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THE SPECTRUM OF CONTRACTIVE OPERATORS ON π_k

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Abstract

In this paper, it is shown that, for a contraction on π_k, the intersection of its spectrum with the exterior of the unit disk is a finite set of isolated eigenvalues, each of which has finite multplicity. Futhermore some relations between its spectrum and the spectrum of its minimal unitary dilation are established.

Throughout this paper, π_k denotes a complete space with an indefinite metric ·), which can be decomposed into $\pi_k = H_- \oplus H_+$, where $(H_\pm, \pm(\cdot, \cdot))$ are lbert spaces and dim $H_- < \infty$. As known, there are some basic facts on the ectrum of selfadjoint or unitary operators on π_k. For example, the spectrum of a fadjoint operator on π_k is real axial symmetry. There are only finite points of the ectrum outside the real axis. In the case of Hilbert spaces, to study contractive rators, S. Nagy and C. Foias have developed the theory of harmonic analysis of rators[2]. One of the basic facts on contractive operators of Hilbert spaces is that spectrum is contained in $\boldsymbol{D}$ where $\boldsymbol{D}$ denotes the unit disk.

In [2, 3], the concept of contractive operators on π_k was introduced, but only a results were obtained. In [4], Yan Shaozong developed the theory of dilation of tractive operators on Krein spaces, found a necessary and sufficient condition for ontractive operator on Krein space to have a unitarydilation and, in parrticular, ated out that any contractive operator on π_k has the minimal unitary dilation. convenience, we give the following definition.

Definition. *A bounded linear operator T on π_k is called a contraction, if* $(Tx,$ $\leqslant (x, x)$ *for any* $x \in \pi_k$.

What can we say about the spectrum of a contraftive operator on π_k? What tions are there between the spectrum of a contration and the spectrum of its tary dilation? There is notany information in [1, 3, 4]. But such two problems very important for us to develop the corresponding harmonic analysis theory on paces. The aims of the present paper are to study the spectrum of the contraction on π_k and find some relations between the spectrum of the contraction and the

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spectrum of its unitary dilation.

§1. The Spectrum of Contractions on π_k

First, we construct a special model of contractions on π_k. This model is derived from the decomposition of contractions in [4]. Here, our decomposition is finer than before.

Lemma 1.1. *Let T be a contraction on π_k and $\pi_k = H_- \oplus H_+$ is a reg decomposition. Then*

i) *$(TH_-)\oplus(TH_-)^\perp$ is also a regular decomposition of π_k and*

$$T=\begin{pmatrix} T_{H_-} & T_1 \\ & T_2 \end{pmatrix},$$

where $T_{H_-}=P_{TH_-}T|_{PH_-}$, $T_1=P_{TH_-}T|_{PH_+}$ and $T_2=P_{(TH_-)^\perp}T|_{PH_+}$.

ii) *There exist Hilbert spaces H_i ($i=0, 1, 2$), $\dim H_1<\infty$, $\dim H_2<\infty$, that $H_+=H_0\oplus H_1$ and $(TH_-)^\perp=H_0\oplus H_2$. Under these decompositions of spaces, of the following form:*

$$T=\begin{pmatrix} T_{H_-} & T_{11}, & T_{12} \\ & T_0 & A \\ & B & C \end{pmatrix}$$

iii) *Set*

$$\tilde{T}=\begin{pmatrix} 0 & 0 & 0 \\ & T_0 & 0 \\ & 0 & 0 \end{pmatrix}.$$

Then $\tilde{T}$ is a contraction on Hilbert space $(\pi_k=H_-\oplus H_+, [\cdot, \cdot])$, where $[x_-+x_+, y_+] = -(x_-, y_-)+(x_+, y_+)$.

Proof i) is a result in [4].

Let $H_0=H_+\cap(TH_-)^\perp$. Since $(H_+\ominus H_0)\cap(TH_-)^\perp=\{0\}$ and $((TH_-)^\perp\ominus$ $\cap H_+=\{0\}$, $\dim(H_+\ominus H_0)\leqslant\dim TH_-$ and $\dim((TH_-)^\perp\ominus H_0)\leqslant\dim H_-$ Corollary I. 3.4 in [1]. This has proved ii).

From [4]. we know that T_2 in i) is a contraction on H_+. So, we have

$$(T_2x, T_2x)=\left(\begin{pmatrix} T_0x \\ Bx \end{pmatrix}, \begin{pmatrix} T_0x \\ Bx \end{pmatrix}\right)=(T_0x, T_0x)+(Bx, Bx)\leqslant(x, x)$$

for any $x\in H_0$. It follows that $(T_0x, T_0x)\leqslant(x, x)$, since $(Bx, Bx)\geqslant 0$. That is, a contraction on H_0. Obviously,

$$[\tilde{T}(x_-+x_0+x_1), \tilde{T}(x_-+x_0+x_1)]=[T_0x_0, T_0x_0]=(T_0x_0, T_0x_0)\leqslant(x_0, x_0)$$
$$\leqslant[x_-+x_0+x_1, x_-+x_0+x_1].$$

Therefore, $\tilde{T}$ is a contraction on Hilbert space $(\pi_k=H_-+H_+, [\cdot, \cdot])$.

From the above lemma one can obtain easily the following theorem.

Theorem 1.2. *Suppose that T is a contraction on π_k, then $\sigma_e(T)\subset D$.*

Using Theorem 1.2, we know that the intersection of $\sigma(T)$ with $C\backslash D$ is a ıntable set of isolatedeigen values of T, each of which has finite multiplicity. hen $\lambda\in(C\backslash D)\cap\sigma(T)$, one sees that x must be a semi-negative vector for any $x\in$ r$(T-\lambda)$. If $\ker(T+\lambda)\perp\ker(T-\mu)$ for any $\lambda\neq\mu$, then we a ffirm that the ;ersection of $\sigma(T)$ with $C\backslash D$ is a finite set from the properties of π_k space. But it not true that $\ker(T-\lambda)\perp\ker(T-\mu)$ for a general contraction on π_k.

Example 1.3. $\pi_2=l_-+l_+$, where both l_- and l_+ are of C^2. Set

$$T=\begin{pmatrix}\begin{pmatrix}2 & 1\\ & 3\end{pmatrix} & 0\\ & 0\end{pmatrix}\begin{matrix}l_-\\ l_+\end{matrix}.$$

ıviously, T is a contraction on π_2. $\operatorname{Ker}(T-2)$ is not orthogonal with $\ker(T-3)$.

Fortunately, we have still the following proposition.

Theorem 1.4. *Suppose T is a contraction. Then the intersection of $\sigma(T)$ with D is a finite set of isolated eigenvalues of T, each of which has finite multiplicity.*

Proof If the statement of the theorem is not true, then we assume there is a ıntable set $\{\lambda_1, \lambda_2, \cdots\}$ of $\sigma_p(T)$ outside the unit disk. First, we show if there is ıeutral vector x in $\ker(T-\lambda)$, then λ belongs to $\sigma_p(U)$ and x belongs to ker $(U-$ where U is a unitary dilation of T. In fact, because $Tx=\lambda x$, we have

$$(Ux, y)=(Tx, y)=(\lambda x, y)$$

any $y\in\pi_k$. It implies that $Ux-\lambda x=h\in H$, where $\pi_k'=\pi_k\oplus H$ is the space of ation of T and H is a Hilbert space [4]. By the hypothesis of $(x, x)=0$, we have

$$(Ux, Ux)=(h+\lambda x, h+\lambda x)=(h, h)=0.$$

erefore, $h=0$, $\lambda\in\sigma_p(U)$ and $x\in\ker(U-\lambda)$.

From the properties of unitary operators on π_k, it follows that the above envalues, whose space of eigenvectors is degenerate, are at most finite. Hence, ›re are countable eigenvalues outside the unit disk, whose space of eigenvectors is ı-degenrate, that is, a complete subspace of π_k. Let $\lambda_1\in\sigma_p(T)\cap(C\backslash D)$ and ker $-\lambda_1)$ be a complete subspace of π_k, then

$$T=\begin{pmatrix}\lambda_1 & A_1\\ & T_1\end{pmatrix}\begin{matrix}\ker(T-\lambda_1)\\ \ker(T-\lambda_1)^\perp.\end{matrix} \tag{1.1}$$

is easy to see that T_1 is also a contraction on $\ker(T-\lambda_1)^\perp$, which is also a π_k space. leed, for any $y\in\ker(T-\lambda_1)^\perp$ we set $x=-\frac{1}{\lambda_1}A_1y$, then

$$\left(T\begin{pmatrix}x\\ y\end{pmatrix}, T\begin{pmatrix}x\\ y\end{pmatrix}\right)=\left(\begin{pmatrix}0\\ T_1y\end{pmatrix}, \begin{pmatrix}0\\ T_1y\end{pmatrix}\right)$$
$$=(T_1y, T_1y)\leqslant(x, x)+(y, y)\leqslant(y, y).$$

Here we have used the fact that $(x, x)\leqslant 0$ for $x\in\ker(T-\lambda_1)$. On the other hand, for any $\lambda\in\sigma_p(T)\cap(C\backslash D)$, where $\lambda_1\neq\lambda$, we have

$$(T-\lambda)\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} (\lambda_1-\lambda)x+Ay \\ (T_1-\lambda)y \end{pmatrix}=0$$

for any $x+y\in\ker(T-\lambda)$. Since $\lambda_1\neq\lambda$, we obtain $\lambda\in\sigma_p(T_1)$ and $y\neq 0$. That is, there is still a countable set $\{\lambda_2, \lambda_3 \cdots\}$ of $\sigma_p(T_1)$ outside the unit disk. Consider T_1 as a contraction on ker $(T-\lambda_1)^\perp$. With no loss of generality, we assume the $\ker(T_1-\lambda_2)$ is a complete subspace of $\ker(T-\lambda_1)^\perp$. Naturally, $\ker(T_1-\lambda_2)^\perp$ $\ker(T-\lambda_1)$ on π_k and T_1 is of the following form:

$$T_1=\begin{pmatrix} \lambda_2 & A_2 \\ & T_2 \end{pmatrix}\begin{matrix}\ker(T_1-\lambda_2) \\ \ker(T_1-\lambda_2)^\perp.\end{matrix}$$

By the hopothesis, the above step can be carried out infinitely. So we have ohc countable semi-negative subspaces $\{\ker(T_i-\lambda_{i+1})\}_{i=1}^{\infty}$ of π_k, which is orthogo with each other. This is impossible for π_k space.

§ 2. The Relations Between the Spectrum of the Contracti and the Spectrum of its Minimal Unitary Dilat.on

Again for convenience, we give the following definition[4].

Definition 2.1. *Suppose that T is a contraction on π_k and H is a Hilbert sp If there exists a unitary operator U from $\pi_k\oplus H$ onto $\pi_k\oplus H$ such taht $T^n=PU$ $n=0, 1, 2, \cdots$, where P is the projection from $\pi_k\oplus H$ onto π_k, then U is calle unitary dilation of T. Denoteit by $T=prU$.*

The existence of a unitary dilation has been proved for any contraction on π one can prove easily the following theorem.

Theorem 2.2. *For every contraction T on π_k there exists a unitary dilation U $\pi_k\oplus H$, which is minimal, i.e. such that*

$$\pi_k\oplus H=\bigvee_{-\infty}^{+\infty} U^n\pi_k.$$

This minimal unitary dilation is determined up to J-unitary transformation, thus can be called" the minimal unitary dilation" of T.

We only point that $\bigvee_{-\infty}^{+\infty} U^n\pi_k\supset\pi_k$ is a complete subspace of $\pi_k\oplus H$, because $\bigvee_{-\infty}^{+\infty} U$ is non-degenerate.

Similar to the case of Hilbert spaces, subspaces $\mathscr{D}_1=(U-T)\pi_k$ and $\mathscr{D}_0^\dagger=(U$ $T^\dagger)\pi_k$will play important roles below. It is obvious that $\overline{U^n\mathscr{D}_0}=U^n\mathscr{D}_0$ and $\overline{U^{\dagger n}\mathscr{D}}$ $U^\dagger\mathscr{D}_0^\dagger$.

Lemma 2. 3. *Suppose that $(U, \pi_k\oplus H)$ is a minimal unitary dilation of contraction T on π_k. Then the space can be decomposed into the orthogonal sum.*

$$\pi_k \oplus H = \cdots \oplus U^{\dagger 2}\mathscr{D}_0^\dagger \oplus U^\dagger \mathscr{D}_0^\dagger \oplus \mathscr{D}_0^\dagger \oplus \pi_k \oplus \mathscr{D}_0 \oplus U\mathscr{D}_0 \oplus \cdots. \quad (2.1)$$

Remark 2.4. In general, for a space with an indefinite metric, the orthogonal m above is nonsense. Below we will show $U^{\dagger n}\mathscr{D}_0^\dagger$ and $U^n\mathscr{D}_0$, $n=0, 1, 2, \cdots$, belong H. In this case, the orthogonal sum of (2.1) is carried out in Hilbert space (H, , ·)) expcept the part of subspace π_k

Proof Let us show first that

$$U^n\mathscr{D}_0 \perp U^m\mathscr{D}_0,\ U^{\dagger n}\mathscr{D}_0^\dagger \perp U^{\dagger m}\mathscr{D}_0^\dagger, \text{ for } m, n \geqslant 0,\ m \neq n \quad (2.2)$$

d

$$U^n\mathscr{D}_0 \perp U^{\dagger m}\mathscr{D}_0^\dagger,\ U^n\mathscr{D}_0 \perp \pi_k,\ U^{\dagger n}\mathscr{D}_0^\dagger \perp \pi_k \text{ for } m, n \geqslant 0; \quad (2.3)$$

even suffices to estabish these relations for $\mathscr{D}_0$ and $\mathscr{D}_0^\dagger$ instead of $\mathscr{D}_0$ and $\mathscr{D}_0^\dagger$. To ove (2.2), without loss of generality we assume $m=0$. We have

$(U^n(U-T)h, (U-T)h') = (U^n h, h') - (U^{n-1}Th, h') - (U^{n+1}h, Th') + U^nTh, Th') = 0$

: any $h, h' \in \pi_k$. Similarly, we have $U^{\dagger m}\mathscr{D}_0^\dagger \perp \mathscr{D}_0^\dagger$. Now, for $h, h' \in \pi_k$, we have also

$$(U^n(U-T)h, U^{\dagger m}(U^\dagger - T^\dagger)h')$$
$$= (U^{n+m+2}h, h') - (U^{n+m+1}h, T^\dagger h') - (U^{n+m+1}Th, h') + (U^{m+n}Th, T^\dagger h') = 0,$$
$$(U^n(U-T)h, h') = (U^{n+1}h, h') - (U^nTh, h') = 0$$

d

$$(U^{\dagger m}(U^\dagger - T^\dagger)h, h') = (U^{\dagger m+1}h, h') - (U^{\dagger m}T^\dagger h. h') = 0.$$

, the orthogonality relations of (2.2) and (2.3) are established,

Using above remark, we know that $\overline{\mathscr{D}}_0^\dagger \oplus U\overline{\mathscr{D}}_0^\dagger \oplus U^{\dagger 2}\overline{\mathscr{D}}_0^\dagger + \cdots\cdots$ and $\overline{\mathscr{D}}_0 \oplus U\overline{\mathscr{D}}_0 \oplus$ $\mathscr{D}_0 \oplus \cdots$ are two closed subspaces of H. Hence they are also two closed positive bspace of $\pi_k \oplus H$. Denote the orthogonal sum on the right hand side of (2.1) by . Applying U torm by torm we obtain

$$U\pi_r^k = \oplus U^\dagger \overline{\mathscr{D}}_0^\dagger \oplus \overline{\mathscr{D}}_0^\dagger \oplus U\overline{\mathscr{D}}_0^\dagger \oplus U\pi_k \oplus U\overline{\mathscr{D}}_0 \oplus U^2\overline{\mathscr{D}}_0 \oplus \cdots. \quad (2.4)$$

$U\overline{\mathscr{D}}_0^\dagger \oplus U\pi_k = \pi_k \oplus \overline{\mathscr{D}}_0$, then $U\pi'_k = \pi'_k$. Therefore π'_k is a subspace of $\pi_k \oplus H$ reducing and containing π_k, and this implies by the minimality of U that

$$\pi'_k = \pi_k \oplus H.$$

ıviously, to complete the proof of Lemma 2.3, we need only to show that $U\mathscr{D}_0^\dagger +$ $\pi_k = \pi_k + \mathscr{D}_0$. Let $x = x_1 + (U-T)x_2$, where $x_i \in \pi_k$, $i=1, 2$. Set $y_1 = T^\dagger x_1 + (1-$ $T)x_2$ and $y_2 = x_1 - Tx_2$. We have

$$x = Uy_1 + U(U^\dagger - T^\dagger)y_2.$$

Conversely, let $x = U(U^\dagger - T^\dagger)y_1 + Uy_2$, where $y_i \in \pi_k$, $i=1, 2$. Set $x_1 = Ty_1 + (1$ $TT^\dagger)y_2$ and $x_2 = y_1 - T^\dagger y_2$. Then we have

$$x = x_1 + (U-T)x_2.$$

ıus, we have proved that $U\mathscr{D}_0^\dagger \oplus U\pi_k = \pi_k \oplus \mathscr{D}_0$. (This implies that $U^\dagger\pi_k \oplus U^\dagger\mathscr{D}_0 =$ $\mathscr{D}_0^\dagger \oplus \pi_k$.)

Using the above decomposition of space $\pi_k \oplus H$, we obtain

Theorem 2.5. *Let T be a contraction on π_k and U be a minimal unitary dilation.*

Suppose $|\lambda|=1$. *Then* $\lambda\in\sigma_p(T)\Leftrightarrow\lambda\in\sigma_p(U)$. *Moreover, the corresponding eigenvectors are the same for* T *and for* U.

Proof Without loss of generality, we assume $\lambda=1$.

Let $Tx=x$ for some $x\in\pi_k$. Because of the relation

$$(Ux-x,\ Ux-x)=2(x,\ x)-2\mathrm{Re}(Tx,\ x)=2\mathrm{Re}(x-Tx,\ x)=0,$$

$Ux-x$ is a neutral vector of $\pi_k\oplus H$. On the other hand, for any $y\in\pi_k$, $(Ux-x,\ y)=(Ux,\ y)-(x,\ y)=(Tx,\ y)-(x,\ y)=0$. Hence $Ux-x\perp\pi_k$, i.e. $Ux-x\in H$. Therefore $Ux=x$.

Conversely, let $Ux=x$ for some $x\in\pi_k\oplus H$. We will show that $x\in\pi_k$. By use the decomposition of Lemma 2.3, x can be expressed uniquely as $x_1+x_2+x_3$, wher

$$x_1=\sum_{n=0}^{\infty}U^{\dagger n}g_n,\ \ g_n\in\mathscr{D}_0^{\dagger},\ \ x_3=\sum_{n=0}^{\infty}U^n f_n,\ f_n\in\mathscr{D}_0$$

and $x_2\in\pi_k$. Under the regular decomposition $\pi_k\oplus H=H_-\oplus\{H_+\oplus H\}$, we take t norm $\|\cdot\|$. Consequently, $\sum_{n=0}^{\infty}\|g_n\|^2<\infty$ and $\sum_{n=0}^{\infty}\|f_n\|^2<\infty$. Since $U(x_1+x_2+x_3)=$ $+x_2+x_3$ and $U\mathscr{D}_0^{\dagger}\oplus U\pi_k=\pi_k\oplus\mathscr{D}_0$, it follows that

$$\sum_{n=0}^{\infty}U^{\ n}g_n=\sum_{n=0}^{\infty}U^{\dagger n}g_{n+1}.$$

This is impossible unless $g_0=g_1=g_2=\cdots\cdots=0$. Hence $x_1=0$. So we can assume t $U^{\dagger}(x_2+x_3)=x_2+x_3$. Using again the decomposition of Lemma 2.3 and $U^{\dagger}\pi_k\oplus U^{\dagger}$ $=\overline{\mathscr{D}}_0^{\dagger}\oplus\pi_k$ we obtain $\sum_{n=0}^{\infty}U^n f_n=\sum_{n=0}^{\infty}U^n f_{n+1}$. It follows immediately that $x_3=0$.

For $|\lambda|>1$ and $\lambda\in\sigma_p(T)$, the above proposition does not hold. Indeed, if U λx, $|\lambda|>1$, then x is a neutral vector. From Example 1.3, we know that x may a negative vector for some $x\in\ker(T-\lambda)$. But we have the following proposition.

Theorem 2.6. *Let* $Tx=\lambda x$, $|\lambda|>1$ *and* x *be a neutral vector. Then,* $\lambda\in\sigma_p($ *and* $Ux=\lambda x$.

Proof In the proof of Theorem 1.4, we have proved this fact.

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