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ON THE REACHABLE SEMIGROUP OF BILINEAR CONTROL SYSTEMS ON LIE GROUP**

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Abstract

This paper studies the reachability and the structure of reachable semigroup of bilinear control systems on Lie group. In the second section some equivalency lemmas are given, which not only simplify the proofs of the main results, but discover some properties of systems also. In the third section some conditions are advanced that the reachable semigroup of system is weakly symmatric by means of the study of one parameter subgroups. This study is discussed by manifold theory and matrix theory, respectively. In the last section, some topological properties of the reachable semigroup are advanced.

§1. Introduction

In this paper the bilinear control system on a Lie group G is described as following

$$\frac{dx}{dt}=A_0(x)+\sum_{i=1}^{n}u^i(t)A_i(x), \tag{1.1}$$

where $x\in G$ and $u^i(t)$, $i\in\{1, \cdots, m\}=:\underline{m}$, are piecewise continuous real value functions on $[0, \infty)$. $A_i(x)$, $i\in\underline{m}$ or $i=0$, are right invariant vector fields on G. Thus, (1.1) is also called right invariant system. For our purpose, it is convenient to write $A_i(x)=A_ix$, and regard A_i as the element of the Lie algebra of G, which is denoted by $\mathfrak{g}$. Without loss of generality, we assume that A_i, $i\in\underline{m}$, are independent vectors in $\mathfrak{g}$. This study is based on the results of [2]. In § 2 we advance some equivalency lemmas which will simplify the proofs of our main results given in this paper. § 3 discribes the conditions that the reachable semigroup from the unit element e of G, denoted by $\mathcal{A}(e)$, is a group by means of manifold theory and matrix theory, respectively. In § 4 some properties related to the reachable semigroups are discussed.

System (1.1) can also be described by a family of vector fields. Let Γ be a topological (metric) space defined as follows. As a set, $\Gamma\subset\mathfrak{g}$, and its topological

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structure is induced by the Euclidean norm of $gl(n, R)$, of which a subspace is isomorphic with $\mathfrak{g}$.

Define the admissible control sot $\Omega(\Gamma)$ to be the set of Γ-valued functions on $[0, \infty)$. Define $\mathscr{T}_n(\Gamma)$ to be the set of differential equations such that

$$\mathscr{T}_\Omega(\Gamma):=\{dx/dt=U(t)x(t)/U(t)\in\Omega(\Gamma)\},$$

where $U(t)\ x(t)$ is a time-varying right invariant vector field, i.e. fixed t, Ux is a right invariant vector field on G, as we have pointed out above. The U can be regarded as a vector of $\mathfrak{g}$. In this study, Γ is often referred to as an affine subspac of $\mathfrak{g}$, i. e.

$$\Gamma:=\{A_0+c^1A_1+\cdots+c^mA_m/c^i\in\boldsymbol{R},\ i\in\underline{m}\}.$$

Furthermore, we define two admissible control sets:

$\Omega_0(\Gamma):=\{U(t)\in\Omega(\Gamma)/U(t)$ is Γ-valued piecewise constant functions$\}$.

$\Omega_1(\Gamma):=\{U(t)\in\Omega(\Gamma)/U(t)$ is Γ-valued piecewise continuous functions$\}$.

Correspondingly, we have $\mathscr{T}_{\Omega_0}(\Gamma)$ and $\mathscr{T}_{\Omega_1}(\Gamma)$, respectively. Instead of (1.1) a bilinear control system can be described by $\mathscr{T}_\Omega(\Gamma)$, or more precisely, $\mathscr{T}_{\Omega_0}(\Gamma)$ ($\mathscr{T}_{\Omega_1}(\Gamma)$.

Notations: $\mathrm{int}(A)$, interior of set A; $bd(A)$ boundary of A; $\mathrm{cl}(A)$ or $\bar{A}$, closure of A $\varnothing$, empty set. $\boldsymbol{Z}$, set of integers; $\boldsymbol{R}$, field of reals; $\boldsymbol{C}$, field of complex; $\exp\langle X\rangle$, th one parameter subgroup of G generated by X of $\mathfrak{g}$; $\exp\mathfrak{P}$, the set $\{\exp(tX)/X\in\mathfrak{P}$ $t\in\boldsymbol{R}\}$.

A trajectory $x(t)$ from $g\in G$ is a piecewise differentiable continuous curve on G which satisfies that $dx/dt=U(t)x(t)$ for some $U(t)\in\Omega(\Gamma)$ and $x(0)=g$. An elemen g of G is called reachable from g_0 if there exists a trajectory $x(t)$ and a real numb $T>0$ such that $x(0)=g_0$ and $x(T)=g$. The set of all elements of G reachable from is called the reachable set and is denoted by $A(g)$. It is easy to varify that $A(g)=A(e)g:=\{xg/x\in A(e)\}$. From [2] $A(e)$ is a path-connected subsemigroup of G thus, it is also called the reachable semigroup of $\mathscr{T}_\Omega(\Gamma)$.

Remark. By our definition of $A(e)$ given above, the unit element e is n necessary to belong to $A(e)$.

Γ is given, then define $\Gamma^s:\Gamma\cup(-\Gamma)$. We call $\mathscr{T}_\Omega(\Gamma^s)$ the symmetrized syste of $\mathscr{T}_\Omega(\Gamma)$. The weakly reachable set of $\mathscr{T}_\Omega(\Gamma)$ is defined by the reachable set of i symmetrized system $\mathscr{T}_\Omega(\Gamma^s)$, and denoted by $WA(g)$, for $g\in G$.

We rewrite the results of [2] in following

Proposition 1. (1) *$WA(e)$ is a Lie subgroup of G. If Γ is given, $\Gamma=X+$ where X is a vector in $\mathfrak{g}$, $L=\Gamma-X$ is a linear subspace of $\mathfrak{g}$, then $WA(e)$ is th smallest connected subsemigroup of G containing $\exp\langle X\rangle$ and $\exp L$.*

(2) *$\mathrm{int}(A(e))$ is not empty in the relative topology of $WA(e)$. Moreover, $\mathrm{int}(A(e))$ is dense in $\mathrm{cl}(A(e))$.*

§ 2. Some Lemmas Relevant to Equivalence of Systems

In this section several lemmas relavant to equivalence of systems are advanced. /e call them equivalency lemmas. Sometimes there are several approaches to study problem. By equivalency lemmas it might be possible for us to choose a simpler ay to deal with the problem that we are interested in.

Γ and $\tilde{\Gamma}$ are two different topological subspaces of $\mathfrak{g}$. $\Omega(\Gamma)$ and $\tilde{\Omega}(\tilde{\Gamma})$ are two ifferent adimissible control sets, and we denote the reachable semigroups of $\mathscr{T}_\Omega(\Gamma)$ nd $\mathscr{T}_{\tilde{\Omega}}(\tilde{\Gamma})$ by $A(e)$ and $\tilde{A}(e)$, respectively.

Definition 2.1. $\mathscr{T}_\Omega(\Gamma)$ and $\mathscr{T}_{\tilde{\Omega}}(\tilde{\Gamma})$ are weakly equivalent to each other if $\overline{A(e)}=\overline{\tilde{A}(e)}$.

Let $\mathfrak{P}$ be a linear subspace of $\mathfrak{g}$ and $A_0\in\mathfrak{g}$. Write $\mathfrak{P}=\operatorname{span}\{A_1, \cdots, A_m/A_i\in$, $i\in\underline{m}\}$ and $\Gamma=A_0+\mathfrak{P}$, which is an affine subspace $\mathfrak{g}$. Therefore, (1.1) can be escribed by $\mathscr{T}_\Omega(\Gamma)$. When Γ is fixed, there exists the smallest closed Lie subgroup H of G, which containing $\exp\mathfrak{P}$. Let η be the Lie algebra of H.

Lemma 2.2. *For any vector $A_0(\in\mathfrak{g})$, $\mathscr{T}_{\Omega_0}(A_0+\mathfrak{P})$ and $\mathscr{T}_{\Omega_0}(A_0+\eta)$ are weakly quivalent to each other.*

Proof The reachable semigroups of $\mathscr{T}_{\Omega_0}(A_0+\mathfrak{P})$ and $\mathscr{T}_{\Omega_0}(A_0+\eta)$ are denoted by $A_{\mathfrak{P}}(e)$ and $A_\eta(e)$, respectively. By definition, $\mathfrak{P}\subset\eta$, thus, $A_{\mathfrak{P}}(e)\subset A_\eta(e)$ and hen $\overline{A_{\mathfrak{P}}(e)}\subset\overline{A_\eta(e)}$.

For each $B\in\mathfrak{P}$, $\exp B=\lim\limits_n((1/n)(A_0+nB))\in\overline{A_{\mathfrak{P}}(e)}$, i.e. $\exp\mathfrak{P}\subset\overline{A_{\mathfrak{P}}(e)}$. Since $\overline{A_{\mathfrak{P}}(e)}$ is a closed semigroup, $\langle\exp\mathfrak{P}\rangle$, the smallest semigroup generated by $\exp\mathfrak{P}$, is contained in $\overline{A_{\mathfrak{P}}(e)}$. Therefore, $H=\overline{\langle\exp\mathfrak{P}\rangle}\subset\overline{A_{\mathfrak{P}}(e)}$. For each $C\in\eta$, $\exp\left(\frac{t}{n}C\right)\in H\subset\overline{A_{\mathfrak{P}}(e)}$ and $\exp\left(\frac{t}{n}A_0\right)\in A_{\mathfrak{P}}(e)$ when $t>0$. Thus, for $t>0$, $\exp t(A_0+C)=\lim\limits_{n\to\infty}\left(\exp\left(\frac{t}{n}A_0\right)\cdot\exp\left(\frac{t}{n}C\right)\right)^n\in\overline{A_{\mathfrak{P}}(e)}$. Notice the condition $\Omega=\Omega_0$, $\overline{A_\eta(e)}$ is the closure of the semigroup generated by the set $\{\exp(t(A_0+C)/t>0, C\in\eta\}$. Recall $\overline{A_{\mathfrak{P}}(e)}$ is a closed semigroup, hence, $\overline{A_\eta(e)}\subset\overline{A_{\mathfrak{P}}(e)}$.

Lemma 2.3. *Let $\Gamma=A_0+\mathfrak{P}$, and $\Gamma^0=\{A_0\}\cup\mathfrak{P}$. $\mathscr{T}_{\Omega_0}(\Gamma)$ and $\mathscr{T}_{\Omega_0}(\Gamma^0)$ are weakly equivalent to each other.*

Proof Let $A_0(e)$ and $A(e)$ be the reachable semigroups of $\mathscr{T}_{\Omega_0}(\Gamma^0)$ and $\Gamma_{\Omega_0}(\Gamma)$, respectively. We show that $\overline{A(e)}\subset\overline{A_0(e)}$. Let $B\in\mathfrak{P}$,

$$\exp(t(A_0+B))=\lim_n\left(\exp\left(\frac{t}{n}A_0\right)\exp\left(\frac{t}{n}B\right)\right)^n\in\overline{A_0(e)}.$$

Thus, $\overline{A(e)}=\overline{A_0(e)}$.

Lemma 2.4. *Let Γ be an arbitrary topological (metric) subspace of $\mathfrak{g}$. $\mathscr{T}_{\Omega_0}(\Gamma)$ is weakly equivalent to $\mathscr{T}_{\Omega_1}(\Gamma)$ (where $\Omega_1(\Gamma)$ is the set of picewise continuous Γ-*

valued functions).

Proof Given $v(t)$ and $u(t)$ in $\Omega_1(\Gamma)$, assume that v and u are defined on $[0, T]$ for a real number $T(>0)$. $\|v-u\|_{\Omega_1} := \sup_{t\in[0,T]} \|v(t)-u(t)\|_T$. Let $x(t)$ and $y(t)$ be the trajectories of $\mathscr{T}_{\Omega_1}(\Gamma)$ with $x(0)=y(0)=e$, which are driven by $v(t)$ and $u(t)$, respectively. Thus, we have

$$dx/dt=u(t)x(t),\ x(0)=e, \tag{2.1}$$

$$dy/dt=v(t)y(t),\ y(0)=e. \tag{2.2}$$

Let $z(t)=x(t)y(t)^{-1}$, then

$$dz/dt=u(t)z(t)-z(t)v(t),\ z(0)=e. \tag{2.3}$$

When $u=v$, the solution of (2.3) is $z(t)=e$. Rowrite (2.3) in the form

$$dz/dt=u(t)z(t)-z(t)u(t)+z(t)(u(t)-v(t)). \tag{1.4}$$

The term $z(u-v)$ is regarded as a perturbation when $\|u-v\|<r$, where r is small positive number. For any $u\in\Omega_1(\Gamma)$ and $r>0$, there exists a $v\in\Omega_0(\Gamma)$ such that $\|u-v\|_{\Omega_1}<r$. Therefore, the solution of (2.4) can remain in an arbitrary unit neighborhood if the perturbation is small enough.

Lemma 2.5. *The system $\mathscr{T}_{\Omega_0}(\Gamma)$ is given. Then $A(e)=WA(e)$ if and only if $e\in A(e)$.*

Proof If $e\in A(e)$, then there exist $t_1, \cdots, t_k>0$ and $X_1, \cdots, X_k\in\Gamma$, such that $\exp(t_kX_k)\cdots\exp(t_1X_1)=e$. Then $\exp(-t_1X_1)=\exp(t_kX_k)\cdots\exp(t_2X_2)\in A(e)$. As for each $t\in\boldsymbol{R}$ there exists $n\in\boldsymbol{Z}$ and $t_0>0$ such that $t=nt_1+t_0$, $\exp(tX_1)=\exp(t_0X_1)(\exp(t_1X_1))^n\in A(e)$, i.e. $\exp\langle X_1\rangle\subset A(e)$. Since $WA(e)$ is the semigroup generated by $\exp\langle X_1\rangle$ and $\exp\mathfrak{P}$, we have $WA(e)\subset A(e)$. This proof is completed.

§3 The Classification of One Parameter Subgroups and Weak Symmetry

For a connected Lie group G with its Lie algebra $\mathfrak{g}$, let $X\in\mathfrak{g}(X\neq 0)$, and define a C^∞-map:

$$\exp(.,\ X): \boldsymbol{R}\to G,\ t\to\exp(tX) \tag{3.1}$$

which is an immersion of $\boldsymbol{R}$ into G. We claim that:

(1) If $X=0$, X and/or $\exp\langle X\rangle$ is of tpye 0.

(2) X and/or $\exp\langle X\rangle$ is of type I if $\exp(.,\ X)$ is an immersion, but not an embedding.

(3) X and/or $\exp\langle X\rangle$ is of type II if $\exp(.,\ X)$ is an irregular embedding.

(4) X and/or $\exp\langle X\rangle$ is of type III if $\exp(.,\ X)$ is a regular embedding.

Remarks. (1) let M and N be C^∞-manifolds, f be a C^∞-map from M to N, f is an immersion if df_m (the differential of f at m) is nonsingular at any $m\in M$. f

an embedding if f is an injective, an embedding f is regular if f: $M \to f(M) \subset N$ a homomorphism.

(2) When $X=0$, $\exp(., X)$ must not be an immersion.

(3) If X is of typo I, then $\exp\langle X\rangle$ is one dimensional period group which can e regarded as the embedding submanifold of one dimensional torus $\boldsymbol{R}/\boldsymbol{Z}$ into G. hus, $\exp\langle X\rangle$ is a compact subgroup (of cause, a closed group too) of G.

(4) Recall the well known theorem given by E. Cartan that a Lie subgrop H of is closed if and only if H is a regular embedding submanifolds to G. Therefore the ne parameter subgroup of type III is closed (but not compact) and that of type II nust not be closed.

Lemma 3.1. *If* $\exp\langle X\rangle$ *is of type* II, *then* $\overline{\exp\langle X\rangle}$ *is a Lie subgroup and* dim $\overline{\text{xp}\langle X\rangle}>1$.

Proof The first conclusion is obvious. We show that dim $\overline{\exp\langle X\rangle}>1$. As xp$\langle X\rangle$ is connected, $\overline{\exp\langle X\rangle}$ is connected too, and it is also a Lie subgroup. Since $\overline{\text{xp}\langle X\rangle}\supset\exp\langle X\rangle$, $T_e\overline{\exp\langle X\rangle}\supset T_e \exp\langle X\rangle$. Then dim $\overline{\exp\langle X\rangle}=\dim T_e\overline{\exp\langle X\rangle}>$ im $T_e\exp\langle X\rangle=1$.

Lemma 3.2. *If* $\exp\langle X\rangle$ *is of type* 0, I *or* II, *then there exist* $t_n\in\boldsymbol{R}$, $n\in\boldsymbol{Z}^+$, *uch that* $\lim t_n=+\infty$ *and* $\lim\exp(t_nX)=e$.

Proof If $\exp\langle X\rangle$ is of type 0 or I, then there exists $T>0$ such that $\exp(TX)=$, let $t_n=nT$, then $\lim t_n=+\infty$ and $\exp(t_nX)=e$. If $\exp\langle X\rangle$ is of type II, then $\overline{\text{xp}\langle X\rangle}$ is a Lie subgroup with dimensisn more than one. Let $\exp^*\langle X\rangle=\{\exp(tX)/ t\|>c>0$ for some $c\}$, then $\exp^*\langle X\rangle$ is dense in $\overline{\exp\langle X\rangle}$, in particular, there exists sequence s_n such that $\lim\limits_n\exp(s_nX)=e$. It is obvious that the sequence is nbounded. Let $t_n=|s_n|$, then it is easy to varify that $\lim t_n=+\infty$ and $\lim\exp(t_nX)=e$.

Lemma 3.3. *For any vector* g *if* X *is not of type* III, *and* $\exp(tX)\in\overline{A(e)}$ $(t>$), *then* $\exp\langle X\rangle\subset\overline{A(e)}$.

Proof By Lemma 3.2 there exists $\{t_n/n=1, 2, \cdots\}$ such that $\lim\exp(t_nX)=e$, nd $\lim t_n=+\infty$. For each $T\in R$, there exists an integer number $N>0$ such that $t_n+T>0$ for any $n>N$, thus, $\exp(TX)=\lim\exp((t_n+T)X)\in\overline{A(e)}$.

Theorem 3.4. *If there exists* $A\in A_0+\eta$ (η *is defined by Lemma* 2.2) *and* A *is not f type* IIf, *then* $A(e)=WA(e)$.

Proof By Lemma 2.1, $\exp(tA)\in\overline{A(e)}$ for each $t>0$, thus $\exp\langle A\rangle\subset\overline{A(e)}$ (by lemma 3.3). Now we claim that $WA(e)=\overline{A(e)}$. In fact, $WA(e)$ is the smallest subsemigroup containning $\exp\langle A\rangle$ and $\exp\mathfrak{P}$ (by Lemma 1.1). But, both $\exp\langle A\rangle$ and $\exp\mathfrak{P}$ are contained in $\overline{A(e)}$ (by Lemma 1.2), thus $WA(e)\subset\overline{A(e)}$. Jurdjevic and Sussmann pointed out (in Lemma 6.3 of [2]) that if $A(e)$ is dense in $WA(e)$, then $WA(e)=A(e)$. The other way to verify that $A(e)=WA(e)$ is to adopt

Proposition 4.3 of this paper, which claims that $\text{int}\,\overline{A(e)} = \text{int}\,A(e)$. $WA(e) = \text{int}\,WA(e) \subset \text{int}\,\overline{A}(e) = \text{int}\,A(e) \subset A(e)$, i.e. $WA(e) = A(e)$.

Corollary 3.5. *If $WA(e)$ is a compact group, then $A(e) = WA(e)$.*

Now we study the one parameter subgroup defined at the beginning of this section by matrix theory. Let $G = \text{GL}(n, R)$. The Lie algebra of G is $\text{gl}(n, R)$. For any $X \in \text{gl}(n, R)$, we give the matrix characteristics of $\exp\langle X\rangle$.

Lemma 3.6. *For any $X \in \text{gl}(n, R)$ if there exists a smooth map f: $\text{GL}(n, R) \to \boldsymbol{R}$ such that the composition of f and $\exp(\cdot, X)$, $f \circ \exp(\cdot, X): \boldsymbol{R} \to \boldsymbol{R}$, $t \mapsto f(\exp(tX))$ is strickly monotone increasing (or decreasing), then X is of type* III.

Proof is omitted. For $X \in \text{gl}(n, R)$, X is called semisimple if X is similar to complex diagonal matrix (cf. [8]).

Lemma 3.7. *If there exists an eigenvalue s of X such that* $\text{Re}\, s \neq 0$, or *X is n… semisimple, then X is of type* III.

Proof We prove only that if there exists an eigenvalue $s = a + bi$ of X such tha… $a, b \neq 0$ then X is of type III. Let $\xi + \eta i$ be the the eigenvector of X related to … Thus

$$\exp(tX)(\xi\eta) = (\xi\eta)\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(at) \end{pmatrix}\exp(at).$$

Let $T = (\xi\eta)$ and f: $\text{GL}(n, R) \to \boldsymbol{R}$, $g \mapsto \det(T'gT)$. Thus,

$$f(\exp(tX)) = \det((T'T)\begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}(\exp(at))) = \exp(2at)\det(T'T),$$

which is strickly monoptone. By Lemma 3.6, X is of type III.

Lemma 3.8. *If $X(\neq 0)$ is semisimple and non-zero eigenvalues of X a… imaginary numbers, then X is of type* I *if and only if its non-zero eigenvalues a… pairwise rationally dependent to each other (i.e. for any s_1, s_2, which are two non-ze… eigenvalues of X, s_1/s_2 is a rational number).*

Proof Let the non-zero eigenvalues of X be $\{iw_1, -iw_1, \cdots, iw_s, -iw_s\}$, the… there exists $Q \in \text{GL}(n, R)$ such that

$$Q^{-1} \cdot \exp(tX) \cdot Q = \begin{bmatrix} \cos(w_1t) & -\sin(w_1t) & & & & & \\ \sin(w_1t) & \cos(w_1t) & & & & & \\ & & \ddots & & & & \\ & & & \cos(w_st) & -\sin(w_st) & & \\ & & & \sin(w_st) & \cos(w_st) & & \\ & & & & & 1 & \\ & & & & & & \cdots \\ & & & & & & & 1 \end{bmatrix}.$$

If X is of type I, then there exists $T(>0)$ such that $Q^{-1} \cdot \exp(TX) \cdot Q =$ … Therefore, for each i, $\sin(w_iT) = 0$, i.e. there are $l_i \in \boldsymbol{Z}$ such that $w_iT = l_i\pi$, so $w_i/w_j = l_i/l_j$ is a rational number. In other hand, when $\{w_i\}$ is pairwise rationally dependent o each other, it is easy to find out a $T(>0)$ such that $\sin(w_iT)=0$ and $\cos(w_iT)=1$. Therefore, $\exp(TX)=e$, X is of type I.

Lemma 3.9. *When X is semisimple and its non-zero eigenvalues are imaginary numbers, X is of type* II *if there is a pair of eigenvalues, e. g. iw_1, iw_2, which are rationally depedent to each other.*

Proof By Lemma 3.8, X must not be of type I. Assume that

$$X=\begin{bmatrix} 0 & -w_1 & & & & & & & & \\ w_1 & 0 & & & & & & & & \\ & & 0 & -w_2 & & & & & & \\ & & w_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & -w_s & & & \\ & & & & & w_s & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \cdots & \\ & & & & & & & & & 0 \end{bmatrix}.$$

and let

$$X_1=\begin{bmatrix} 0 & -w_1 & & & \\ w_1 & 0 & & & \\ & & 0 & & \\ & & & \cdots & \\ & & & & 0 \end{bmatrix}, \cdots, X_s=\begin{bmatrix} 0 & \cdots & & & & \\ & & 0 & -w_s & & \\ & & w_s & 0 & & \\ & & & & \cdots & \\ & & & & & 0 \end{bmatrix}.$$

Then $X=X_1+\cdots+X_s$. Since $\{X_i/i=1, \cdots, s\}$ generates a compact commutative s-imensional Lie group K, and $\exp\langle X\rangle$ is a non-compact subgroup of K (as $\exp\langle X\rangle$ is not of type I), it must not be closed, therefore, X is of type II.

Summarizing above lemmas, we give following

Theorem 3. 10. *Let $X\in \mathrm{gl}(n, R)$ and $X\neq 0$. $\exp\langle X\rangle$ is not of type* III *in* GL(*n*, *R*) *if and only if X is semisimple and its non-zero eigenvalues are imaginary numbers. In this case,* $\exp\langle X\rangle$ *is of type* I *if and only if each pair of eigenvalues of X is rationally dependent to each other.*

§4. Some Topological Properties of $A(e)$

In this section we assume that $A(e)=WA(e)$, otherwise all results given here are meaningless or trivial. We often denote $A(e)$ by A and $WA(e)$ by S.

Lemma 4. 1*I. f $x\in \mathrm{cl}(A)$, $y\in \mathrm{int}(A)$, then $xy\in \mathrm{int}(A)$, and $yx\in \mathrm{int}(A)$. In other words,* $\mathrm{cl}(A)\cdot\mathrm{int}(A)\subset\mathrm{int}(A)$ *and* $\mathrm{int}(A)\cdot\mathrm{cl}(A)\subset\mathrm{int}(A)$.

Proof As $y\in \mathrm{int}(A)$, there exists an open neighborhood V of e, such that Vy is contained in $A(e)$ and $V^{-1}=V$. As $x\in\overline{A(e)}$, there exists $x_1\in xV\subset A(e)$. Hence, $x\in$

x_1V, $xy \in x_1VyyA(e)$. Thus, xy is contained in $\mathrm{int}(A(e))$.

Corollary 4.2. (1) $\mathrm{bd}(A)\cdot\mathrm{int}(A)\subset\mathrm{int}(A)$, $\mathrm{int}(A)\cdot\mathrm{bd}(A)\subset\mathrm{int}(A)$.

(2) $\mathrm{int}(A)^{-1}\subset S\backslash\bar{A}$.

(3) $H\subset\mathrm{bd}(A)$.

(4) *Both* $\mathrm{int}(A)$ *and* $\mathrm{cl}(A)$ *are semigroup.*

Proof (1) It is trivial by Lamma 4.1.

(2) For $x\in\mathrm{int}(A)$, if $x^{-1}\notin S\backslash\bar{A}$, i.e. $x^{-1}\in\mathrm{cl}(A)$, then $e=x^{-1}x\in\mathrm{int}(A)$. By Lemma 2.5, we have $A(e)=WA(e)$, it is contrary to the assumption of this secti Hence, for any $x\in\mathrm{int}(A)$, $x^{-1}\in S\backslash\bar{A}$.

(3) It is known to us that H is a subgroup of S and $H\subset\mathrm{cl}(A)$. If $x\in H$ a $x\in\mathrm{int}(A)$, then $x^{-1}\in H$ and $x^{-1}\in S\backslash\bar{A}$ (by (2)), that is a contradiction.

(4) It is trivial.

Lemma 4.3. *We consider the system as following*

$$dx/dt=-x(t)\Big(A_0+\sum_{i=1}^{m}u_i(t)A_i\Big) \qquad (4$$

and denote its reachable semigroup by $D(e)$. (4.3) *is a left invarent system. It is difficult to varift that:*

(1) *If* $x(t)$, $y(t)$ *are the trajectories of* (1.1) *and* (4.3), *respectively, with same input* $u(t)$ *and the initial condition* $x(0)=y(0)=e$, *then* $y(t)=(x(t))^{-1}$ *for e* $t\in\boldsymbol{R}$,

(2) $D(e)=(A(e))^{-1}$,

(3) $\mathrm{int}(D(e))=(\mathrm{int}(A(e)))^{-1}$.

Theorem 4.4. $\mathrm{int}(\bar{A})=\mathrm{int}(A)$.

Proof It is obvious that $\mathrm{int}(A)\subset\mathrm{int}(\bar{A})$. For the inverse inclusion, let $\mathrm{int}(\bar{A})$. As $\mathrm{int}(A)$ is dense in $\mathrm{cl}(A)$, there exists an open unit neighborhood V su that $V^{-1}=V$, $Vx\subset\mathrm{cl}(A)$, and $\mathrm{int}(A)\cap Vx$ is dense in Vx. We denote $D=A^{-1}$, th $\mathrm{int}(D)=\mathrm{int}(A)^{-1}$. By $\mathrm{int}(D)\cap V=(\mathrm{int}(A)\cap V)^{-1}\neq\varnothing$, $(\mathrm{int}(D)\cap V)x$ is an op neighberhood of x. Let $g\in(\mathrm{int}(A)\cap Vx)\cap(\mathrm{int}(D)\cap V)x$, then tnere exists $\mathrm{int}(D)\cap V$ such that $g=zx$, hence $x=z^{-1}g$. Since $z^{-1}\in(\mathrm{int}(D))^{-1}=\mathrm{int}(A)$, $\mathrm{int}(A)\cap Vx\subset\mathrm{int}(A)$, by Lemma 4.1, $x\in\mathrm{int}(A)$. Therefore, $\mathrm{int}(\bar{A})\subset\mathrm{int}(A)$.

Corollary 4.5. (1) $\overline{\mathrm{bd}(A)}=\mathrm{bd}(A)=\mathrm{bd}(\bar{A})$,

(2) $\overline{S\backslash A}=\overline{S\backslash\mathrm{cl}(A)}$,

(3) *For each* $x\in\mathrm{bd}(A)$ *and a unit neighcerhood* U, $xU\cap \mathit{int}(A)\neq\varnothing, xU\cap S\backslash\mathrm{cl}($ $\neq\varnothing$.

According to foregoing discussion, $\mathrm{bd}(A)$ divides S into two parts, $\mathrm{int}(A)$ $S\backslash\bar{A}$. $\mathrm{int}(A)$ is a semigroup. It is a natural question: Is $S\backslash\bar{A}$ a semigroup too? geneaaal, it is not true. But we have

Theorem 4.6. $S\backslash\bar{A}$ *is a semigroup if and only if* $(S\backslash\bar{A})^{-1}=\mathrm{int}(A)$, i.e. *if and* nly if $(S\backslash\bar{A})=\mathrm{int}(D(e))$.

Proof (only if) If there exists $x\in S\backslash\bar{A}$ such that $x^{-1}\in S\backslash\bar{A}$, then $xx^{-1}=e\in S\backslash$. It is contrary to the fact $e\in\bar{A}$. Thus, $(S\backslash\bar{A})^{-1}\subset\bar{A}$. $(S\backslash\bar{A})^{-1}=\mathrm{int}((S\backslash\bar{A})^{-1})\subset$ t$(\bar{A})=\mathrm{int}(A)$. By Corellary 4.2(2), $(S\backslash\bar{A})=(\mathrm{int}(A))^{-1}=\mathrm{int}(D)$.

(if) In fact, it is easy to verify that $S\backslash\bar{A}$ is a semigroup when $(S\backslash\bar{A})^{-1}\subset\bar{A}$.

Theorem 4.7. *$S\backslash\bar{A}$ is a semigroup iff* $\mathrm{bd}(A)$ *is a group.*

Proof (only if) It is trivial to verify that $\mathrm{cl}(S\backslash\bar{A})$ is a semigroup when $S\backslash\bar{A}$ $\mathrm{bd}(A)=\mathrm{cl}(S\backslash\bar{A})\cap\mathrm{cl}(A)$, hence, $\mathrm{bd}(A)$ is a semigroup. Again, let $x\in\mathrm{bd}(A)$. If $^{1}\in\mathrm{int}(A)$, then $e=x^{-1}x\in\mathrm{int}(A)$, which implies $A(e)=WA(e)$, it is contrary to r assumption; if $x^{-1}\in S\backslash\bar{A}$, then $x^{-1}x\in S\backslash\bar{A}$, but we know that $e\in\bar{A}$, therefore, ^{1} must be in $\mathrm{bd}(A)$, i.e. $\mathrm{bd}(A)$ is a subgroup.

(if) When $\mathrm{bd}(A)$ is a group, we show that $(S\backslash\bar{A})^{-1}\subset\mathrm{int}(A)$, or equivalently $\backslash\bar{A})\cap(S\backslash A)^{-1}=\varnothing$.

Let V be a small path-connected open unit neighbornood such that $V\cap\mathrm{bd}(A)$ d $V\cap(S\backslash\bar{A})$ are path-connected, and $V^{-1}=V$. If there exists $x\in V\cap(S\backslash\bar{A})\cap$ $\backslash\bar{A})^{-1}$, then both x and x^{-1} are in $S\backslash\bar{A}$. As $V\cap\mathrm{int}(A)\neq\varnothing$, there is y such that $^{1}\in\mathrm{int}(A)\cap V$, thus y must be in $(S\backslash\bar{A})\cap V$. Let $r(t)$ be a path such that $r(0)=$ $r(1)=y$ and $r(t)\in(S\backslash\bar{A})\cap V$ $(f\in[0,1])$. Thus $r^{-1}(t)$ is a path such that $r^{-1}(0)$ $x^{-1}\in S\backslash\bar{A}$, $r^{-1}(1)=y^{-1}\in\mathrm{int}(A)$. Then there is $c\in(0,1)$ such that $r^{-1}(c)\in\mathrm{bd}(A)$. ıt that $\mathrm{bd}(A)$ is a group implies that $r(c)\in\mathrm{bd}(A)$. It is a contradiction. Thus, V $(S\backslash\bar{A})\cap(S\backslash\bar{A})^{-1}=\varnothing$, i.e. $V\cap(S\backslash\bar{A})=V\cap\mathrm{int}(D)$.

Let $W=\mathrm{int}(A)\dot{\cup}\mathrm{bd}(A)\dot{\cup}\mathrm{int}(D)$ ($\dot{\cup}$ denotes the union of sets which have no nmon element). It is easy to show that $\mathrm{bd}(A)=\mathrm{bd}(D)$. Thus $W=(\mathrm{int}(A)\cup$ $(A))\cup(\mathrm{int}(D)\cup\mathrm{bd}(D))=\mathrm{cl}(A)\cup\mathrm{cl}(D)$, W is closed. On the other hand, $V=(V$ $\mathrm{int}(A))\dot{\cup}(V\cap\mathrm{bd}(A))\dot{\cup}(V\cap(S\backslash\bar{A})=(V\cap\mathrm{int}(A))\dot{\cup}(V\cap\mathrm{bd}(A))\dot{\cup}(V\cap\mathrm{int}(D))\dot{\cup}$ W. By Lemma 4.1 and its dual results that $\mathrm{bd}(D)\cdot\mathrm{int}(D)\subset\mathrm{int}(D)$, $\mathrm{int}(D)$. $(D)\subset\mathrm{int}(D)$. For each $x\in\mathrm{bd}(A)$, let $V_x=xV\cap Vx$, then $V_x\subset W$. V_x is open, $(A)\subset\bigcup_{x\in bd(A)}V_x$. Therefore, $W=\mathrm{int}(A)\cup\mathrm{int}(D)\cup(\bigcup_{x\in bd(A)}V_x)$, W is open, too. As S connected, $W=S$, i.e. $\mathrm{int}(A)\cup\mathrm{bd}(A)\cup(S/\bar{A})=\mathrm{int}(A)\cup\mathrm{bd}(A)\cup\mathrm{int}(D)$. Thus, $\bar{A}=\mathrm{int}(D)$ is a semigroup.

References

] Brcckett, R. W., System theorem on group manifolds and coset space, *SIAM J. Ctrl.*, **10**: 2(1972), **265** —283.

] Jurdjevic, V. & Sussmann, H. J., Control system on Lie grop, *J. Diff. Eq.*, (1972), 313—329.

] Sussmann, H. J. & Jurdjevic, V., Controllability of nonlinear system, *J. Diff. Eq.*, (1972) 95—116;

[4] Sussmann, H. J., Lie brackets, real analyticity and geometric control, *Differential geometric control theory*, Birkhauser (1983), 1—116.

[5] Hermann, R. & Krener, A. J., Nonlinear controllability and observability, *IEEE. Trans.*, **AC22**

(1977), 728—740.

[6] Macdonald, I. G., Algebraic structure of Lie groups, in Representation theory of Lie groups, Oxfold, 1977.

[7] Warner, F. W., Foundations of differentiable manifolds and Lie groups, GTM94, Springer-Verlag, 1983.

[8] Humphrey, J., Introduction to Lie algebras and repesentation theory, Springer-Verlag, New York-Heiderberg-Berlin, 1972.

[9] Hermann, H., Lie groups for physicists, 1966, The Benjamin/cummings publishing company, INC.