$: $p\in X\}$ countablo subcover.

Two Results Concerning Calibre ω_1 and Calibre (ω_1, ω)

ieorem 2. *If there exists a first countable* T_2 *space which has calibre* (ω_1, ω) *not have calibre* ω_1, *then there exists a first countable* T_2 *space with calibre* $(\omega_1,$ *ch is not pc-lindelöf.*

Proof (1) Let S be a first countable T_2 space which has calibre (ω_1, ω) but oes not have calibre ω_1. Let T be the Sorgenfrey line. $Y=S\times T$ is the product pace. Y is first countable T_2 space. If $\mathscr{W}=\{u_\alpha\times v_\alpha: \alpha<\omega_1\}$ is a family of basic open ets, $\mathscr{V}=\{v_\alpha: \alpha<\omega_1\}$, there must be $t\in T$ such that $\mathrm{ord}(t, \mathscr{V})=\omega_1$. Let $A=\{\alpha: t\in$ $v_\alpha\}$, $\mathscr{U}_A=\{u_\alpha: \alpha\in A\}$. Then there exists $s\in S$ such that ord $(s, \mathscr{U}_A)\geqslant\omega$. Let $y=(s, t)$ $\in Y$. It is easy to see that $\mathrm{ord}(y, \mathscr{W})\geqslant\omega$. This shows that Y has calibre (ω_1, ω). But Y has not calibre ω_1 since S is the continuous image of Y.

(2) S is ccc with character $\chi(S)=\omega$, then $|S|\leqslant^{c(s)\chi(s)}=2^\omega$. Let $S=\{s_x: x\in E\}$, $(E\subset R)$. $X=E\cup Y$. We select a decreasing neighbourhood base $\{W_n(s): n<\omega\}$ for every $s\in S$. For $n<\omega$, if $y=(s, t)\in Y$, we define $V_n(y)=W_n(s)\times[t, t+1/n)$, if $x\in E$, we define $V_n(x)=\{x\}\cup[W_n(s_x)-\{s_x\}]\times(x-1/n, x)$. With the base $\{V_n(y): y\in Y, n<\omega\}\cup\{V_n(x): x\in E, n<\omega\}$, the topology on X is first countable and T_2. Y is an open dense subspace of X. By (1), X has calibre (ω_1, ω).

(3) Let $\{U_\xi: \xi<\omega_1\}$ be a point countable family of open sets of S. Then $\{U_\xi\times T: \xi<\omega_1\}$ is a point countable family of open sets of Y. Let $U=\bigcup_{\xi<\omega_1} U_\xi$. $U\times T$ is an open subspace of Y, hence $U\times T$ has calibre (ω_1, ω) but does not have calibre ω_1. For all $\xi<\omega_1$, we choose $s_{x_\xi}\in U_\xi$ and $n_\xi<\omega$ such that $W_{n_\xi}(s_{x_\xi})\subset U_\xi$. Since

$$V_{n_\xi}(x_\xi)=\{x_\xi\}\cup(W_{n_\xi}(s_{x_\xi})-\{s_{x_\xi}\})\times(x_\xi-1/n_\xi, x_\xi),$$

we have $V_{n_\xi}(x_\xi)\cap Y\subset U_\xi\times T\subset U\times T$.

(4) Let $Z=\{x_\xi: \xi<\omega_1\}\cup(U\times T)$. The subspace Z of X has calibre (ω_1, ω) because $U\times T$ is open and dense in Z. $\{V_{n_\xi}(x_\xi): \xi<\omega_1\}\cup\{U\times T\}$ is a point countable opencover of Z which has not countable subcover.

Theorem 3. *If there exists a first countable T_2 space with calibre ω_1 which is not separable, then there exists a first countable non-*lindelöf T_2 space with calibre ω_1.*

Proof Let S be a first countable non-separable T_2 space with calibre ω_1, T be the Sorgenfrey line, $Y=S\times T$. It is easy to see $|[Y]^\omega|=2^\omega$. Let $\{A_\alpha: \alpha<2^\omega\}$ be an enumeration of $[Y]^\omega$ such that for every $A\in[Y]^\omega$, $|\{\alpha: A_\alpha=A\}|=2^\omega$. Let P be the projection from Y to S. Obviously, $P(cl_Y A_\alpha)\subset cl_S(P(A_\alpha))$ for all $\alpha<2^\omega$. $|P(A_\alpha)|\leqslant\omega$ and S is not separable. There exists $s_\alpha\in S-cl_S(P(A_\alpha))$. Choose a decreassing neighbourhood base $\{W_n(s_\alpha): n<\omega\}$ of s_α such that $W_0(s_\alpha)\cap cl_S(P(A_\alpha))=\varnothing$. Let $u_{n\alpha}=W_n(s_\alpha)-\{s_\alpha\}$.

Let $X=\{x_\alpha: \alpha<2^\omega\}$ be the set of all reals, $Z=X\cup Y$ be the disjoint union of X and Y. We define a topology on Z generated by the base $\mathscr{B}$ described as following:

If $y=(s, t)\in Y$, $V_n(y)=W_n(s)\times[t, t+1/n)\in\mathscr{B}$ for all $n<\omega$. Here $\{W_n(s): n<\omega\}$ is a decreasing neighbourhood base of s in S.

If $x\in X$, $V_n(x_\alpha)=\{x_\alpha\}\cup u_{n\alpha}\times(x-1/n, x)\in\mathscr{B}$ for all $n<\omega$. It is not difficult to check that the topology generated by $\mathscr{B}$ is first countable and T_2. Y is open and in Z, and X is a closed discrete subset of Z. The subspace Y is homeomorphic T, the product space of S and T. Therefore Z has calibre ω_1.

ow we claim that Z is not *Lindelöf. For every $\alpha<2^\omega$, let $G_\alpha=(Y-cl_Z(A_\alpha))\cup$ $\mathscr{G}=\{G_\alpha: \alpha<2^\omega\}$.

) $\mathscr{G}$ covers Z. Let $y=(s, i)\in Y$. Since $Y=\{y\}\cup[\bigcup_{n<\omega}(Y-V_n(y))]$ and Y is ntable, there exist $n<\omega$ and $\alpha<2^\omega$ such that $cl_Y(A_\alpha)$ is contained in $Y-V_n(y)$. $y\in cl_Y(A_\alpha)=cl_Z(A_\alpha)\cap Y$ and $y\in G_\alpha$.

) G_α is open for all $\alpha<2^\omega$. It is sufficient to show that $x_\alpha\in \text{Int } G_\alpha$. Note that $)\cap Y=cl_Y(A_\alpha)$, so $Y-cl_Z(A_\alpha)=Y-cl_Y(A_\alpha)$. Since $u_{1\alpha}\cap cl_S(P(A_\alpha))=\varnothing$, $u_{1\alpha}\times$ $, x_\alpha)\subset u_{1\alpha}\times T$ is disjoint with $cl_S(P(A_\alpha))\times T$. But $cl_Y(A_\alpha)\subset P^{-1}(cl_S(P(A_\alpha)))$ $P(A_\alpha))\times T$, we have $V_1(x_\alpha)\subset(Y-cl_Y(A_\alpha))\cup\{x_\alpha\}=G_\alpha$.

$\{st\ (z, \mathscr{G}): z\in Z\}$ has not countable subcover. Let $A\in[Y]^\omega$ and $A=A_\alpha$. $A\cap G_\alpha=\varnothing$ and $x_\alpha\bar{\in} U\{st(y, \mathscr{G}): y\in A\}$. Because $\{\alpha: A_\alpha=A\}$ is uncountable, $\{st(y, \mathscr{G}): y\in A\}$ is uncountable. On the other hand, $st(x_\alpha\ \mathscr{G})\cap X=\{x_\alpha\}$ for $\in X$. This shows that for any $M\in[Z]^\omega$, $U\{st(z, \mathscr{G}): z\in M\}\neq Z$.

is known that the space $\mathscr{X}$ in [2] is 0–dimensional first countable Baire cco) with calibre (ω_1, ω)) non–separable space. Assuming CH, $\mathscr{X}$ has not calibre ssuming $MA+\neg CH$, $\mathscr{X}$ is ω_2 Baire and has calibre ω_1 ([5]). As the ries of Theorems 2 and 3, we have

orollary 4. (CH) *There exists a first countable T_2 space with calibre (ω_1, ω) is not pc–lindelöf.*

orollary 5. (MA+$\neg$CH) *There exists a first countable T_2 space with calibre ch is not *Lindelöf.*

1 the other hand, the result of Efimov [3] tells us that CH implies that every untable T_2 space with calibre ω_1 is separable. Then the statement in Corollary lependent of ZFC. But we do not know wheather the statement in Corollary depen–dent of ZFC.

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