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A MINIMUM-RATIO-TEST-FREE APPROACH TO LINEAR PROGRAMMING**

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Abstract

Two non-simplex-type pivotal algorithms are given in this paper which realize constructively the Farkas Lemma and the strong duality theorem of the linear programming on purely combinatorial pivoting rules, i. e. they involve no process of minimum-ratio-test and work purely on smallest subscript principle in accordance with the signs of the quantities concerned.

Farkas Lemma can be set in various way, the one we take here is: Let A^0 be an $\times n$ real matrix where $m<n$ and $r(A)=m$, b^0 an m-dimensional real column ctor, P a non-empty subset of $\{1, 2, \cdots, n\}$. Exactly one of the following two tements holds:

(a) There exists an n-dimensional real vector x satisfying:

$$\begin{cases} A^0x \geqslant b^0, \\ x_i \geqslant 0, \ \forall i \in P. \end{cases}$$

(b) There exists an m-dimensional real vector u satisfying:

$$\begin{cases} ua^i \geqslant 0, \ \forall i \in P, \\ ua^i = 0, \ \forall i \in \{1, 2, \cdots, n\} \backslash P, \\ ub^0 < 0 \end{cases}$$

ere a^i is the th column vector of A^0.

It is quite obvious that statements (a) and (b) can not hold simultaneously. A 1-simplex-type pivotal algorithm[4] Iwhich realizes constructively the above Farkas nma will be given in the following; its pivoting rules involve no process of nimum-ratio-test and work purely on smallest subcript principle[1].

Algorith mI

Input and Working Units: A^0, b^0; A an $m\times n$ working matrix with a^i as its ith umn vector, b an m-dimensional working column vector, $(i_1, i_2, \cdots, i_m)$ an m-

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dimensional working row vector (standing for "basic indices").

Output: Either a vector x satisfying the condition set in statement (a) or a vector u satisfying the condition set in statement (b)

Step 0: Assume that A^0 takes α^i as its ith column vector and $\alpha^{i_1^0}, \alpha^{i_2^0}, \cdots, \alpha^{i_m^0}$ is a set of independent column vectors of A^0.

0.1: Let

$$F_0 \equiv [\alpha^{i_1^0}, \alpha^{i_2^0}, \cdots, \alpha^{i_m^0}],$$
$$T_0 \equiv [A^0, b^0],$$
$$T \equiv [A, b];$$

Set

$$T := F_0^{-1} T_0,$$
$$(i_1, i_2, \cdots, i_m) := (i_1^0, i_2^0, \cdots, i_m^0).$$

(T is called a tableau; $i_1, i_2, \cdots, i_m$ are called basic indices with respect to T

0.2:If

$$\exists i_l \in \{i_1, i_2, \cdots, i_m\} \cap P$$

Such that

$$\exists \bar{j} \in \{1, 2, \cdots, n\} \setminus (\{i_1, i_2, \cdots, i_m\} \cup P) \text{ and } a_l^{\bar{j}} \neq 0,$$

then take $a_l^{\bar{j}}$ as the pivoting element in T and transform T, through pivo operation, to an up-dated one; set $i_l := \bar{j}$ and go back to Step 0.2. Otherwise, g to Step 1.

Step 1: If $\forall i_l \in \{i_1, i_2, \cdots, i_m\} \cap P$, we have $b_l \geqslant 0$, let x be such an n-dimensi vector:

$$x_k = \begin{cases} 0 \text{ for } k \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}, \\ b_l \text{ for } k = i_l \in \{i_1, i_2, \cdots, i_m\} \end{cases}$$

(it is easy to see that now x satisfies the condition set in statement (a)).

Stop. Otherwise, assume

$$i_{\bar{l}} = \min \{i_l \mid b_l < 0 \text{ and } i_l \in \{i_1, i_2, \cdots, i_m\} \cap P\};$$

($i_{\bar{l}}$ is now called to be ready for becoming non-basic.)

go on to Step 2.

Step 2: If $\forall j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$, we have $a_{\bar{l}}^j \geqslant 0$, let u be the $\bar{l}$ th vector of $[\alpha^{i_1}, \alpha^{i_2}, \cdots, \alpha^{i_m}]^{-1}$ (it is not difficult to see that now u satisfies the c ition set in statement(b)). Stop. Otherwise, assume

$$\bar{j} = \min\{j \mid a_{\bar{l}}^j < 0 \text{ and } j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$$

($\bar{j}$ is now called to be ready for becoming basic), then take $a_{\bar{l}}^{\bar{j}}$ as the piv element in T and transform T, through pivoting operation, to an up-dated one $i_{\bar{l}} := \bar{j}$ and go back to Step 1.

Proof We are going to prove sketchily in the following that the above algorithm can cause on cycling, i. e. can not produce two identical sets of basic idices; therefore, the finiteness of the algorithm is secured.

If the algorithm causes cycling, then during period of cycling, any basic (non-asic) index if once becoming non-basic (basic) must once again become basic (non-asic); of all these indices, let g be the greatest one. Suppose, on the one hand, that is basic with respect to some tableau $\tilde{T}$ and is ready for becoming non-basic; and ippose, on the other hand, that g is non-basic with respect to some other tableau and is ready for becoming basic. Now, assume that, with respect to the tableau T, basic index i_s is ready for becoming non-basic. With respect to $\tilde{T}$, let $\tilde{\lambda}=(\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_n, \tilde{\lambda}_{n+1})$ be such a vector:

$$\tilde{\lambda}_k=\begin{cases}0 \text{ for } k\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\},\\ b_l \text{ for } k=i_l\in\{i_1, i_2, \cdots, i_m\},\\ -1 \text{ for } k=n+1\end{cases}$$

id let δ be the sth row vector of T (obviously, $\tilde{\lambda}$ is orthogonal to every row vector $\tilde{T}$; therefore, $\tilde{\lambda}$ is orthogonal to δ). Now, it can be shown (mainly due to the sma-est subscript principle set for pivoting) that instead of $\delta\cdot\tilde{\lambda}=0$ a contradiction $\delta\cdot\tilde{\lambda}$ 0 can be derived.

The strong duality theorem of the linear programming can be set in various ays, the one we take here is: Let A^0 be an $m\times n$ real matrix where $m<n$ and $r(A)$ m, b^0 an m-dimensional real column vector, c^0 an n-dimensional row vector, P a n-empty subset of $\{1, 2, \cdots, n\}$. Then, for the following two linear programming:

LP: $\max c^0x$

$$\text{s. t. }\begin{cases}A^0x=b^0\\ x_i\geqslant 0,\ \forall i\in P\end{cases}$$

DLP: $\min ub^0$

$$\text{s. t. }\begin{cases}ua^i\geqslant c_i^0,\ \forall i\in P\\ ua^i=o,\ \forall i\in\{1, 2, \cdots, n\}\setminus P,\end{cases}$$

where a^i is the ith column vector of A^0.

iher at least one of them is infeasible or they have optimal solution x and u such at $c^0x=u^0b$. (It is well-known that if one of the two programmings is infeasible, en the other one is either infeasible or unbounded).

A non-simplex-type pivotal algorithm[2,3,5] II, which realizes constructively e above strong duality theorem will be given in the following, its pivoting rules volve no process of minimum-ratio-test and work purely on smallest subsc pt principle[1].

Algorithm II

Input and Working Units: A^0, b^0, c^0; A an $m\times n$ working matrix with a^i as its ... column vector, c an n-dimensional working row vector, $(i_1, i_2, \cdots, i_m)$ an m-dimensional working row vector, (standing for "basic indices"), h a working real value unit.

Output: Either

(1) an unbounded-augmenting vector β of the LP, (hence, the DLP is infeasible and the LP is either infeasible or unbounded) or an unbounded-augmenting vector γ of the DLP; (hence the LP is infeasible and the DLP is either infeasible or unbounded)

Or

(2) an optimal solution x of the LP and an optimal solution u of the DLP (such that $c^0x=ub^0$) together with an optimal value h.

Step 0: Assume that A^0 takes α^i as its column vector and $\alpha^{i_1^0}, \alpha^{i_2^0}, \cdots, \alpha^{i_m^0}$ is a of independent column vectors of A^0.

0.1: Let

$$F_0\equiv[\alpha^{i_1^0}, \alpha^{i_2^0}, \cdots, \alpha^{i_m^0}],$$

$$T_0=\begin{bmatrix} 1 & -c^0 & 0 \\ 0 & A^0 & b_0 \end{bmatrix},$$

$$\pi_0\equiv(c^0_{i_1^0}, c^0_{i_2^0}, \cdots, c^0_{i_m^0}),$$

$$T\equiv\begin{bmatrix} 1 & c & h \\ 0 & A & b \end{bmatrix}.$$

Set

$$T:\equiv\begin{bmatrix} 1 & \pi_0F_0^{-1} \\ 0 & F_0^{-1} \end{bmatrix}\cdot T_0$$

$$(i_1, i_2, \cdots, i_m):=(i_1^0, i_2^0, \cdots, i_m^0).$$

(T is called a tableau; $i_1, i_2, \cdots, i_m$ are called basic indices with respect to T)

0.2: If

$\exists i_l\in\{i_1, i_2,\cdots,i_m\}\cap P$, and $\exists \tilde{j}\in\{1, 2, \cdots, n\}\setminus(\{i_1, i_2, \cdots\}\cup P)$, we have $a_l^{\tilde{j}}\neq 0$, then take $a_l^{\tilde{j}}$ as tee pivoting element in T and transform T, through pivot operation, to an up-dated one; set $i_l:=\tilde{j}$ and go back to Step 0.2. Otherwise, go to Step 0.3.

0.3; If

$$\exists\tilde{j}\in\{1, 2, \cdots, n\}\setminus(\{i_1, i_2, \cdots i_m\}\cup P) \text{ and } c_{\tilde{j}}\neq 0,$$

then let β be such an n-dimensional vector:

if $c_{\tilde{j}}<0$, let

$$\beta_k=\begin{cases} 1 \text{ for } k=\tilde{j} \\ 0 \text{ for } k(\neq\tilde{j})\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\}, \\ -a_l^{\tilde{j}} \text{ for } k=i_l\in\{i_1, i_2, \cdots, i_m\}; \end{cases}$$

if $c_{\tilde{j}}>0$, let

$$\beta_k=\begin{cases} -1 \text{ for } k=\tilde{j}, \\ 0 \text{ for } k(\neq\tilde{j})\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\}, \\ a_l^{\tilde{j}} \text{ for } k=i_l\in\{i_1, i_2, \cdots, i_m\}; \end{cases}$$

Stop. (It is not difficult to prove that now β is an unbound-augmenting vector of LP; therefore the DLP is infeasible and the LP is either infeasible or unbounded.)

Otherwise, go on to Step 1.

Step 1: If

$b_l \geqslant 0,\ \forall i_l \in \{i_1, i_2, \cdots, i_m\} \cap P$; and $c_j \geqslant 0,\ \forall j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$;

x be such an n-dimensional vector:

$$x_k = \begin{cases} 0 \text{ for } k \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}, \\ b_l \text{ for } k = i_l \in \{i_1, i_2, \cdots, i_m\}, \end{cases}$$

l let

$$u = \pi F^{-1}, \text{ where } \pi \equiv (c_{i_1}, c_{i_2}, \cdots, c_{i_m}) \text{ and } F = (\alpha^{i_1}, \alpha^{i_2}, \cdots, \alpha^{i_m}).$$

Stop. (Now, it is known that x and u are optimal, and h is the optimal ue of both the LP and DLP).

Otherwise, let

$$t = \min\{i_l, j \mid i_l \in \{i_1, i_2, \cdots, i_m\} \cap P \text{ and } b_l < 0;\ j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\} \text{ and } c_j < 0\},$$

$= i_{\bar{l}}$ ($i_{\bar{l}}$ is called to be ready for becoming non-basic "actively"),

n go to Step 2;

$= \tilde{j}$ ($\tilde{j}$ is called to be ready for becoming basic "actively"),

n go to Step 3.

Step 2: If $a_{\bar{l}}^j \geqslant 0, \forall j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$, let γ be the $\bar{l}$ th row vector of ^{1} where $F \equiv (\alpha^{i_1}, \alpha^{i_2}, \cdots, \alpha^{i_m})$; stop. (It is not difficult to see that γ is now an bounded-augmenting vector of the DLP; therefore the LP is infeasible and the P is either infeasible or unbounded).

Otherwise, let

$$\tilde{j} = \min\{a_{\bar{l}}^j < 0 \text{ and } j \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}\}$$

is called to be ready for becoming basic "passively");

e $a_{\bar{l}}^{\tilde{j}}$ as the pivoting element in T and transform T, through pivoting operation, an up-dated one; set $i_{\bar{l}} := \tilde{j}$ and go back to Step 1.

Step 3: If $a_l^{\tilde{j}} \leqslant 0, \forall i_l \in \{i_1, i_2, \cdots, i_m\} \cap P$, let β be such an n-dimensional vector:

$$\beta_k = \begin{cases} 1 \text{ for } k = \tilde{j}, \\ 0 \text{ for } k (\neq \tilde{j}) \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}, \\ -a_l^{\tilde{j}} \text{ for } k = i_l \in \{i_1, i_2, \cdots, i_m\}. \end{cases}$$

Stop. (It is not difficult to varify that now β is an unbounded-augmenting tor of the LP; therefore the DLP is infeasible and the LP is either infeasible or bounded).

Otherwise, let

$$i_{\bar{l}} = \min\{i_l \mid a_l^{\tilde{j}} > 0 \text{ and } i_l \in \{i_1, i_2, \cdots, i_m\} \cap P\}$$

($i_{\bar{l}}$ is called to be ready for becoming non-basic "passively");

take $\tilde{j}\alpha$ as th pivoting element in T and transform T, through pivoting operation, to an up-date one; set $i_{\bar{l}} := \tilde{j}$ and go back to Step 1.

Proof We are going to prove that the above algorithm can cause no cycling, i. e. can not produce two indentical sets of basic indices; therefore, the fini-teness of the algorithm is secured.

If the algorithm causes cycling, then during period of cycling, any basic (non-basic) index if once becoming non-basic (basic) must once again become basic (non-basic); of all these indices let g be the greatest one. At least one of the follow four cases would occur:

(a) g is, on the one hand, non-basic with respect to some tableau $\tilde{T}$ anc ready for becoming basic "actively"; and, on the other hand, g is basic with resp to some other tableau T and is ready for becoming non-basic "passively".

(b) g is, on the one hand, basic with respect to some tableau $\tilde{T}$ and is ready becoming non-basic "actively"; and, on the other hand, g is non-basic with resp some other tableau T and is ready for becoming basic "passively".

(c) g is, on the one hand, non-basic with recpect to some tableau $\tilde{T}$ and is re for becoming basic "passively"; and, on the other hand, g is basic with respec to some other tableau T and is ready for becoming non-basic "passively".

(d) g is, on the one hand, non-basic with respect to some tableau $\tilde{T}$ an ready for becoming basic "actively"; and, on the other hand, g is basic with res to some other tableau T and is ready for becoming non-basic "actively".

We are going to show sketchily in the following that each case leads to a con diction.

(1) Suppose case (a) occurs. Assume that, with respect to tableau T, a r basic index f is ready for becoming basic "actively". Let $\tilde{\delta}$ be the first row vecto the tableau $\tilde{T}$, and with respect to T, let $\lambda=(\lambda_0, \lambda_1, \cdots, \lambda_n, \lambda_{n+1})$ be such an $(n+$ dimensional vector:

$$\lambda_k=\begin{cases}-c_f & \text{for } k=0,\\ 1 & \text{for } k=f,\\ 0 & \text{for } k(\neq f)\in\{1,\ 2,\ \cdots,\ n,\ n+1\}\setminus\{i_1,\ i_2,\ \cdots\ i_m\},\\ -a_l^f & \text{for } k=i_l\in\{i_1,\ i_2,\ \cdots,\ i_m\}.\end{cases}$$

(Obviously, λ is orthogonal to every row vector of the tableau T; therefore, orthogonal to $\tilde{\delta}$.) Now, it can be shown (mainly due to the smallest subscript p ciple set for pivoting) that instead of $\tilde{\delta}\cdot\lambda=0$ a contradiction $\tilde{\delta}\cdot\lambda>0$ can be deri

(2) Suppose case (b) occurs. Assume that, with respect to tableau T, a index i_s is ready for becoming non-basic "actively". Let δ be the $(s+1)$th row v of the tableau T, and with respect to $\tilde{T}$, let $\tilde{\lambda}=(\tilde{\lambda}_0,\ \tilde{\lambda}_1,\ \cdots,\ \tilde{\lambda}_n,\ \tilde{\lambda}_{n+1})$ be such an $(n+2)$-dimensional vector:

$$\tilde{\lambda}_k=\begin{cases}\tilde{h} & \text{for } k=0,\\ \tilde{b}_l & \text{for } k=i_l\in\{i_1, i_2, \cdots, i_m\},\\ 0 & \text{for } k\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\},\\ -1 & \text{for } k=n+1.\end{cases}$$

)bviously, $\tilde{\lambda}$ is orthogonal to every row vector of the tableau $\tilde{T}$; therefore, $\tilde{\lambda}$ is thogonal to δ). Now, it can be shown (mainly due to the smallest subscript inciple set for pivoting) that instead of $\delta\cdot\tilde{\lambda}=0$ a contradiction $\delta\cdot\tilde{\lambda}>0$ can be rived.

(3) Suppose case (c) occurs. Assume that, with respect to tableau $\tilde{T}$, abasic dex i_s is ready for becoming non-basic "activelyr. We also assume that, with spect to tableau T, a non-basic index f is ready for becoming basic "actively". ɔw, let $\tilde{\delta}$ be the $(s+1)$th row vector of the tableau $\tilde{T}$, and with respect to T, let $=(\lambda_0, \lambda_1, \cdots, \lambda_n, \lambda_{n+1})$ be such an $(n+2)$-dimensional vector:

$$\lambda_k=\begin{cases}-c_f & \text{for } k=0,\\ 1 & \text{for } k=f,\\ 0 & \text{for } k(\neq f)\in\{1, 2, \cdots, n, n+1\}\setminus\{i_1, i_2, \cdots, i_m\},\\ -a_l^f & \text{for } k=i_l\in\{i_1, i_2, \cdots, i_m\}.\end{cases}$$

'bviously, λ is orthogonal to every row vector of the tableau T; therefore, λ is hogonal to $\tilde{\delta}$). Now, it can be shown (mainly due to the smallest subscript inciple set for pivoting) that instead of $\tilde{\delta}\cdot\lambda=0$ a contradiction $\tilde{\delta}\cdot\lambda>0$ can be rived.

(4) Suppose case (d) occurs. Let $\tilde{\delta}$ and δ be the first row vectors of tableau $\tilde{T}$ d T respectively, and with respect to $\tilde{T}$ and T, let $\tilde{\lambda}=(\tilde{\lambda}_0, \tilde{\lambda}_1, \cdots, \tilde{\lambda}_n, \tilde{\lambda}_{n+1})$ and $\lambda=$ $, \lambda_1, \cdots, \lambda_n, \lambda_{n+1})$ be respectively two $(n+2)$-dimensional vectors as follows:

$$\tilde{\lambda}_k\equiv\begin{cases}\tilde{h} & \text{for } k=0\\ \tilde{b}_l & \text{for } k=i_l\in\{i_1, i_2, \cdots, i_m\},\\ 0 & \text{for} k\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\},\\ -1 & \text{for } =k=n+1;\end{cases}$$

$$\lambda_k=\begin{cases}h & \text{for } k=0\\ b_l & \text{for } k=i_l\in\{i_1, i_2, \cdots, i_m\},\\ 0 & \text{for } k\in\{1, 2, \cdots, n\}\setminus\{i_1, i_2, \cdots, i_m\},\\ -1 & \text{for } k=n+1.\end{cases}$$

bviously, $\tilde{\lambda}$ is orthogonal to every row vector of the tablesu $\tilde{T}$ and λ is orthogonal every row vector of the tableau T; therefore, $\tilde{\lambda}$ and λ are orthogonal to $\tilde{\delta}$ and δ).

Now from $(\tilde{\delta}-\delta)\cdot\tilde{\lambda}=0$, it can be verified (mainly due to the smallest subscript principle set for pivoting) that the last component of $\tilde{\delta}-\delta$ must be non-positive, this conclusion would therefore lead to a contradiction (mainly due to the smallest subscript principle set for pivoting): instead of $(\tilde{\delta}-\delta)\cdot\lambda$ being zero, $(\tilde{\delta}-\delta)\cdot\lambda$.

becomes positive.

The minimum-ratio-testfree approach taken in this paper when being technically modified can be further applied to linear feasible problems and linear programming problems with arbitrary upper and lower bound constraints on the variables without necessarily transforming them to the standard forms set in this paper. These have been discussed in [6, 7] and [8].

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