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ON THE VISCOSITY SPLITT NG METHOD FOR INITIAL BOUNDARY VALUE PROBLEMS OF THE NAVIAR-STOKES EQUATIONS**

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Abstract

The viscosity splitting method for the Navier-Stokes equations on two dimensional multi-connected domains is considered. The equation is split into an Euler equation and a non-stationary Stokes equation within each time step. The author proves the convergence theorem as he has done for the problem on simply connected domains, and the rate of convergence is improved from less than 1/4 to 1.

§ 1. Introduction

We consider initial boundary value problem of the Navier-Stokes equation twodimensional viscous, incompressiable flow

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)\,u+\frac{1}{\rho}\nabla p=\nu\Delta u+f, \qquad (1$$

$$\nabla\cdot u=0, \qquad (1$$

$$u|_{x\in\partial\Omega}=0, \qquad (1$$

$$u|_{t=0}=u_0(x), \qquad (1$$

where $u=(u_1, u_2)^T$ is velocity, p is pressure, $f=(f_1, f_2)^T$ is body force, supersc T stands for transpose of a vector, positive constants ρ, ν are the density viscosity respectively

$$\nabla=\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right),$$

$\Delta=\nabla^2$, Ω is a domain with boundary $\partial\Omega$ in plane $\boldsymbol{R}^2$, and $\nabla\cdot u_0=0$.

The question considered in this paper is: in solving (1.1)—(1.4), is it possi to split equation (1.1) into two equations at each time step, one is an Euler equa which has no viscosity term, and the other one is a Stokes equation which has convection term: The motivation of this consideration is the calculation for flow w high Reynold's number[1].

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Beale and Majda[2] studied the corresponding initial value problem, it was roved that the approximate solutions converge to the true solution with rate $O(k)$ s time step k tends to zero. Douglis and Fabes[3] also studied the initial value roblem but with different approach, they gave polynomial approximate solutions r the Euler equation, then proved an existence theorem by means of the L^p norm timate. With the same scheme, Alessandrini, Douglis and Fabes considered the itial boundary value problem (1.1)—(1.4) and proved convergence theorems in [1]. We considered this initial boundary value problem too [5,6], where Ω was sumed to be a bounded simply connected domain. In our papers a modified Chorin's cheme was applied, where one more step for boundary value correction was used ke [1], but nonhomogeneous Stokes problems were solved instead of solving omogeneous ones. This modification seems necessary for convergence. We proved at this scheme converges with rate $O(k^{(s-1)/2})$, where $1<s<3/2$.

The purpose of this paper is to consider the same method as [5, 6] with respect to ulti-connected domains, moreover we will prove a better estimate $O(k)$ for the rate convergence. Because there is no one to one correspondence between vorticity and locity for these domains, argument in this paper is more complicated.

Now let us give a brief statement of our main results. Let Ω be a bounded main in $\boldsymbol{R}^2$. We assume that its boundary $\partial\Omega$ consists of $N+1$ sufficiently smooth, nple closed curves $\Gamma_0, \Gamma_1, \cdots\cdots, \Gamma_N$, $N\geqslant 0$, where $\Gamma_j (j=1, \cdots, N)$ are inside of Γ_0 d outside of one another. Denote by $x=(x_1, x_2)$ a point in $\boldsymbol{R}^2$. Let T be any positive mber, then problem (1.1)—(1.4) admits a solution u, p on closed domain $\bar{\Omega}\times$ $, T]$ provided functions u_0, f satisfy a fairly weak assumption, and the solution u unique, p is unique up to a scalar function of t which may be added to p.[7]

The usual notations $H^s(\Omega)$, $W^{m,p}(\Omega)$ for Sobolev spaces and $\|\cdot\|_s$, $\|\cdot\|_{m,p}$ for their rms are applied throughout this paper, and space $L^2(\Omega)=H^0(\Omega)$. We introduce a sed subspace $V\subset L^2(\Omega)$, such that $\theta\in V$ iff there is a $\varphi\in H^2(\Omega)$ and constants c_j, 1, $\cdots$, N, such that

$$-\Delta\varphi=\theta, \tag{1.5}$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_{x\in\partial\Omega}=0, \tag{1.6}$$

$$\varphi|_{x\in\Gamma_0}=0,\ \varphi|_{x\in\Gamma_j}=c_j,\ j=1, \cdots, N, \tag{1.7}$$

ere n is the unit outward normal vector. Let P be the orthogonal projection from (Ω) to V.

Denote $\nabla\wedge=\left(\frac{\partial}{\partial x_2}, \ -\frac{\partial}{\partial x_1}\right)$, let $\omega=-\nabla\wedge u$ for an arbitrary $u\in(H^1(\Omega))^2$, set $\theta=P\omega$, then $\theta\in V$. Determine function φ according to (1.5)—(1.7), set $v=(\nabla\wedge\varphi)^T$

and denote $v=\Theta u$. Θ is also a projection operator

$$\Theta: (H^1(\Omega))^2 \to (H_0^1(\Omega))^2 \cap X,$$

where

$$X = \text{closure in } (L^2(\Omega))^2 \text{ of } \{u \in (C_0^\infty(\Omega))^2;\ \nabla\cdot u=0\}.$$

The following scheme is considered: We devide the interval $[0, T]$ into equal subintervals with length k. Then we solve $\tilde{u}_k(t)$, $\tilde{p}_k(t)$, $u_k(t)$, $p_k(t)$, $i=0, 1, \cdots$ on each interval $[ik, (i+1)k)$, according to the following procedure:

First step, solve a problem on interval $[ik, (i+1)k)$ as

$$\frac{\partial \tilde{u}_k}{\partial t} + (\tilde{u}_k\cdot\nabla)\tilde{u}_k + \frac{1}{\rho}\nabla\tilde{p}_k = f, \tag{1.8}$$

$$\nabla\cdot\tilde{u}_k = 0, \tag{1.9}$$

$$\tilde{u}_k\cdot n|_{x\in\partial\Omega} = 0, \tag{1.10}$$

$$\tilde{u}_k(ik) = u_k(ik-0). \tag{1.11}$$

Second step, projection, construct $\Theta\,\tilde{u}_k((i+1)k-0)$.

Third step, solve a problem on interval $[ik, (i+1)k)$ as

$$\frac{\partial u_k}{\partial t} + \frac{1}{\rho}\nabla p_k = \nu\Delta u_k + \frac{1}{k}(I-\Theta)\tilde{u}_k((i+1)k-0), \tag{1.12}$$

$$\nabla\cdot u_k = 0, \tag{1.13}$$

$$u_k|_{x\in\partial\Omega} = 0, \tag{1.14}$$

$$u_k(ik) = \Theta\tilde{u}_k((i+1)k-0). \tag{1.15}$$

In (1.8)—(1.15), I is the operator of identity, $u_k(-0)=u_0$, and the spatial variab x is omitted since there is no confusion. We always assume that f, u_0 and th solution u of (1.1)—(1.4) are sufficiently smooth throughout this paper.

Our main result is the following

Theorem. *If u is the solution of problem (1.1)—(1.4), $\tilde{u}_k$, u_k is the solution problem (1.8)—(1.15), $0\leqslant s<3/2$, then*

$$\max(\|u_k(t)\|_{s+1}, \|\tilde{u}_k(t)\|_{s+1}) \leqslant M,\ 0\leqslant t\leqslant T, \tag{1.16}$$

$$\max(\|u(t)-u_k(t)\|_1, \|u(t)-\tilde{u}_k(t)\|_1) \leqslant M'k,\ 0\leqslant t\leqslant T, \tag{1.17}$$

where constants M, M' depend only on the domain Ω, constants ν, s, T, know functions f, u_0 and the solution u of (1.1)—(1.4) (in fac t, M' is independent of s)

We discuss some properties of the operator Θ and Stokes operator A in sectio 2. In section 3 we consider a special case, i. e. the case when the convection term (1.1) is dropped. We give some estimates for the Euler equation in section 4, a some estimates for problem (1.8)—(1.15) in section 5. Finally, the main theorem proved in section 6.

§ 2. Preliminaries

In this paper we always denote by C a generic constant which depends only on the domain Ω and constants ν, s, T, by C_0 a generic constant which depends only on the domain Ω, constants ν, s, T, the known functions f, u_0, and the solution u of (1.1)—(1.4), by C_1, C_2, $\cdots$, M_0, M_1, $\cdots$ some other generic constants which are determined according to special requirements.

Let $\omega\in L^2(\Omega)$, then $\theta=P\omega\in V$. Since θ is given, constants c_j, $j=1, \cdots, N$ are determined by (1.5)—(1.7), which are written as $c_j=c_j(\omega)$.

Lemma 1. $|c_j(\omega)|\leqslant C\|\omega\|_0$, $j=1, \cdots, N$.

Proof We consider a boundary value problem with part of the boundary conditions (1.6) (1.7).

$$-\Delta\varphi=P\omega,$$

$$\varphi|_{x\in\Gamma_0}=0,\ \left.\frac{\partial\varphi}{\partial n}\right|_{x\in\Gamma_j}=0,\ j=1, \cdots, N,$$

which is well posed. From the L^2 norm estimate for the solutions of elliptic boundary value problems[8]

$$\|\varphi\|_2\leqslant C\|P\omega\|_0.$$

But P is an orthogonal projection

$$\|P\omega\|_0\leqslant\|\omega\|_0. \tag{2.1}$$

By the trace theorem[9]

$$|c_j(\omega)|\leqslant C\|\varphi\|_1.$$

Then this lemma follows.

Lemma 2. *The operator P maps $H^s(\Omega)$ in $H^s(\Omega)$ for any $s\geqslant 0$, and*

$$\|P\omega\|_s\leqslant C\|\omega\|_s. \tag{2.2}$$

Proof Denote by $(\cdot, \cdot)$ the inner product of $L^2(\Omega)$. We construct functional

$$R(\theta)=(\theta, \theta)/2-(\theta, \omega),$$

then $P\omega$ is the solution of the following problem:

$$R(P\omega)=\min_{\theta\in V} R(\theta).$$

We consider a subset $V_\omega\subset V$, such that $\theta\in V_\omega$ iff there is a function $\varphi\in H^2(\Omega)$ which satisfies (1.5)—(1.7) and $c_j=c_j(\omega)$. Then $P\omega$ is also the solution of

$$R(P\omega)=\min_{\theta\in V_\omega} R(\theta). \tag{2.3}$$

Let $\quad Y_\omega=\{\varphi\in H^1(\Omega);\ \varphi|_{x\in\Gamma_0}=0,\ \varphi|_{x\in\Gamma_j}=c_j(\omega),\ j=1, \cdots, N\}.$

If $\theta\in L^2(\Omega)$, then $\theta\in V_\omega$ iff there is a $\varphi\in Y_\omega$, such that

$$(\nabla\varphi, \nabla v)=(\theta, v), \quad \forall v\in H^1(\Omega).$$

Let v be a Lagrangian multiplier, and consider a functional

$$R_1(\theta, \varphi, v)=(\theta, \theta)/2-(\theta, \omega)+(\nabla\varphi, \nabla v)-(\theta, v)$$

in the set $L^2(\Omega)\times Y_\omega\times H^1(\Omega)$. Then (2.3) is equivalent to: find $P\omega$, φ, v such that $R_1'(P\omega, \varphi, v)=0$, that is

$$(P\omega-\omega-v, \theta)=0, \quad \forall\theta\in L^2(\Omega),$$

$$(\nabla v, \nabla \chi) = 0, \quad \forall \chi \in H_0^1(\Omega),$$
$$(\nabla \varphi, \nabla w) - (P\omega, w) = 0, \quad \forall w \in H^1(\Omega).$$

Thus $P\omega$, φ, v is the weak solution of the following boundary value problem:

$$\Delta v = 0,$$
$$-\Delta \varphi = P\omega = \omega + v,$$
$$\left.\frac{\partial \varphi}{\partial n}\right|_{x \in \partial\Omega} = 0,$$
$$\varphi|_{x \in \Gamma_0} = 0, \quad \varphi|_{x + \Gamma_j} = c_j(\omega), \quad j = 1, \cdots, N.$$

Eliminating v we get $\Delta^2 \varphi = -\Delta \omega$.

By thy L^2 norm estimate for the solutions of elliptic boundary val problems [8], if m is an integer and $m \geqslant 2$, $\omega \in H^m(\Omega)$, then $\varphi \in H^{m+2}(\Omega)$, and

$$\|\varphi\|_{m+2} \leqslant C\left(\|\Delta\omega\|_{m-2} + \sum_{j=1}^{N} |c_j(\omega)| \right).$$

Thanks to Lemma 1

$$\|\varphi\|_{m+2} \leqslant C \|\omega\|_m. \tag{2.}$$

Therefore

$$\|P\omega\|_m \leqslant \|\varphi\|_{m+2} \leqslant C\|\omega\|_m. \tag{2.}$$

For $0 < s < m$, by (2.1) (2.5) and the interpolation theorem [8], (2.2) is obtained.

Lemma 3. *The operator Θ maps $(H^{s+1}(\Omega))^2$ in $(H^{s+1}(\Omega))^2$ for any $s \geqslant 0$, an*

$$\|\Theta u\|_{s+1} \leqslant C\|u\|_{s+1}. \tag{2.}$$

Proof By (2.4) and the definition of operator Θ,

$$\|\Theta u\|_{m+1} \leqslant \|\varphi\|_{m+2} \leqslant C\|\omega\|_m \leqslant C\|u\|_{m+1}$$

for any $m \geqslant 2$. Then (2.6) follows from the interpolation theorem like Lemma 2.

We now consider a decomposition of the space $X \cap (H^1(\Omega))^2$, equipped with nor $\|\cdot\|_1$. We construct a subspace $X_0 \subset X \cap (H^1(\Omega))^2$, such that $\bar{u} \in X_0$ iff there a function $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$, such that $\bar{u} = (\nabla \wedge \varphi)^T$. Consider the followi boundary value problem [10]:

$$\Delta \varphi^{(i)} = 0, \quad i = 1, \cdots, N,$$
$$\varphi^{(i)}|_{x \in \Gamma_j} = \delta_{ij}, \quad i, j = 1, \cdots, N.$$

Let $u^{(i)} = (\nabla \wedge \varphi^{(i)})^T$, then $u^{(i)} \in X \cap (H^1(\Omega))^2$. Set $\{u^{(i)}\}$ is linearly independent, ar is orthogonal to space X_0 with respect to the inner product of L^2. We orthonormali it, still denoted by $\{u^{(i)}\}$, such that

$$(u^{(i)}, u^{(j)}) = \delta_{ij}, \quad i, j = 1, \cdots, N.$$

Lemma 4. *An arbitrary element u in $X \cap (H^1(\Omega))^2$ can be decomposed uniquel as*

$$u = \bar{u} + \sum_{j=1}^{N} \lambda_j u^{(j)}, \tag{2.7}$$

where $\bar{u} = (\nabla \wedge \varphi)^T$, and φ is the solution of

$$-\Delta \varphi = \omega = -\nabla \wedge u, \tag{2.8}$$

$$\varphi|_{x\in\partial\Omega}=0. \tag{2.8}$$

Proof By the L^2 norm estimate of the solutions of elliptic boundary value problem[6] $\varphi\in H_0^1(\Omega)\subset H^2(\Omega)$. Let the stream function corresponding to u be ψ, then $-\Delta\psi=\omega$, hence $\Delta(\psi-\varphi)=0$. We may assume that $\psi|_{x\in\Gamma_0}=0$. Since $\psi-\varphi$ are constants on Γ_j, $j=1, 1, \cdots, N$, it can be developed uniquely as a linear compo-sition of $\varphi^{(j)}$

$$\psi-\varphi=\sum_{j=1}^{N}\lambda_j\varphi^{(j)}.$$

Applying operator $\nabla\wedge$ to it, we get (2.7). By the orthogonality of $\bar{u}$ and $u^{(j)}$, we konw the expression (2.7) is unique.

In what follows we consider some properties of the Stokes operator [11]. Set

$$G=\{\nabla p;\ p\in H^1(\Omega)\},$$

then we have the Helmholtz decomposition

$$(L^2(\Omega))^2=X\oplus G.$$

Let P' be the continuous projection from $(L^2(\Omega))^2$ to X associated with this decomposition. In virtue of [7], we have the following

Lemma 5. *The operator P' maps $(H^s(\Omega))^2$ in $(H^s(\Omega))^2$ for any $s\geqslant 0$, and*

$$\|P'f\|_s\leqslant C\|f\|_s.$$

The Stokes operator is defined as $A=-P'\Delta$, with domain

$$D(A)=X\cap\{u\in(H^2(\Omega))^2;\ u|_{x\in\partial\Omega}=0\}.$$

It is known that $\{e^{-tA}, t\geqslant 0\}$ extends uniquely to a bounded holomorphic semigroup in X, and inequality

$$\|A^\alpha e^{-tA}\|\leqslant Ct^{-\alpha},\quad \alpha\geqslant 0,\ t>0, \tag{2.9}$$

holds. We denote by $D(A^\alpha)$ the domain of operator A^α, then

$$D(A^\alpha)=[X,\ D(A)]_\alpha=X\cap[(L^2(\Omega))^2,\ D(-\Delta)]_\alpha \tag{2.10}$$

for $0\leqslant\alpha\leqslant 1$, where $[\cdot,\ \cdot]_\alpha$ are the in termediate spaces[8],

$$D(-\Delta)=\{u\in(H^2(\Omega))^2;\quad u|_{x\in\partial\Omega}=0\}.$$

In $D(A^\alpha)$, $\alpha\geqslant 0$, the norm $\|A^\alpha u\|_0$ and $\|u\|_{2\alpha}$ are equivalent, namely

$$\|A^\alpha u\|_0\leqslant C\|u\|_{2\alpha}, \tag{2.11}$$

and

$$\|u\|_{2\alpha}\leqslant C\|A^\alpha u\|_0, \tag{2.12}$$

for any $u\in D(A^\alpha)$. Strictly speaking, constant C depends only on the domain Ω and constant α.

Lemma 6. *If $0\leqslant s<1/2$, $u\in X\cap(H^s(\Omega))^2$, then $u\in D(A^{s/2})$; if $1\leqslant s<3/2$, $u\in D(A)\cap(H^{s+1}(\Omega))^2$, then $u\in D(A^{(s+1)/2})$.*

Proof If $0\leqslant s<1/2$, then $H^s(\Omega)=H_s^0(\Omega)$[8]. By (2.10)

$$D(A^{s/2})=X\cap[(L^2(\Omega))^2,\ D(-\Delta)]_{s/2}$$
$$\supset X\cap[(L^2(\Omega))^2,\ (H_0^2(\Omega))^2]_{s/2}=X\cap(H_0^s(\Omega))^2.$$

If $1<s<3/2$, $u\in D(A)\cap(H^{s+1}(\Omega))^2$, by the definition of operator A, $Au\in X$. By Lemma 5, $Au\in X\cap(H^{s-1}(\Omega))^2$. By the first part of this Lemma, $Au\in D(A^{(s-1)/2})$. Hence $u\in D(A^{(s+1)/2})$.

§ 3. Some Estimates for Solutions of the Stokes Problem

In this section we consider the linear counterpart of (1.1)—(1.4), that is

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\nabla p=\nu\Delta u+f, \tag{3}$$

$$\nabla\cdot u=0, \tag{3}$$

$$u|_{x\in\partial\Omega}=0, \tag{3}$$

$$u|_{t=0}=u_0(x). \tag{3}$$

We assume that f, u_0 and solution u are sufficiently smooth as before. We introd vorticity $\omega=-\nabla\wedge u$ and stream function ψ, such that $u=(\nabla\wedge\psi)^T$, then equat (3.2) is satisfied automatically. We may take ψ such that $\psi|_{x\in\Gamma_0}=0$. Let τ be unit tangent vector along $\partial\Omega$, such that n, τ form a right-handed system. Then solution of (3.1)—(3.4) satisfies

$$\frac{\partial\omega}{\partial t}=\nu\Delta\omega-\nabla\wedge f, \tag{3}$$

$$-\Delta\psi=\omega, \tag{3}$$

$$\left.\frac{\partial\psi}{\partial n}\right|_{x\in\partial\Omega}=0, \tag{3}$$

$$\psi|_{x\in\Gamma_0}=0,\ \psi|_{x\in\Gamma_j}=C_j,\ j=1,\ \cdots,\ N, \tag{3}$$

$$\omega|_{t=0}=\omega_0=-\nabla\wedge u_0, \tag{3}$$

$$\int_{\Gamma_j}\left(\nu\frac{\partial\omega}{\partial n}+f\cdot\tau\right)ds=0,\quad j=1,\ \cdots,\ N, \tag{3.}$$

where c_j are unknown scalar functions with independent variable t.

Lemma 7. *If* $u\in(H_0^1(\Omega))^2$, $\omega=-\nabla\wedge u$, *then*

$$\|u\|_1\leqslant C\|\omega\|_0. \tag{3.}$$

Proof Let ψ be the stream function corresponding to u and

$$\psi|_{x\in\Gamma_0}=0,$$

then

$$\left.\frac{\partial\psi}{\partial n}\right|_{x\in\Gamma_j}=0,\quad j=1,\ \cdots,\ N,$$

and

$$-\Delta\psi=\omega.$$

By the L^2 norm estimate of elliptic equations

$$\|\psi\|_2\leqslant C\|\omega\|_0.$$

Then $u=(\nabla\wedge\psi)^T$ yields (3.11).

Lemma 8. *If* ω *is the solution of* (3.5)—(3.10), *then*

$$\frac{d}{dt}\|\omega\|_0^2 \leqslant \frac{1}{2\nu}\|f\|_0^2. \tag{3.12}$$

Proof Differentiating (3.6) with respect to t, we get

$$-\Delta\frac{\partial\psi}{\partial t}=\frac{\partial\omega}{\partial t}. \tag{3.13}$$

ubstituting it into (3.5), we get

$$-\Delta\frac{\partial\psi}{\partial t}=\nu\Delta\omega-\nabla\wedge f.$$

Iultiplying it with $\frac{\partial\psi}{\partial t}$, and integrating it on domain Ω. by Green's formula and oundary condition (3.7) we get

$$\begin{aligned}\left(\nabla\frac{\partial\psi}{\partial t},\ \nabla\frac{\partial\psi}{\partial t}\right)&+\nu\left(\nabla\omega,\ \nabla\frac{\partial\psi}{\partial t}\right)\\ &=\nu\int_{\partial\Omega}\frac{\partial\omega}{\partial n}\frac{\partial\psi}{\partial t}ds+\left(f,\ \nabla\wedge\frac{\partial\psi}{\partial t}\right)\\ &\quad+\int_{\partial\Omega}f\frac{\partial\psi}{\partial t}\tau ds.\end{aligned} \tag{3.14}$$

Iultiplying equation (3.13) with ω, and integrating it on domain Ω, by Green's ormula and boundary condition (3.7) we get

$$\left(\nabla\frac{\partial\psi}{\partial t},\ \nabla\omega\right)=\left(\frac{\partial\omega}{\partial t},\ \omega\right). \tag{3.15}$$

t is known that $\frac{\partial\psi}{\partial t}$ are constants along Γ_j. By boundary conditions (3.8) (3.10) nd (3.14) (3.15) we obtain

$$\left(\nabla\frac{\partial\psi}{\partial t},\ \nabla\frac{\partial\psi}{\partial t}\right)+\nu\left(\frac{\partial\omega}{\partial t},\ \omega\right)=\left(f,\ \nabla\wedge\frac{\partial\psi}{\partial t}\right).$$

Hence

$$\left\|\nabla\frac{\partial\psi}{\partial t}\right\|_0^2+\frac{\nu}{2}\frac{d}{dt}\|\omega\|_0^2\leqslant\frac{1}{4}\|f\|_0^2+\left\|\nabla\frac{\partial\psi}{\partial t}\right\|_0^2,$$

which is (3.12).

Lemma 9. *If u is the solution of* (3.1)—(3.4),

$$u_0\in D(A)\cap(H^{s+1}(\Omega))^2,\quad 0\leqslant s<3/2,$$

then

$$\|u(t)\|_{s+1}\leqslant C(\|u_0\|_{s+1}+\max_{0<\tau<T}\|f(\tau)\|_1),\ 0\leqslant\tau\leqslant T.$$

Proof By means of Stokes operator, u can be expressed as

$$u(t)=e^{-\nu tA}u_0+\int_0^t e^{-\nu(t-\tau)A}P'f(\tau)d\tau. \tag{3.16}$$

We estimate the terms in (3.16). By Lemma 6, $u_0\in D(A^{(s+1)/2})$; by (2.9) (2.11) (2.12), we get

$$\begin{aligned}\|e^{-\nu tA}u_0\|_{s+1}&\leqslant C\|A^{(s+1)/2}e^{-\nu tA}u_0\|_0=C\|e^{-\nu tA}A^{(s+1)/2}u_0\|_0\\ &\leqslant C\|A^{(s+1)/2}u_0\|_0\leqslant C\|u_0\|_{s+1}.\end{aligned}$$

Take a positive constant r, $s-1<r<1/2$, then by Lemma 6, $P'f(\tau)\in D\ (A^{r/2})$, $\forall\tau\in[0, T]$. By (2.9)(2.11)(2.12) and Lemma 5, we get

$$\left\|\int_0^t e^{-\nu(t-\tau)A}P'f(\tau)d\tau\right\|_{s+1}$$
$$\leqslant C\int_0^t \|A^{(s+1)/2}e^{-\nu(t-\tau)A}P'f(\tau)\|_0 d\tau$$
$$=C\int_0^t \|A^{(s+1-r)/2}e^{-\nu(t-\tau)A}A^{r/2}P'f(\tau)\|_0 d\tau$$
$$\leqslant C\int_0^t (\nu(t-\tau))^{-(s+1-r)/2}\|f(\tau)\|_r d\tau$$
$$\leqslant C\max_{0\leqslant\tau\leqslant T}\|f(\tau)\|_1.$$

Now we apply the scheme (1.8)—(1.15) to problem (3.1)—(3.4), and gi some useful estimates. For this case, equation (1.8) becomes

$$\frac{\partial\tilde{u}_k}{\partial t}+\frac{1}{\rho}\nabla\tilde{p}_k=f, \qquad (3.1$$

where the term $(\tilde{u}_k\nabla)\tilde{u}_k$ is dropped.

Lemma 10. *If u_k is the solution of problem* (3.17)(1.9)—(1.15), *then*

$$u_k(t)=e^{-\nu tA}u_0+\sum_{i=0}^{[t/k]}e^{-\nu(t-ik)A}\int_{ik}^{(i+1)k}\Theta f(\tau)d\tau$$
$$+\sum_{i=0}^{[t/k]-1}\int_{ik}^{(i+1)k}e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k}P'(I-\Theta)f(\zeta)d\zeta\, d\tau$$
$$+\int_{[t/k]k}^{t}e^{-\nu(t-\tau)A}\frac{1}{k}\int_{[t/k]k}^{([t/k]+1)k}P'((I-\Theta)f(\zeta)d\zeta\, d\tau, \qquad (3.1$$

where [ ] *deno es the integral part of a number.*

Proof We prove

$$u_k(jk-0)=e^{-\nu jkA}u_0+\sum_{i=0}^{j-1}e^{-\nu(j-i)kA}\int_{ik}^{(i+1)k}\Theta f(\tau)d\tau$$
$$+\sum_{i=0}^{j-1}\int_{ik}^{(i+1)k}e^{-\nu(jk-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k}P'(I-\Theta)f(\zeta)d\zeta\, d\tau \qquad (3.1$$

by induction. (3.19) is obviously valid for $j=0$. Now we assume that (3.19) is va for a certain j.

Applying P' to equation (3.17) we get

$$\frac{\partial\tilde{u}_k}{\partial t}=P'f. \qquad (3.2$$

Integrating it on interval $[jk, (j+1)k)$, and using initial condition (1.11), we g

$$\tilde{u}_k((j+1)k-0)=u_k(jk-0)+\int_{jk}^{(j+1)k}P'f(\tau)d\tau.$$

Substituting it into equation (1.12) and initial condition (1.15), we get

$$\frac{\partial u_k}{\partial t}+\frac{1}{\rho}\nabla p_k=\nu\Delta u_k+\frac{1}{k}(I-\Theta)\left(u_k(jk-0)+\int_{jk}^{(j+1)k}P'f(\tau)d\tau\right), \qquad (3.21)$$

$$u_k(jk)=\Theta(u_k(jk-0)+\int_{jk}^{(j+1)k}P'f(\tau)d\tau). \tag{3.22}$$

pplying operator P' to equation (3.21), we get

$$\frac{\partial u_k}{\partial t}=-\nu Au_k+\frac{1}{k}P'(I-\Theta)\Big(u_k(jk-0)+\int_{jk}^{(j+1)k}P'f(\tau)d\tau\Big). \tag{3.23}$$

ıt $(P')^2=P'$, $\Theta P'=\Theta$, integrating equation (3.23) on interval $[jk, t)$, and using itial condition (3.22), we obtain

$$u_k(t)=e^{-\nu(t-jk)A}\Big(\Theta u_k(jk-0)+\int_{jk}^{(j+1)k}\Theta f(\tau)d\tau\Big)$$
$$+\int_{jk}^{t}e^{-\nu(t-\tau)A}\frac{1}{k}P'(I-\Theta)\Big(u_k(jk-0)+\int_{jk}^{(j+1)k}f(\zeta)d\zeta)d\tau\Big).$$

ut $u_k(jk-0)\in(H_0^1(\Omega))^2\cap X$, therefore $\Theta u_k(jk-0)=u_k(jk-0)$, hence

$$u_k(t)=e^{-\nu(t-jk)A}\Big(u_k(jk-0)+\int_{jk}^{(j+1)k}\Theta f(\tau)d\tau\Big)$$
$$+\int_{jk}^{t}e^{-\nu(t-\tau)A}\frac{1}{k}\int_{jk}^{(j+1)k}P'(I-\Theta)f(\zeta)d\zeta\,d\tau.$$

/e substitute (3.19) into it and obtain (3.18). Let $t\to(j+1)k-0$, then (3.19) is erified for $j+1$. This completes the indution and (3.18) is poved at the same time.

Lemma 11. *If* $u_0\in D(A)\cap(H^{s+1}(\Omega))^2$, $0\leqslant s<3/2$, *then*

$$\|u_k(jk-0)\|_{s+1}\leqslant C(\|u_0\|_{s+1}+\sup_{0<\tau<jk}\|f(\tau)\|_1),\ j=0,1,\cdots.$$

Proof We estimate the terms in (3.19). Like the proof of Lemma 9, take a ositive constant r, $s-1<r<1/2$, then the second term

$$\Big\|\sum_{i=0}^{j-1}e^{-\nu(j-i)kA}\int_{ik}^{(i+1)k}\Theta f(\tau)d\tau\Big\|_{s+1}$$
$$\leqslant C\Big\|\sum_{i=0}^{j-1}A^{(s+1)/2}e^{-\nu(j-i)kA}\int_{ik}^{(i+1)k}\Theta f(\tau)d\tau\Big\|_0$$
$$=C\Big\|\sum_{i=0}^{j-1}A^{(s+1-r)/2}e^{-\nu(j-i)kA}A^{r/2}\int_{ik}^{(i+1)k}\Theta f(\tau)d\tau\Big\|_0$$
$$\leqslant C\sum_{i=0}^{j-1}(\nu(j-i)k)^{-(s+1-r)/2}\int_{ik}^{(i+1)k}\|A^{r/2}\Theta f(\tau)\|_0d\tau$$
$$\leqslant C\sup_{0\leqslant\tau<jk}\|f(\tau)\|_r\sum_{i=0}^{j-1}(\nu(j-i)k)^{-(s+1-r)/2}k$$
$$\leqslant C\sup_{0\leqslant\tau<jk}\|f(\tau)\|_1\int_0^{jk}(\nu(jk-\tau))^{-(s+1-r)/2}d\tau$$
$$\leqslant C\sup_{0\leqslant\tau\ jk}\|f(\tau)\|_1.$$

'he estimate of the third term is similar, and the first term has been estimated in he proof of Lemma 9.

Lemma 12. *If* $u_0\in D(A)\cap(H^{s+1}(\Omega))^2$, $0\leqslant s<3/2$, *u is the solution of problem* 3.1)—(3.4), u_k, $\tilde{u}_k$ *is the solution of problem* (3.17) (1.9)—(1.15), *then*

$$\max_{0\leqslant t\leqslant T}(\|u(t)-u_k(t)\|_{s+1},\ \|u(t)-\tilde{u}_k(t)\|_{s+1})\leqslant C_0k. \tag{3.24}$$

Proof By (3.16) (3.18) we have

$$
\begin{aligned}
u(t)-u_k(t) =& \sum_{i=0}^{[t/k]-1} \int_{ik}^{(i+1)k} (e^{-\nu(t-\tau)A}-e^{-\nu(t-ik)A})\Theta f(\tau)\,d\tau \\
&+\int_{[t/k]k}^{t} (e^{-\nu(t-\tau)A}-e^{-\nu(t-[t/k]k)A})\Theta f(\tau)\,d\tau \\
&-\int_{t}^{([t/k]+1)k} e^{-\nu(t-[t/k]k)A}\Theta f(\tau)\,d\tau \\
&+\sum_{i=0}^{[t/k]-1} \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k} P'(I-\Theta)(f(\tau)-f(\zeta))\,d\zeta\,d\tau \\
&+\int_{[t/k]k}^{t} e^{-\nu(t-\tau)A}\frac{1}{k}\int_{[t/k]k}^{([t/k]+1)k} P'(I-\Theta)(f(\tau)-f(\zeta))\,d\zeta\,d\tau. \quad (3.
\end{aligned}
$$

We estimate the terms in (3.25). With regard to the first term

$$
\begin{aligned}
I_1 =& \left\|\sum_i \int_{ik}^{(i+1)k} (e^{-\nu(t-\tau)A}-e^{-\nu(t-ik)A})\Theta f(\tau)\,d\tau\right\|_{s+1} \\
\leqslant& C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+1)/2}e^{-\nu(t-\tau)A}(I-e^{-\nu(t-ik)A})\Theta f(\tau)\,d\tau\right\|_0 \\
=& C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+3)/2}e^{-\nu(t-\tau)A}\int_0^{t-ik} e^{-\nu\varphi A}d\zeta\,\Theta f(\tau)\,d\tau\right\|_0 .
\end{aligned}
$$

We take a constant s_1, $s<s_1<3/2$, then by Lemma 6 and (2.11)

$$
\begin{aligned}
I_1 \leqslant& C\left\|\sum_i \int_{ik}^{(i+1)k} A^{1+(s-s_1)/2}e^{-\nu(t-\tau)A}\int_0^{t-ik} e^{-\nu\zeta A}\,d\zeta\, A^{(s_1+1)/2}\Theta f(\tau)\,d\tau\right\|_0 \\
\leqslant& C\sum_i \int_i^{t}{}^{(i+1)k} (t-\tau)^{-1+(s_1-s)/2}\int_0^{t-ik} \|A^{(s_1+1)/2}\Theta f(\tau)\|_0\,d\zeta\,d\tau \\
\leqslant& C\max_{0<\tau<T}\|\Theta f(\tau)\|_{s_1+1}k\int_0^t (t-\tau)^{-1+(s_1-s)/2}d\tau.
\end{aligned}
$$

By Lemma 3

$$
I_1 \leqslant Ck\max_{0<\tau<T}\|f(\tau)\|_{s_1+1}.
$$

With regard to the fourth term, we take a positive constant r, $s-1<r<1/2$, Lemma 6 and (2.12)

$$
\begin{aligned}
I_2 =& \left\|\sum_i \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k} P'(I-\Theta)(f(\tau)-f(\zeta))\,d\zeta\,d\tau\right\|_{s+1} \\
=& \left\|\sum_i \int_{ik}^{(i+1)k} e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k}\int_\zeta^t P'(I-\Theta)f'(\xi)\,d\xi\,d\zeta\,d\tau\right\|_{s+1} \\
\leqslant& C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+1)/2}e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k}\int_\zeta^t P'(I-\Theta)f'(\xi)\,d\xi\,d\zeta\,d\tau\right\|_0 \\
=& C\left\|\sum_i \int_{ik}^{(i+1)k} A^{(s+1-r)/2}e^{-\nu(t-\tau)A}\frac{1}{k}\int_{ik}^{(i+1)k}\int_\zeta^t A^{r/2}P'(I-\Theta)f'(\xi)\,d\xi\,d\zeta\,d\tau\right\|_0 .
\end{aligned}
$$

By inequalities (2.9), (2.11)

$$
I_2 \leqslant C\sum_i \int_{ik}^{(i+1)k} (\nu(t-\tau))^{-(s+1-r)/2}\frac{1}{k}\int_{ik}^{(i+1)k}\int_{ik}^{(i+1)k} \|P'(I-\Theta)f'(\tau)\|_r\,d\xi\,d\zeta\,d\tau.
$$

By Lemma 5

$$
I_2 \leqslant C\int_0^t (t-\tau)^{-(s+1-r)/2}d\tau\cdot\max_i\int_{ik}^{(i+1)k} \|(I-\Theta)f'(\xi)\|_r\,d\xi.
$$

Lemma 3

$$I_2 \leqslant Ck \cdot \max_{0<\xi<T} \|f'(\xi)\|_1.$$

e can estimate the rest terms in a similar way. Thus the estimate (3.24) for $t) - u_k(t)$ is obtained.

Now we estimate $u(t) - \tilde{u}_k(t)$. Because u is sufficiently smooth,

$$\|u(t) - u(ik)\|_{s+1} \leqslant C_0 k$$

$t \in [ik, (i+1)k)$. From equation (3.20) and using Lemma 5,

$$\|\tilde{u}_k(t) - \tilde{u}_k(ik)\|_{s+1}$$

$$= \left\| \int_{ik}^{t} P'f(\tau) d\tau \right\|_{s+1} \leqslant \int_{ik}^{t} \|P'f(\tau)\|_{s+1} d\tau \leqslant Ck \cdot \max_{ik<\tau<t} \|f(\tau)\|_{s+1}.$$

initial condition (1.11)

$$\|u(ik) - \tilde{u}_k(ik)\|_{s+1} = \|u(ik-0) - u_k(ik-0)\|_{s+1}.$$

sing the triangle inequality, we get

$$\|u(t) - \tilde{u}_k(t)\|_{s+1} \leqslant \|u(ik-0) - u_k(ik-0)\|_{s+1} + C_0 k.$$

hen the estimate (3.24) for $u(t) - \tilde{u}_k(t)$ follows.

§ 4. Some Estimates for Solutions of the Euler Equation

In this section we consider initial boundary value problem of the Euler equation rresponding to (1.1)—(1.4), that is

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \frac{1}{\rho} \nabla p = f, \tag{4.1}$$

$$\nabla \cdot u = 0, \tag{4.2}$$

$$u \cdot n |_{x \in \partial\Omega} = 0, \tag{4.3}$$

$$u|_{t=0} = u_0(x). \tag{4.4}$$

t was proved in [10] that (4.1)—(4.4) admits a unique solution provided the ata are suitably regular, where the sense of uniqueness is the same as that for 1.1)—(1.4). We assume as before that functions f, u_0 and solution u are sufficiently nooth. By introducing vorticity ω and stream function ψ as section 3, we have

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = F = -\nabla \wedge f, \tag{4.5}$$

$$-\Delta \psi = \omega, \tag{4.6}$$

$$u = (\nabla \wedge \psi)^T, \tag{4.7}$$

$$\psi|_{x \in \Gamma_0} = 0, \ \psi|_{x \in \Gamma_j} = c_j, \ j = 1, \cdots, N, \tag{4.8}$$

$$\omega|_{t=0} = \omega_0 = -\nabla \wedge u_0, \tag{4.9}$$

$$\left(\frac{\partial u}{\partial t}, u^{(j)} \right) + ((u \cdot \nabla) u - f, u^{(j)}) = 0, \ j = 1, \cdots, N, \tag{4.10}$$

$$(u|_{t=0} - u_0, u^{(j)}) = 0, \ j = 1, \cdots, N. \tag{4.11}$$

By Lemma 4, we have unique decomposition

$$u(t)=\bar{u}(t)+\sum_{i=1}^{N}\lambda_i(t)u^{(i)},\ \bar{u}(t)\in X_0. \tag{4.12}$$

Lemma 13. *There exists a constat $k_0>0$ which depends only on the domain Ω,*

$$\max_{0<t<T}\|\omega(t)\|_0 \ \text{ and } \ \max_{0<t<T}\|f(t)\|_0,$$

such that

$$\sum_{i=1}^{N}\lambda_i^2(t)\leqslant 2\Big(\sum_{i=1}^{N}\lambda_i^2(0)+1\Big), \tag{4.13}$$

for $0\leqslant t\leqslant k_0$.

Proof Setting $u'=u-\bar{u}$, multiplying (4.10) by $\lambda_j(t)$, and finding the s[illegible] with respect to j, we get

$$\Big(\frac{\partial(\bar{u}+u')}{\partial t},\ u'\Big)+(((\bar{u}+u')\cdot\nabla)(\bar{u}+u')-f,\ u')=0.$$

We notice that $\bar{u}$ is perpendicular to u', and

$$(((\bar{u}+u')\cdot\nabla)u',\ u')=0,$$

thus

$$\frac{1}{2}\frac{d}{dt}(u',\ u')+(((\bar{u}+u')\cdot\nabla)\bar{u}-f,\ u')=0$$

holds. Substitute (4.12) into it and obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}\lambda_i^2(t)+\sum_{i=1}^{N}\xi_i(t)\lambda_i(t)+\sum_{i,j=1}^{N}\xi_{ij}(t)\lambda_i(t)\lambda_j(t)=0, \tag{4.14}$$

where

$$\xi_i(t)=((\bar{u}\cdot\nabla)\bar{u},\ u^{(i)})-(f,\ u^{(i)})=-((\bar{u}\cdot\nabla)u^{(i)},\ \bar{u})-(f,\ u^{(i)}),$$

$$\xi_{ij}(t)=((u^{(i)}\cdot\nabla)\bar{u},\ u^{(j)}).$$

In virtue of the L^2 norm estimate of the solution of (2.8), we have

$$\|\bar{u}\|_1\leqslant C\|\omega\|_0.$$

Therefore

$$|\xi_i(t)|,\ |\xi_{ij}(t)|\leqslant C_1.$$

where and hereafter C_1 is a generic constant which depends only on the doamin Ω,

$$\max_{0<t<T}\|\omega(t)\|_0 \ \text{ and } \ \max_{0<t<T}\|f(t)\|_0.$$

From (4.14) we have

$$\frac{d}{dt}\sum_{i=1}^{N}\lambda_i^2(t)\leqslant C_1\Big(\sum_{i=1}^{N}\lambda_i^2(t)+1\Big).$$

Hence

$$\sum_{i=1}^{N}\lambda_i^2(t)\leqslant\sum_{i=1}^{N}\lambda_i^2(0)+C_1\Big(\int_0^t\sum_{i=1}^{N}\lambda_i^2(t)dt+t\Big).$$

Using Gronwall inequality we obtain

$$\sum_{i=1}^{N}\lambda_i^2(t)\leqslant e^{C_1 t}\Big(\sum_{i=1}^{N}\lambda_j^2(0)+C_1 t\Big).$$

Taking k_0 small enough such that $e^{C_1k_0}\leqslant 2$, $C_1k_0\leqslant 1$, we get (4.13).

Lemma 14. *If $1<s<3/2$, $s_1=1+s/3$, and if $\|u_0\|_{s_1+1}\leqslant M_1$, then there is a constant $k_0>0$, which depends only on the domain Ω, constants s, T, M_1 and*

$$\max_{0<t<T} \|f(t)\|_{s_1+1},$$

h that

$$\|u\|_{s+1} \leqslant C_2(\|u_0\|_{s_1+1}+1), \tag{4.15}$$

· $0 \leqslant t \leqslant k_0$, *where constant* C_2 *depends only on the domain* Ω, *constants* s, T *and*

$$\max_{0<t<T} \|f(t)\|_{s_1+1}.$$

Proof In the following we always denote by C_2 a generic constant which sesses the above property. Integraitng equation (4.5) along characteristic curves, obtain

$$\omega(x, t) = \omega_0(\xi(x, 0; t)) + \int_0^t F(\xi(x, \zeta; t), \zeta)d\zeta, \tag{4.16}$$

ere $\xi(y, t; \tau)$ satsfies

$$\frac{\partial}{\partial t}\xi(y, t; \tau) = u(\xi(y, t; \tau), t),$$

$$\xi(y, \tau; \tau) = y,$$

th $y=(y_1, y_2) \in \Omega$. Applying operator ∇ to it, we get an initial value problem isfied by Jacobean matrices which are the derivatives of $\xi(y, t; \tau)$ with respect to

$$\frac{\partial}{\partial t}\frac{\partial \xi(y, t; \tau)}{\partial y} = \frac{\partial u(\xi(y, t; \tau), t)}{\partial x}\frac{\partial \xi(y, t; \tau)}{\partial y},$$

$$\frac{\partial \xi(y, \tau_0; \tau)}{\partial y} = I,$$

ıere $\frac{\partial u}{\partial x}$ is 2×2 Jacobian matrix, and I is the unit matrix of second erder. Denote

$$v = \frac{\partial u}{\partial x},$$

$$\eta(y, t; \tau) = \frac{\partial \xi(y, t; \tau)}{\partial y},$$

obtain by integrating

$$\eta(y, t; \tau) = e^{\int_\tau^t v(\xi(y,\zeta;\tau),\zeta)d\zeta}. \tag{4.17}$$

pplying operator ∇ to (4.16), we get

$$\nabla\omega = \nabla\omega_0(\xi(x, 0; t))\eta(x, 0; t) + \int_0^t \nabla F(\xi(x, \zeta; t), \zeta)\eta(x, \zeta; t)d\zeta. \tag{4.18}$$

The usual notations $C^{m,\delta}(\bar{\Omega})$ and $\|\cdot\|_{C^{m,\delta}(\bar{\Omega})}$ are used for the spaces of functions ıose derivatives up to m-th order satisfy the Hölder condition, and the norms. As result of the imbedding theorem,[9]

$$\|\omega_0\|_{C^{0,s_1-1}(\bar{\Omega})} \leqslant C_2\|\omega_0\|_{s_1} \leqslant C_2 M_1. \tag{4.19}$$

was proved in [10] that there are constants C_3 and $\delta>0$, depending only on the domain Ω, constants s, T, $\|\omega_0\|_{C^{0,s_1-1}(\bar{\Omega})}$, and

$$\max_{0<t<T} \|F(t)\|_{C^{0,s_1-1}(\bar{\Omega})},$$

such that

$$\|\omega(t)\|_{C^{0,\delta}(\bar{\Omega})}\leqslant C_3.$$

Taking note of inequality (4.19), in what follows we always denote by C_3 a generic constant which depends only on the domain Ω, constants s, T, M_1, and

$$\max_{0\leqslant t\leqslant T}\|f(t)\|_{s_1+1}.$$

One sees that the constant k_0 in Lemma 13 depends only on C_3. By (2.8) and Schauder's estimate for elliptic equations[12], we get

$$\|\bar{u}(t)\|_{C^{1,\delta}(\bar{\Omega})}\leqslant C_3.$$

By Lemma 13 and decomposition (4.12),

$$\|u(t)\|_{C^{1,\delta}(\bar{\Omega})}\leqslant C_3 \qquad (4.$$

for $0\leqslant t\leqslant k_0$. Reduce k_0, if necessary, such that $C_3k_0\leqslant 1$, then by (4.17)(4.20)

$$|\eta(y,\ t;\ \tau)|\leqslant C_2. \qquad (4.$$

Then set $p=2/(2-s_1)$, by (4.18) (4.21)

$$\left(\int_\Omega|\nabla\omega|^p\,dx\right)^{1/p}\leqslant C_2\left(\int_\Omega|\nabla\omega_0(\xi(x,\ 0;\ t))|^p\,dx\right)^{1/p}$$
$$+C_2\int_0^t\left(\int_\Omega|\nabla F(\xi(x,\ \zeta;\ t),\ \zeta)|^p\,dx\right)^{1/p}d\zeta. \qquad (4.$$

As a consequence of $\nabla\cdot u=0$, the map $x\to\xi(x,\ \zeta;\ t)$ is measure preserving, hence

$$\int_\Omega\nabla\omega_0(\xi(x,\ 0;\ t))|^p\,dx=\int_\Omega|\nabla\omega_0(y)|^p\,dy.$$

In virture of the imbedding theorem

$$\|\omega_0\|_{1,p}\leqslant C_2\|\omega_0\|_{s_1}.$$

Another term in (4.22) can be treated in the same way, hence

$$\|\nabla\omega\|_{0,p}\leqslant C_2\|\omega_0\|_{s_1}+C_2t.$$

On the analogy of this estiamate, using (4.16), we may estimate $\|\omega\|_{0,p}$, therefor

$$\|\omega\|_{1,p}\leqslant C_2\|\omega_0\|_{s_1}+C_2t. \qquad (4.$$

To obtain the estimate of $\|\omega\|_s$ we should estimate[9]

$$I=\left(\int_\Omega\int_\Omega\frac{|\nabla\omega(x,\ t)-\nabla\omega(x',\ t)|^2}{|x-x'|^{2s}}\,dx\,dx'\right)^{1/2}. \qquad (4$$

We substitute (4.18) into (4.24) and begin with considering the first term, th to estimate

$$I_1=\left(\int_\Omega\int_\Omega\frac{|\nabla\omega_0(\xi(x,\ 0;\ t))\eta(x,\ 0;\ t)-\nabla\omega_0(\xi(x',\ 0;\ t))\eta(x',\ 0;\ t)|^2}{|x-x'|^{2s}}\,dx\,dx'\right)^{1/}$$

Through some calculation, we can get

$$I_1\leqslant C_2\left\{\|\omega_0\|_s+\left(t\int_0^t\|\omega_0\|_{1,p}^2\|u\|_{1+\sigma,q}^2d\zeta\right)^{1/2}\right\}, \qquad (4.$$

where $p=2/(2-s_1)$, $q=2/(s_1-1)$, $\sigma=s-2/q$. In virtue of the imbedding theore

$$\|u\|_{1+\sigma,q}\leqslant C_2\|u\|_{2,p},\ \ \|\omega_0\|_{1,p}\leqslant C_2\|\omega_0\|_{s_1}. \qquad (4.$$

By Lemma 4 and the L^p norm estimate for the boundary value problem of elliptic equations[13],

$$\|\bar{u}\|_{2,p}\leqslant C_2\|\omega\|_{1,p}.$$

Lemma 13 and decomposition (4.12)

$$\|u\|_{2,p}\leqslant C_2\Big(\|\omega\|_{1,p}+\sum_{j=1}^{N}|\lambda_j(0)|+1\Big).\tag{4.27}$$

substitute (4.12) into initial condition (4.11), then get

$$\lambda_i(0)=(u_0,\ u^{(i)}),$$

ce

$$|\lambda_i(0)|\leqslant C\|u_0\|_0.\tag{4.28}$$

(4.23)(4.26)—(4.28)

$$\|u\|_{1+\sigma,q}\leqslant C_3.$$

uce k_0, if necessary, and let $t\leqslant k_0$, then by (4.25) we get

$$I_1\leqslant C_2\|\omega_0\|_{s_1}.$$

The estimate associated with the second term of (4.18) is all the same. We get

$$I\leqslant C_2\|\omega_0\|_{s_1}+C_2t.$$

ibine it with (4.23), then the H^s estimate

$$\|\omega\|_s\leqslant C_2\|\omega_0\|_{s_1}+C_2t$$

ows. By Lemma 4 and the estimate for the solutions of elliptic boundary value blems

$$\|\bar{u}\|_{s+1}\leqslant C_2\|\omega_0\|_{s_1}+C_2t.$$

ally, (4.15) follows from decomposition (4.12), Lemma 13 and (4.28).

Now we consider problem (4.5)(4.9) in general terms, where u is assumed to be arbitrary, suffiiciently smooth function and $u(\cdot,\ t)\in X$. Let $\psi\in H^2(\Omega)$ be the eam function corresponding to u_0 and $\psi|_{x\in\Gamma_0}=0$. We construct characteristic ves $\xi(x,\ t;\ \tau)$ like Lemma 14. Let $\Psi(y)=\Psi(\xi(y,\ 0;\ t))$, $\theta=-\Delta\Psi$, then we have

Lemma 15. *If $\omega(t)$ is the solution of (4.5)(4.9), and $u_0\in D(A)$, then*

$$\|\theta(t)-\omega(t)\|_0\leqslant C_4t\|\omega_0\|_0+\int_0^t\|F(\tau)\|_0d\tau,\tag{4.29}$$

re constant C_4 depends only on the domain Ω and function u.

Proof By (4.17) it can be directly verified that[5]

$$-\Delta_y\Psi(y)=\omega_0(\xi(y,\ 0;\ t))+R_1,\tag{4.30}$$

ere

$$\Delta_y=\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2},\ y=(y_1,\ y_2),$$

$$\|R_1\|_0\leqslant C_4t\|\psi\|_2.$$

the proof of Lemma 7

$$\|\psi\|_2\leqslant C_4\|\omega_0\|_0.$$

us

$$\|R_1\|_0\leqslant C_4t\|\omega_0\|_0.\tag{4.31}$$

From (4.16) and the property of measure preserving of mapping $x\to\xi(x,\ \zeta;\ t)$, we get

$$\|\omega(t)-\omega_0(\xi(\cdot,0;t))\|_0$$
$$=\left\|\int_0^t F(\xi(\cdot,\zeta;t),\zeta)d\zeta\right\|_0$$
$$\leqslant\int_0^t \|F(\xi(\cdot,\zeta;t),\zeta)\|_0 d\zeta=\int_0^t \|F(\zeta)\|_0 d\zeta.$$

By (4.30)

$$\theta(t)=\omega(t)+R_1+R_2,$$

where

$$\|R_2\|_0\leqslant\int_0^t \|F(\tau)\|_0 d\tau. \qquad (4$$

Then (4.29) follows from (4.31) (4.32).

§5. Some Estimates for the Viscosity Splitting Methc

In this section we give some estimates for the solutionos of schemo (1. (1.15). We always denote by u, ω the solution of problem (1.1)—(1.4), and b $\tilde{\omega}_k$ the vorticity corresponding to u_k, $\tilde{u}_k$.

Lemma 16. *If* $1<s<3/2$, $s_1=1+s/3$, $u_0\in D(A)$, *and there is a constant such that*

$$\|\tilde{u}_k(t)\|_1\leqslant M_0,\quad 0\leqslant t\leqslant T, \qquad ($$

and there are constants C_2, $k_0>0$, *such that*

$$\|\tilde{u}_k(t)\|_{s+2}\leqslant C_2(\|\tilde{u}_k(ik)\|_{s+1}+1),\ ik\leqslant t<(i+1)k,\ i=0,1,\cdots, \qquad ($$

as $0<k\leqslant k_0$, *then*

$$\max_{0\leqslant t\leqslant T}\|\tilde{u}_k(t)\|_{s+1}\leqslant M_2, \qquad ($$

as $0<k\leqslant k_0$, *where constant* M_2 *depends only on the domain* Ω, *constants* C_2, M_0, ! ν, *and functions* f, u_0.

Proof We denote by C_5 a generic costant depending only on the domai constants C_2, T, s, ν, and functions f, u_0. Set

$$f_1(\tau)=f(\tau)-(\tilde{u}\cdot\nabla)\tilde{u},$$

then by Lemma 11

$$\|u_k(jk-0)\|_{s_1+1}\leqslant C(\|u_0\|_{s_1+1}+\sup_{0\leqslant\tau<jk}\|f_1(\tau)\|_1).$$

The norm of the nonlinear term has an upper bound

$$\|(\tilde{u}_k\cdot\nabla)\tilde{u}_k\|_1\leqslant C(\|\tilde{u}_k\|^2_{1,4}+\|\tilde{u}_k\|_{0,\infty}\|\tilde{u}_k\|_2).$$

We take a constant q, $1<q<s$, then owing to the imbedding theorem

$$\|f_1(\tau)\|_1\leqslant\|f(\tau)\|_1+C(\|\tilde{u}_k\|^2_{3/2}+\|\tilde{u}_k\|_q\|\tilde{u}_k\|_2),$$

and by the interpolation inequality[8]

$$\|f_1(\tau)\|_1\leqslant\|f(\tau)\|_1+C(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s}+\|\tilde{u}_k\|_1^{1-(q-1)/s}\|\tilde{u}_k\|_{s+1}^{(q-1)/s}\|\tilde{u}_k\|_1^{1-1/s}\|\tilde{u}_k\|_{s+1}^{1/s}$$
$$=\|f(\tau)\|_1+C(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s}+\|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s}).$$

Hence

$$\|u_k(jk-0)\|_{s1+1}\leqslant C_5+C_5\sup_{0\leqslant\tau<jk}(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s}+\|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s}). \quad (5.4)$$

n assumption (5.2) of this lemma and initial condition (1.11) we obtain

$$\|\tilde{u}_k(t)\|_{s+1}\leqslant C_5+C_5\max_{0<t<T}(\|\tilde{u}_k\|_1^{2-1/s}\|\tilde{u}_k\|_{s+1}^{1/s}+\|\tilde{u}_k\|_1^{2-q/s}\|\tilde{u}_k\|_{s+1}^{q/s})+C_2.$$

ing maximum value of the right hand side and using assumption (5.1) of this na, we get

$$\max_{0\leqslant t<T}\|\tilde{u}_k\|_{s+1}\leqslant C_5+C_5(M_0^{2-1/s}\max_{0<t<T}\|\tilde{u}_k\|_{s+1}^{1/s}+M_0^{2-q/s}\max_{0<t<T}\|\tilde{u}_k\|_{s+1}^{q/s})+C_2.$$

n (5.3) follows.

If we replace $(\tilde{u}_k\cdot\nabla)\tilde{u}_k$ in equation (1.8) by $(u\cdot\nabla)u$, then it becomes a linear ntion

$$\frac{\partial\tilde{u}_k}{\partial t}+\frac{1}{\rho}\nabla\tilde{p}_k=f-(u\cdot\nabla)u. \quad (5.5)$$

solutions of problem (5.5), (1.9)—(1.15) are denoted by $\tilde{u}^*$, $\tilde{p}^*$, $\tilde{\omega}^*$, u^*, p^*, ω^*. Lemma 12, for any $0\leqslant s'<3/2$,

$$\max_{0<t<T}(\|u(t)-u^*(t)\|_{s'+1},\ \|u(t)-\tilde{u}^*(t)\|_{s'+1})\leqslant C_0k. \quad (5.6)$$

In the following from Lemma 17 to Lemma 23, we fix constant s, $1<s<3/2$, and me $\|\tilde{u}_k\|_{s+1}\leqslant M_3$, denote by C_6 a generic constant depending only on the domain onstants s, ν, T, M_3, functions f, u_0, and the solution u of (1.1)—(1.4).

Lemma 17. *As* $ik\leqslant t<(i+1)k$

$$\|\tilde{u}^*(t)-\tilde{u}_k(t)\|_1\leqslant C_6\max_{ik\leqslant\tau\leqslant t}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+C_6k. \quad (5.7)$$

Proof By Lemma 4, we have decomposition

$$\tilde{u}^*(t)-\tilde{u}_k(t)=\bar{u}_k(t)+\sum_{j=1}^N\lambda_j(t)u^{(j)}, \quad (5.8)$$

1

$$\|\bar{u}(t)\|_1\leqslant C\|\tilde{\omega}^*(t)-\tilde{\omega}_k(t)\|_0. \quad (5.9)$$

m equation (4.10) we have

$$\left(\frac{\partial\tilde{u}^*}{\partial t},\ u^{(j)}\right)+((u\cdot\nabla)u-f,\ u^{(j)})=0,$$

$$\left(\frac{\partial\tilde{u}_k}{\partial t},\ u^{(j)}\right)+((\tilde{u}_k\cdot\nabla)\tilde{u}_k-f,\ u^{(j)})=0.$$

subtraction,

$$\left(\frac{\partial(\tilde{u}^*-\tilde{u}_k)}{\partial t},\ u^{(j)}\right)+(((u-\tilde{u}_k)\cdot\nabla)u+(\tilde{u}_k\cdot\nabla)(u-\tilde{u}_k),\ u^{(j)})=0.$$

stitute (5.8) into it and get

$$\frac{d}{dt}\lambda_j(t)+\sum_{l=1}^N a_{jl}(t)\lambda_l(t)+g_j(t)=0,$$

where

$$a_{jl}(t)=((u^{(l)}\cdot\nabla)u+(\tilde{u}_k\cdot\nabla)u^{(l)},\ u^{(j)}),$$

$$g_j(t)=(((u-\tilde{u}^*+\bar{u})\cdot\nabla)u+(\tilde{u}_k\cdot\nabla)(u-\tilde{u}^*+\bar{u}),\ u^{(j)}).$$

Owing to the imbedding theorem

$$\|\tilde{u}_k\|_{1,\infty}\leqslant CM_3.$$

By (5.9) and (5.6) with $s'=0$, we get

$$|a_{jl}(t)|\leqslant C_6,$$

$$|g_j(t)|\leqslant C_6\|\tilde{\omega}^*(t)-\tilde{\omega}_k(t)\|_0+C_6k.$$

Set $t=ik$, by initial condition (1.11) and Lemma 7

$$\|\tilde{u}^*(ik)-\tilde{u}_k(ik)\|_1=\|u^*(ik-0)-u_k(ik-0)\|_1$$
$$\leqslant C\|\omega^*(ik-0)-\omega_k(ik-0)\|_0.$$

By (5.8)

$$|\lambda_j(ik)|\leqslant C\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0.$$

Applying Gronwall inequality we get

$$|\lambda_j(t)|\leqslant C_6\max_{ik\leqslant\tau\leqslant t}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+C_6k. \quad (5.$$

Then (5.7) follow from (5.8)—(5.10).

We constuct characteristic curves $\xi(x,\ t;\ \tau)$ like Lemma 14 where u is solution of (1.1)—(1.4). Let $\Psi(y)=(\psi^*-\psi_k)\ (\xi(y,\ ik;\ (i+1)k),\ (ik-0),U=(\Psi)^T$, $\theta=-\Delta\Psi$, then $\theta\in V$.

Lemma 18.

$$\|U-(\tilde{u}^*-\tilde{u}_k)(\xi(\cdot,\ ik;\ (i+1)k),\ ik)\|_0\leqslant C_6k\|(\tilde{u}^*-\tilde{u}_k)(ik)\|_0.$$

Proof By initial condition (1.11)

$$(\tilde{u}^*-\tilde{u}_k)(x,\ ik)=(\nabla\wedge(\psi^*-\psi_k)(x,\ ik-0))^T.$$

According to the definition of U

$$U(y)=\left(\nabla\wedge(\psi^*-\psi_k)\ (x,\ ik-0)\,|_{x=\xi(y,ik;(i+1)k)}\cdot\left(\frac{\partial\xi}{\partial y}\right)^*\right)^T,$$

where

$$\left(\frac{\partial\xi}{\partial y}\right)^*=\begin{pmatrix}\dfrac{\partial\xi_2}{\partial y_2} & -\dfrac{\partial\xi_2}{\partial y_1}\\ -\dfrac{\partial\xi_1}{\partial y_2} & \dfrac{\partial\xi_1}{\partial y_1}\end{pmatrix}.$$

We have

$$\left\|I-\left(\frac{\partial\xi}{\partial y}\right)^*\right\|=O(k),$$

since u is smooth enough. The mapping $y\to\xi(y,\ ik;\ (i+1)k)$ is measure preser so

$$\|U-(\tilde{u}^*-\tilde{u}_k)(\xi(\cdot,\ ik;\ (i+1)k);\ ik)\|_0^2$$
$$=\int_\Omega\left|\nabla\wedge(\psi^*-\psi_k)(\xi(y,\ ik;\ (i+1)k),\ ik-0)\left(\left(\frac{\partial\xi}{\partial y}\right)^*-I\right)\right|^2dy$$
$$=O(k^2)\int_\Omega|\nabla\wedge(\psi^*-\psi_k)(x,\ ik-0)|^2dx$$
$$=O(k^2)\int_\Omega|(\tilde{u}^*-\tilde{u}_k)(x,\ ik)|^2dx.$$

Lemma 19.

$$\|U-(\tilde{u}^*-\tilde{u}_k)(ik)\|_0\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+k).$$

Proof By the triangular inequality

$$\|U-(\tilde{u}^*-\tilde{u}_k)(ik)\|_0\leqslant J_1+J_2,$$

ere

$$J_1=\|U-(\tilde{u}^*-\tilde{u}_k)(\xi(\cdot,\ ik;\ (i+1)k),\ ik)\|_0,$$

$$J_2=\|(\tilde{u}^*-\tilde{u}_k)(\xi(\cdot,\ ik;\ (i+1)k),\ ik)-(\tilde{u}^*-\tilde{u}_k)(ik)\|_0.$$

egrating along characteristic curves we get

$$\begin{aligned}&(\tilde{u}^*-\tilde{u}_k)(\xi(y,\ ik;\ (i+1)k),\ ik)-(\tilde{u}^*-\tilde{u}_k)(y,\ ik)\\&=-\int_{ik}^{(i+1)k}\frac{d}{d\tau}(\tilde{u}^*-\tilde{u}_k)(\xi(y,\ \tau;\ (i+1)k),\ ik)d\tau\\&=-\int_{ik}^{(i+1)k}\frac{\partial(\tilde{u}^*-\tilde{u}_k)}{\partial x}(\xi(y,\ \tau;(i+1)k),\ ik)\frac{\partial\xi}{\partial\tau}\,d\tau\\&=-\int_{ik}^{(i+1)k}\frac{\partial(\tilde{u}^*-\tilde{u}_k)}{\partial x}(\xi(y,\ \tau;\ (i+1)k),\ ik)u(\xi(y,\ \tau;\ (i+1)k),\ \tau)d\tau.\end{aligned}$$

ce u is bounded and the mapping is measure preserving, we get

$$\begin{aligned}J_2&\leqslant C_6\int_{ik}^{(i+1)k}\left\|\frac{\partial(\tilde{u}^*-\tilde{u}_k)}{\partial x}(\xi(\cdot,\ \tau;\ (i+1)k),\ ik)\right\|_0d\tau\\&\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{u}^*-\tilde{u}_k\|_1.\end{aligned}$$

Lemma 18

$$J_1+J_2\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{u}^*-\tilde{u}_k\|_1.$$

is the conclusion follows from Lemma 17.

Lemma 20.

$$\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_0\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)|_0+k).$$

Proof We apply the Helmholz projection operaor P' to equations (1.8) and 5) and get

$$\frac{\partial\tilde{u}_k}{\partial t}=P'(f-(\tilde{u}_k\cdot\nabla)\tilde{u}_k),$$

$$\frac{\partial\tilde{u}^*}{\partial t}=P'(f-(\ \ \cdot\nabla)u).$$

refore

$$(\tilde{u}^*-\tilde{u}_k)(t)=(\tilde{u}^*-\tilde{u}_k)(ik)-\int_{ik}^{t}P'((u\cdot\nabla)u-(\tilde{u}_k\cdot\nabla)\tilde{u}_k)d\tau.$$

the triangular inqeuality

$$\begin{aligned}&\|(u\cdot\nabla)u-(\tilde{u}_k\cdot\nabla)\tilde{u}_k\|_0\\&\leqslant\|(u\cdot\nabla)(u-\tilde{u}_k)\|_0+\|((u-\tilde{u}_k)\cdot\nabla)\tilde{u}_k\|_0\leqslant C_6\|u-\tilde{u}_k\|_1.\end{aligned}$$

inequalities (5.6) with $s'=0$ and (5.7)

$$\begin{aligned}&\|(u\cdot\nabla)u-(\tilde{u}_k\cdot\nabla)\tilde{u}_k\|_0\\&\leqslant C_6\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+C_6k.\end{aligned}$$

P' is an orthogonal projection operator, hence

$$\|(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)-(\tilde{u}^*-\tilde{u}_k)(ik)\|_0$$
$$\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+k).$$

The desired inequality follows from Lemma 19.

Lemma 21.

$$\|(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1$$
$$\leqslant C(\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_0$$
$$+\|\theta-(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0. \qquad (5$$

Proof By Lemma 4, we have decomposition

$$U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)=\bar{u}+\sum_{j=1}^{N}\lambda_j u^{(j)}, \qquad (5$$

then

$$\|\bar{u}\|_1\leqslant C\|\theta-(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0. \qquad (5$$

Since (5.12) is an orthogonal projection, we have

$$|\lambda_j|\leqslant\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_0,\quad j=1,\cdots,N. \qquad (5$$

Using (5.12) angain, we get

$$\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1\leqslant\|\bar{u}\|_1+C\sum_{j=1}^{N}|\lambda_j|.$$

By (5.14)

$$\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1$$
$$\leqslant\|\bar{u}\|_1+\|U-(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_0. \qquad (5$$

Since P is an orthogonal projection operator, we have

$$\|(I-P)(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0$$
$$\leqslant\|\theta-(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0,$$

thus

$$\|\theta-P(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0$$
$$\leqslant 2\|\theta-(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0. \qquad (5$$

By Lemma 7

$$\|\Theta(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)-U\|_1$$
$$\leqslant C\|P(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)-\theta\|_0. \qquad (5$$

Then (5.11) follows from (5.13)(5.15)—(5.17).

Lemma 22.

$$\|(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1$$
$$\leqslant C_6k\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)\tilde{\omega}_k(\tau)\|_0+k).$$

Proof The following equations are similar to (4.5):

$$\frac{\partial\tilde{\omega}^*}{\partial t}+u\cdot\nabla\omega=F,$$
$$\frac{\partial\tilde{\omega}_k}{\partial t}+\tilde{u}_k\cdot\nabla\tilde{\omega}_k=F,$$

so

$$\frac{\partial\tilde{\omega}^*-\tilde{\omega}_k)}{\partial t}+u\cdot\nabla(\tilde{\omega}^*-\tilde{\omega}_k)=u\cdot\nabla(\tilde{\omega}^*-\omega)-(u-\tilde{u}_k)\cdot\nabla\tilde{\omega}_k. \qquad (5.18)$$

r Lemma 15

$$\|\theta-(\widetilde{\omega}^*-\widetilde{\omega}_k)((i+1)k-0)\|_0$$
$$\leqslant C_6 k\|\widetilde{\omega}^*(ik)-\widetilde{\omega}_k(ik)\|_0$$
$$+\int_{ik}^{(i+1)k}\|u\cdot\nabla(\widetilde{\omega}^*-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0 d\tau. \quad (5.19)$$

e estimate the integrand. As $ik\leqslant t<(i+1)k$, by ineqnality (5.6) with $s'=1$, we tain

$$\|u\cdot\nabla(\widetilde{\omega}^*-\omega)\|_0\leqslant C_6\|\widetilde{u}^*-u\|_2\leqslant C_6 k.$$

Hölder inequality

$$\|(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0^2$$
$$=\int_\Omega|(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k|^2 dx$$
$$\leqslant\Big(\int_\Omega|\nabla\widetilde{\omega}_k|^p dx\Big)^{2/p}\Big(\int_\Omega|u-\widetilde{u}_k|^q dx\Big)^{2/q}$$
$$\leqslant\|\widetilde{\omega}_k\|_{1,p}^2\|u-\widetilde{u}_k\|_{0,q}^2,$$

ere $p=2/(2-s)$, $q=2/(s-1)$. By the imbedding theorem

$$\|\widetilde{\omega}_k\|_{1,p}\leqslant C\|\widetilde{\omega}_k\|_s,$$
$$\|u-\widetilde{u}_k\|_{0,q}\leqslant C\|u-\widetilde{u}_k\|_1\leqslant C(\|u-u^*\|_1+\|\widetilde{u}^*-\widetilde{u}_k\|_1).$$

inequalities (5.6) with $s'=0$ and (5.7)

$$\|u(t)-\widetilde{u}_k(t)\|_{0,q}\leqslant C_6(\max_{ik\leqslant\tau\leqslant t}\|\widetilde{\omega}^*(\tau)-\widetilde{\omega}_k(\tau)\|_0+k).$$

nce

$$\|u\cdot\nabla(\widetilde{\omega}_k-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0$$
$$\leqslant C_6\max_{ki\leqslant\tau\leqslant t}\|\widetilde{\omega}^*(\tau)-\widetilde{\omega}_k(\tau)\|_0+k). \quad (5.20)$$

(5.19)

$$\|\theta-(\widetilde{\omega}^*-\widetilde{\omega}_k)((i+1)k-0)\|_0$$
$$\leqslant C_6 k\sup_{ik\leqslant\tau<(i+1)k}\|\widetilde{\omega}^*(\tau)-\widetilde{\omega}_k(\tau)\|_0+k).$$

en the desired inequality follows from Lemma 20 and Lemma 21.

Lemma 23. *If* $u_0\in(H_0^1(\Omega))^2$, *and if* $\|\widetilde{u}_k\|_{s+1}\leqslant M_3$ *as stated above, then*

$$\max(\|u(t)-u_k(t)\|_1,\ \|u(t)-\widetilde{u}_k(t)\|_1)\leqslant C_6 k,\quad 0\leqslant t\leqslant T.$$

Proof Multiplying equation (5.18) with $\widetilde{\omega}^*-\widetilde{\omega}_k$, integrating on domain Ω, l taking note of

$$(u\cdot\nabla(\widetilde{\omega}^*-\widetilde{\omega}_k),\ \widetilde{\omega}^*-\widetilde{\omega}_k)=0,$$

obtain

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0^2=(u\cdot\nabla(\widetilde{\omega}^*-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k,\ \widetilde{\omega}^*-\widetilde{\omega}_k)$$
$$\leqslant\|u\cdot\nabla(\widetilde{\omega}^*-\omega)-(u-\widetilde{u}_k)\cdot\nabla\widetilde{\omega}_k\|_0\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0.$$

(5.20)

$$\frac{1}{2}\frac{d}{dt}\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0^2\leqslant C_6\max_{ik\leqslant\tau\leqslant t}\|\widetilde{\omega}^*(\tau)-\widetilde{\omega}_k(\tau)\|_0+k)\|\widetilde{\omega}^*-\widetilde{\omega}_k\|_0.$$

Therefere either $\|\widetilde{\omega}^*-\widetilde{\omega}\|_0=0$ or

$$\frac{d}{dt}\|\widetilde{\omega}^*-\widetilde{u}_k\|_0\leqslant C_6(\max_{ik\leqslant\tau\leqslant t}\|\widetilde{\omega}^*(\tau)-\widetilde{\omega}_k(\tau)\|_0+k).$$

Thanks to the Gronwall inequality, we obtain

$$\|\tilde{\omega}^*(t)-\tilde{\omega}_k(t)\|_0\leqslant e^{C_6 k}(\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0+C_6k^2),\quad ik\leqslant t<(i+1)k. \tag{5.21}$$

By Lemma 22

$$\|(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1 \leqslant C_6k(\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0+k). \tag{5.22}$$

From (1.12)—(1.15), u^*-u_k is the solution of

$$\frac{\partial(u^*-u_k)}{\partial t}+\frac{1}{\rho}\nabla(p^*-p_k) =\nu\Delta(u^*-u_k)+\frac{1}{k}(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0),$$

$$\nabla\cdot(u^*-u_k)=0,$$

$$(u^*-u_k)|_{x\in\partial\Omega}=0,$$

$$(u^*-u_k)(ik)=\Theta(\tilde{u}^*-\tilde{u}_k)((i+1)k-0).$$

By Lemma 8

$$\frac{d}{dt}\|\omega^*-\omega_k\|_0^2\leqslant\frac{1}{2\nu}\left\|\frac{1}{k}(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\right\|_0^2.$$

Substituting (5.22) into it, we get

$$\frac{d}{dt}\|\omega^*-\omega_k\|_0^2\leqslant C_6(\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0^2+k^2).$$

Integrating on interval (ik, t), we get

$$\|\omega^*(t)-\omega_k(t)\|_0^2 \leqslant\|\omega^*(ik)-\omega_k(ik)\|_0^2 +C_6k(\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0^2+k^2),\quad ik\leqslant t<(i+1)k. \tag{5.23}$$

We may assume that $C_6\geqslant 1$ in inequality (5.21). If

$$\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0\leqslant C_6k,$$

then

$$\|\tilde{\omega}^*(t)-\tilde{\omega}_k(t)\|_0\leqslant C_6k.$$

By initial condition (1.15)

$$\|\omega^*(ik)-\omega_k(ik)\|_0=\|P(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0 \leqslant\|(\tilde{\omega}^*-\tilde{\omega}_k)((i+1)k-0)\|_0\leqslant C_6k.$$

Substituting them into (5.23), we obtain

$$\|(\omega^*-\omega_k)((i+1)k-0)\|_0\leqslant C_6k. \tag{5.24}$$

Similarly if

$$\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0>C_6k$$

in inequality (5.21), then

$$\|(\omega^*-\omega_3)((i+1)k-0)\|_0^2\leqslant(1+C_6k)\|\tilde{\omega}^*(ik)-\tilde{\omega}_k(ik)\|_0^2.$$

By initial condition (1.11)

$$\|(\omega^*-\omega_k)((i+1)k-0)\|_0^2\leqslant(1+C_6k)\|(\omega^*-\omega_k)(ik-0)\|_0^2. \tag{5.25}$$

From (5.24) (5.25), we always have

$$\|(\omega^*-\omega_k)((i+1)k-0)\|_0^2$$
$$\leqslant\max\{(1+C_6k)\|(\omega^*-\omega_k)(ik-0)\|_0^2,\ C_6k^2\}.$$

king note of $\omega^*(-0)=\omega_k(-0)$, we can prove by induction that

$$\|(\omega^*-\omega_k)(ik-0)\|_0\leqslant C_6e^{c_6T}k.$$

initial condition (1.11)

$$\|(\tilde{\omega}^*-\tilde{\omega}_k)(ik)\|_0\leqslant C_6k.$$

(5.21) and (5.23)

$$\|(\tilde{\omega}^*-\tilde{\omega}_k)(t)\|_0\leqslant C_6k,\quad \|(\omega^*-\omega_k)(t)\|_0\leqslant C_6k.$$

Lemma 17

$$\|\tilde{u}^*(t)-\tilde{u}_k(t)\|_1\leqslant C_6k.$$

cause $u^*(t)-u_k(t)\in(H_0^1(\Omega))^2$, by Lemma 7

$$\|u^*(t)-u_k(t)\|_1\leqslant C_6k.$$

ing (5.6) again, we get the desired desult.

Lemma 24. *If* $\|\tilde{u}_k(t)\|_{s+1}\leqslant M_2$ *as* $ik\leqslant t<(i+1)k$ *for a certain* $i\geqslant 0$ *and* $0\leqslant s<2$, *then* $\|u_k(t)\|_{s+1}\leqslant M_4$ *on the same interval, where constant* M_4 *depends only on the nain* Ω, *constants* ν, s, T, M_2. *functions* f, u_0. *and the solution* u *of problem* (1.1)—.4).

Proof By (1.12)—(1.15) and Lemma 9

$$\|u_k(t)\|_{s+1}\leqslant C\Big(M_2+\frac{1}{k}\|(I-\Theta)\tilde{u}_k((i+1)k-0)\|_1\Big). \tag{5.26}$$

ice $(I-\Theta)u\equiv 0$,

$$\|(I-\Theta)\tilde{u}_k((i+1)k-0)\|_1$$
$$\leqslant\|(I-\Theta)(\tilde{u}^*-\tilde{u}_k)((i+1)k-0)\|_1+\|(I-\Theta)(u-\tilde{u}^*)((i+1)k-0)\|_1$$

(5.6), Lemma 3 and Lemma 22

$$\frac{1}{k}\|(I-\Theta)\tilde{u}_k((i+1)k-0)\|_1$$
$$\leqslant C_7\sup_{ik\leqslant\tau<(i+1)k}\|\tilde{\omega}^*(\tau)-\tilde{\omega}_k(\tau)\|_0+k+1),$$

re constant C_7 depends only on the domain Ω, constants ν, s, T, M_2, functions f, and the solution u of problem (1.1)—(1.4). we have

$$\|\tilde{\omega}_k(\tau)\|_0\leqslant\|\tilde{u}_k(\tau)\|_{s+1}\leqslant M_2.$$

l the upper bound of $\|\tilde{\omega}^*(\tau)\|_0$ can be obtained from (5.6) and the upper bound $\|u(t)\|_1$. Thus (5.26) gives the desired estimate.

§6. Proof of the Theorem

Let $s_0=s$, $s_l=s_{l-1}/3+1$, $l=1, 2, \cdots$, we denote

$$H=\bigcap_{l=0}^{\infty}(H^{s_l+1}(\Omega))^2.$$

It is the assumption of this theorem that u is sufficiently smooth, so we may assume $u_0 \in H$. By Lemma 14 and Lemma 24. $u_k(t) \in H$ and $\tilde{u}_k(t) \in H$.

Set

$$m = \max_{0<t<T} \|u(t)\|_1.$$

Let $M_0 = 2m$. We determine constant C_2 according to Lemma 14, then determine constant M_2 according to Lemma 16, then determine constant C_5 according to (5.4). and let

$$M_1 = C_5 + C_5(M_0^{2-1/s} M_2^{1/s} + M_0^{2-q/s} M_2^{q/s}). \quad (6.$$

We determine constant k_0 according to Lemma 14, and let

$$M_3 = \max(C_2 M_1 + C_2,\ M_2). \quad (6.$$

Then we determine constant C_6 according to Lemma 23, reduce constant k_0, necessary, such that $C_6 k_0 \leqslant m$.

We claim that with the determined constants, if $k \leqslant k_0$. then

$$\|\tilde{u}_k(t)\|_1 \leqslant M_0,\ \|u_k(t)\|_1 \leqslant M_0,\ \|\tilde{u}_k(t)\|_{s+1} \leqslant M_2, \text{ and}$$

$$\|u(t) - u_k(t)\|_1 \leqslant C_6 k,\ \|u(t) - \tilde{u}_k(t)\|_1 \leqslant C_6 k. \quad (6$$

It is proved by induction. Two cases are considered simultaneously: (i) $j =$ (ii) $j > 0$ and the above assertion is valid for $0 \leqslant t < jk$. If $j > 0$, then by (6.1)(5.

$$\|u_k(jk - 0)\|_{s+1} \leqslant M_1. \quad (6$$

(6.4) also holds for $j = 0$ evidently. By Lemma 14 and (6.2), $\|\tilde{u}_k(t)\|_{s+1} \leqslant M_3$ $jk \leqslant t < (j+1)k$. By Lemma 23, (6.3) holds for all $0 \leqslant t < (j+1)k$, in virtue of way by which we take k_0, $\|\tilde{u}_k(t)\|_1 \leqslant M_0$ and $\|u_k(t)\|_1 \leqslant M_0$ on the same interval. Lemma 14 and Lemma 16, $\|\tilde{u}_k(t)\|_{s+1} \leqslant M_2$ on this interval. Thus the inductior complete.

Using Lemma 24 we obtain the upper bund of $\|u_k(t)\|_{s+1}$ Therefore, inequali (1.16)(1.17) are proved as $k \leqslant k_0$.

To prove the theorem, we should consider the case $k > k_0$. But there are at n T/K_0 steps. By Lemma 14 and Lemma 24 we can get the upper bound of $\|u_k\|_{s+1}$ $\|\tilde{u}_k\|_{s+1}$ step by step. And (1.16), (1.17) always hold if we take M, M' large enou

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