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MULTIPLE SOLUTIONS OF NONLINEAR FREDHOLM INTEGRAL EQUATIONS IN BANACH SPACES**

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Abstract

In this paper, the author uses the topological degree theory to investigate the multiple solutions of nonlinear Fredholm integral equations in Banach spaces. Two new theorems are obtained and two examples are given.

In [1], Vaughn established some existence theorems of the solutions an maximal solutions for nonlinear Volterra integral equations in a Banach spice b means of the monotne technique. Now, in this paper, we shall use the topologic degree theory to discuss the multiple solutions of nonlinear Fredholm integr equations in a Banach space, i.e. we consider the following integral eqoation

$$x(t)=\int_I H(t,\ s,\ x(s))ds, \qquad (1$$

where $I=[a,\ b]$ and $H\in C[I\times I\times E,\ E]$, E is a real Banach space (i. e. H is continuous mapping from $I\times I\times E$ into E).

Let P be a cone in E (see [4]), and consequently, P defines a partial ordering E. Let $P_I=\{x\in C[I,\ E]\,|\,x(t)\geqslant\theta$ for all $t\in I\}$, where $C[I,\ E]$ denotes the Bana space of all continuous mappings $x: I\to E$ with norm

$$\|x\|=\max_{t\in I}\|x(t)\|$$

and θ is the zero element of E. It is clear that P_I is a cone of space $C[I,\ C]$, and it definesa partial ordering in $C[I,\ E]$. Obviously, the normality of P implies t normality of P_I.

Lemma 1. *If P is solid (i. e. the interior $\mathring{P}\neq\phi$), then P_I is also solid, and $\mathring{P}_I$ $\{x\in C[I,\ E]\,|\,x(t)\gg\theta$ for all $t\in I\}$.*

Proof Let $Q=\{x\in C[I,\ E]\,|\,x(t)\gg\theta$ for all $t\in I\}$ and we need to prove $\mathring{P}_I=$ If $x_0\in\mathring{P}_I$, then there exifts an $r>0$ such that

$$x\in C[I,\ E],\ \|x-x_0\|_C<r\Rightarrow x\in P_I,\ \text{i. e. } x(t)\geqslant\theta \text{ for all } t\in I. \qquad ($$

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or any $s\in I$ and $z\in E$ with $\|z-x_0(s)\|<r$, setting $x(t)=x_0(t)-x_0(s)+z$ in(2), we et $x(t)=x_0(t)-x_0(s)+z\geqslant\theta$ for all $t\in I$ and, in particular, $z=x(s)\geqslant\theta$, hence $x_0(s)\in\mathring{P}$. Since s is arbitrary in I, we obtain $x_0\in Q$ and consequently, $\mathring{P}_I\subset Q$.

Conversely, let $y_0\in Q$. We choose a fixed $u_0\in\mathring{P}$. Then for any $t'\in I$ there exists a $\varepsilon'=\varepsilon'(t')>0$ such that

$$y_0(t')\geqslant 2\varepsilon' u_0. \tag{3}$$

nce $y_0(t)$ is continuous in I, we can find an open interval

$$J(t',\ \delta')=(t-\delta',\ t'+\delta')$$

ich that

$$\varepsilon' u_0+[y_0(t)-y_0(t')]\geqslant\theta \text{ for } t\in J(t',\ \delta'). \tag{4}$$

follows from (3) and (4) that

$$y_0(t)\geqslant\varepsilon' u_0 \quad \text{for} \quad t\in J(t',\ \delta').$$

ow, using the Heine-Borel finite convering theorem, we see that there exists a nite collection $\{J(t_i,\ \delta_i)\}\ (i=1,\ 2,\ \cdots,\ m)$, which covers I, and

$$y_0(t)\geqslant\varepsilon_i u_0 \quad \text{for} \quad t\in J(t_i,\ \delta_i),\ i=1,\ 2,\ \cdots,\ m,$$

here $\varepsilon_i>0\ (i=1,\ 2,\ \cdots,\ m)$ are constants. Consequently,

$$y_0(t)\geqslant\varepsilon_0 u_0 \quad \text{for} \quad t\in I, \tag{5}$$

here $\varepsilon_0=\min\{\varepsilon_1,\ \varepsilon_2,\ \cdots,\ \varepsilon_m\}>0$. Since $u_0\in\mathring{P}$, there exists $\eta>0$ such that

$$(\varepsilon_0/2)u_0+y(t)-y_0(t)\geqslant\theta \quad \text{for} \quad t\in I \tag{6}$$

henever $y\in C[I,\ E]$ with

$$\|y-y_0\|_C=\max_{t\in I}\|y(t)-y_0(t)\|<\eta.$$

onsequently, from (5) and(6) we get

$$y(t)\geqslant(\varepsilon_0/2)u_0\gg\theta \text{ for all } t\in I' \text{ and so } y\in P_I$$

henever $\|y-y_0\|_C<\eta$; this implies $y_0\in\mathring{P}_I$, and hence $Q\subset\mathring{P}_I$. Therefore $\mathring{P}_I=Q$ is roved.

Finally, letting $z(t)\equiv u_0$ for $t\in I$, wefind $z\in Q=\mathring{P}_I$, hence $\mathring{P}_I\neq\phi$and so P_I is solid.

In the following, we define operator A by

$$Ax(t)=\int_I H(t,\ s,\ x(s))ds. \tag{7}$$

Lemma 2. *Let $H\in C[I\times I\times E,\ E]$ be uniformly continuous and bounded on $I\times\times\overline{B}_R$ for any $R>0$, where $B_R=\{x\in E\,|\,\|x\|<R\}$. Suppose that there exists an $L>0$ ith $L(b-a)<1/2$ such that*

$$\alpha(H(t,\ s,\ B))\leqslant L\alpha(B) \text{ for } t,\ s\in I \text{ and bounded } B\subset E, \tag{8}$$

here α denotes the Kuratowski's measure of noncompactness.

Then $A: C[I,\ E]\to C[I,\ E]$ is a strict set contraction, i. e. there exists a constant $\leqslant k<1$ such that $\alpha(A(S))\leqslant k\alpha(S)$ for any bounded $S\subset C[I,\ E]$.

Proof By the uniform continuity of H and (8) (see[3], p. 23), we have

$$\alpha(H(I\times I\times B))=\max_{t,s\in I}\alpha(H(t,\ s,\ B))\leqslant L\alpha(B) \text{ for bounded } B\subset E. \tag{9}$$

Since H is uniformly continuous and bounded on $I\times I\times \overline{B}_R$ for any $R>0$, A is continuous and bounded from $C[I, E]$ into $C[I, E]$. Now, let $S\subset C[I, E]$ be bounded. Then we can find an $R>0$ such that $S\subset \overline{T}_R=\{x\in C[I, E], \|x\|_C\leqslant R\}$. By the uniform continuity and boundness of H on $I\times I\times \overline{B}_R$, it is easy to see that the functions $\{Ax\,|\,x\in S\}$ are uniformly bounded and equicontinuous, hence (see [2], Lemma 1·4·1)

$$\alpha(A(S))=\sup_{t\in I}\alpha(A(S(t))), \tag{10}$$

where

$$A(S(t))=\{Ax(t)\,|\,x\in S,\ t \text{ is fixed}\}.$$

Using the formula

$$(b-a)^{-1}\cdot\int_I x(t)\,dt\in\overline{co}\{x(t)\,|\,t\in I\}\quad \text{for } x\in C[I, E],$$

and observing (9), we get

$$\begin{aligned}\alpha(A(S(t)))&=\alpha\left(\left\{\int_I H(t, s, x(s))\,ds\,|\,x\in S\right\}\right)\\ &\leqslant(b-a)\alpha(co\{H(t, s, x(s))\,|\,t, s\in I, x\in S\})\\ &=(b-a)\alpha(\{H(t, s, x(s))\,|\,t, s\in I, x\in S\})\\ &\leqslant(b-a)\alpha(H(I\times I\times B))\leqslant(b-a)L\alpha(B),\end{aligned} \tag{11}$$

where $B=\{x(s)\,|\,s\in I,\ x\in S\}\subset\overline{B}_R$. For any given $\varepsilon>0$, there exists a partition $S=\bigcup_{j=1}^{n}S_j$ such that

$$\text{diam }(S_j)<\alpha(S)+\varepsilon,\ j=1, 2, \cdots, n. \tag{12}$$

Choosing $x_j\in S_j (j=1, 2, \cdots, n)$ and a partition

$$a=t_0<t_1<\cdots<t_{i-1}<t_i<\cdots<t_m=b$$

such that

$$\|x_j(t)-x_j(s)\|<\varepsilon \text{ for } j=1, 2, \cdots, n;\ t, s\in I_i=[t_{i-1}, t_i],\quad i=1, 2, \cdots, m. \tag{13}$$

Obviously, $B=\bigcup_{i=1}^{m}\bigcup_{j=1}^{n}B_{ij}$, where $B_{ij}=\{x(s)\,|\,s\in I_i,\ x\in S_j\}$. For any two $u, v\in B_{ij}$, we have $u=x(t)$, $v=y(s)$ for some $t, s\in I_i$ and $x, y\in S_j$. It follows from (12) and (13) that

$$\begin{aligned}\|u-v\|&\leqslant\|x(t)-x_j(t)\|+\|x_j(t)-x_j(s)\|+\|x_j(s)-y(s)\|\\ &\leqslant\|x-x_j\|_C+\varepsilon+\|x_j-y\|_C\\ &\leqslant 2\,\text{diam}(S_j)+\varepsilon\leqslant 2\alpha(S)+3\varepsilon.\end{aligned}$$

Consequently,

$$\text{diam}(B_{ij})\leqslant 2\alpha(S)+3\varepsilon,\ i=1, 2, \cdots, m;\ j=1, 2, \cdots, n,$$

and so

$$\alpha(B)\leqslant 2\alpha(S)+3\varepsilon.$$

Since ε is arbitrary, we find

$$\alpha(B)\leqslant 2\alpha(S). \tag{14}$$

It then follows from (10), (11) and (14) that $\alpha(A(S))\leqslant k\alpha(S)$ with $k=2(b-a)L<1$, ''is shows that A is a strict set contraction.

Theorem 1. *Let P be a normal solid cone in the real Banach space E. Suppose at:*

(a) *$H\in C[I\times I\times E, E]$ is uniformly continuous and bounded on $I\times I\times \overline{B}_R$ for any >0 and there exists an $L>0$ with $L(b-a)<1/2$ such that*

$$\alpha(H(t, s, B))\leqslant L\alpha(B) \text{ for } t, s\in I \text{ and bounded } B\subset E. \tag{15}$$

(b) $\|H(t, s, x)\|/\|x\|\to 0$ *as* $\|x\|\to 0$ *uniformly in* $t, s\in I$. (16)

(c) $\|H(t, s, x)\|/\|x\|\to 0$ *as* $\|x\|\to\infty$ *uniformly in* $t, s\in I$. (17)

(d) *there exist $x_0\in \mathring{P}$ and $k\in C[I\times I, R^1]$ such that*

$$H(t, s, x)\geqslant k(t, s)x_0 \text{ for } x\geqslant x_0 \tag{18}$$

nd

$$\int_I k(t, s)ds>1 \text{ for } t\in I. \tag{19}$$

Then equation (1) has at least three different solutions $x_i(t)$ $(i=1, 2, 3)$ in $C[I,]$ such that $x_1(t)\equiv\theta$ $(t\in I)$, $x_2(t)\gg x_0$ for all $t\in I$ and $x_3(t)\not\equiv\theta$ with $x_3(t)<x_0$ for me $t\in I$.

Proof First, by Lemma 2, $A:C[I, E]\to C[I, E]$ is a strict set contraction. It clear from (16) and the continuity of H that $H(t, s, \theta)\equiv\theta$ $(t, s\in I)$, and so $(t)\equiv\theta$ is the trivial solution of equation (1).

Now, from (16) and (17) we can find two numbers r and R_0 such that

$$0<r<\|x_0\|/N<R_0 \tag{20}$$

ıd

$$\|H(t, s, x)\|\leqslant[2(b-a)]^{-1}\|x\|, \quad t, s\in I, \ \|x\|\leqslant r \text{ or } \|x\|\geqslant R_0, \tag{21}$$

here N denotes the normal constant of P, i. e. $\theta\leqslant x\leqslant y$ implies $\|x\|\leqslant N\|y\|$. ›nsequently,

$$\|H(t, s, x)\|\leqslant[2(b-a)]^{-1}\|x\|+M, \ t, s\in I, \ x\in E, \tag{22}$$

here

$$M=\sup\{\|H(t, s, x)\| \mid t, s\in I, x\in \overline{B}_{R_0}\}.$$

ıoose

$$R>\max\{2M(b-a), R_0\} \tag{23}$$

ıd set $\Omega_1=\{x\in C[I, E] \mid \|x\|_C<r\}$, $\Omega_2=\{x\in C[I, E] \mid \|x\|_C<R\}$, $\Omega_3=\{x\in C[I] \mid \|x\|<R$ and $x(t)\gg x_0$ for all $t\in I\}$. Obviously, Ω_1 and Ω_2 are open sets of space $[I, E]$. By Lemma 1, $\Omega_3=\{x\in C[I, E] \mid \|x\|<R$ and $x-x_0\in\mathring{P}_I\}$, andso Ω_3 is also ı open set of space $C[I, E]$. It is clear that $\overline{\Omega}_1\in\{x\in CI[, E] \mid \|x\|\leqslant r\}$, $\overline{\Omega}_2=\{x\in [I, E] \mid \|x\|\leqslant R\}$ and $\overline{\Omega}_3\subset\{x\in C[I, E] \mid \|x\|\leqslant R, x(t)\geqslant x_0$ for all $t\in I\}$. Moreover, om (20) we find

$$\Omega_1\subset\Omega_2, \ \Omega_3\subset\Omega_2, \ \Omega_2\cap\Omega_3=\varnothing. \tag{24}$$

Now, (21), (22) and (23)i mply that

$$x \in \bar{\Omega}_1 \Rightarrow \|Ax\|_C \leqslant \max_{t\in I} \int_I [2(b-a)]^{-1} \|x(s)\| ds \leqslant \|x\|_C/2 < r$$

and

$$x \in \bar{\Omega}_2 \Rightarrow \|Ax\|_C \leqslant \|x\|_C/2 + M(b-a) \leqslant R/2 + M(b-a) < R,$$

hence

$$A(\bar{\Omega}_1) \subset \Omega_1, \ A(\bar{\Omega}_2) \subset \Omega_2. \tag{25}$$

For $x \in \Omega_3$, we have $\|x\|_C \leqslant R$ and $x(t) \geqslant x_0$ for all $t \in I$, and so $\|Ax\|_C < R$, and, by (18) and (19),

$$Ax(t) \geqslant \int_I k(t, s) x_0 ds \geqslant \gamma x_0,$$

where

$$\gamma = \min_{t\in I} \int_I k(t, s) ds > 1,$$

which implies $Ax(t) \gg x_0$ for all $t \in I$, and therefore

$$A(\bar{\Omega}_3) \subset \Omega_3. \qquad ($$

It follows from (25) and (26) that the topological degree of the strict set contract fields (see [4])

$$\deg(\mathrm{id} - A, \Omega_i, \theta) = 1 (i = 1, 2, 3), \qquad ($$

where id denotes the identical operator. Consequently, A has a fixed point x_2 in which satisfies $x_2(t) \gg x_0$ for all $t \in I$. On the other hand, (27) implies

$$\deg(\mathrm{id} - A, \Omega_2/(\bar{\Omega}_1 \cup \bar{\Omega}_3), \theta) = \deg(\mathrm{id} - A, \Omega_2, \theta)$$
$$- \deg(\mathrm{id} - A, \Omega_1, \theta) - \deg(\mathrm{id} - A, \Omega_3, \theta) = -1 \neq 0,$$

and so A has a fixed point x_3 in $\Omega_2/(\bar{\Omega}_1 \cup \bar{\Omega}_3)$, and our theorem is proved.

Remark 1. If E is finite dimensional, then any $H \in C I \times [I \times E, E]$ satis the condition (a) of Theorem 1.

Example 1. Consider the system of nonlinear integral equations

$$\begin{cases} x_1(t) = \int_0^1 H_1(t, s, x_1(s), x_2(s)) ds, \\ x_2(t) = \int_0^1 H_2(t, s, x_1(s), x_2(s)) ds, \end{cases} \qquad ($$

where

$$H_1(t, s, x_1, x_2) = (2 + ts) \sqrt[3]{x_1 + x_2} \ln(1 + x_1^2 + x_2^2),$$
$$H_2(t, s, x_1, x_2) = \frac{(2 - ts) \sqrt[3]{x_1 x_2} \operatorname{arc\,tg}^2(x_1^2 + x_2^2)}{1 + \operatorname{arc\,tg}^2(x_1^2 + x_2^2)}.$$

Conclusion *System* (28) *has at least three continuous solutions* $\{x_{1i}(t), x_{2i}(t)$ $= 1, 2, 3)$ *such that* $x_{11}(t) \equiv 0$, $x_{21}(t) \equiv 0)$ $(0 \leqslant t \leqslant 1)$; $x_{12}(t) > 1$,

$$x_{22}(t) > \frac{\pi^3}{64} \left[2 \ln 2 \left(1 + \frac{\pi^2}{16} \right) \right]^{-3/2}$$

$(0 \leqslant t \leqslant 1)$ *and* $x_{13}(t) < 1$ *or*

$$x_{23}(t) < \frac{\pi^3}{64} \left[2 \ln 2 \left(1 + \frac{\pi^2}{16} \right) \right]^{-3/2}$$

for some $t \in [0, 1]$.

Proof Let $I=[0, 1]$, $E=R^2$, $P=\{x=(x_1, x_2)\in R^2|x_1\geqslant 0, x_2\geqslant 0\}$ and $H(t, s,$ $)=(H_1(t, s, x_1, x_2), H_2(t, s, x_1, x_2))$, $x=(x_1, x_2)$. It is not hard to see that all onditions of Theorem 1 are satisfied with $x_0=(1, \varepsilon_0)$, where

$$\varepsilon_0=\frac{\pi^3}{64}\left[2\ln 2\left(1+\frac{\pi^2}{16}\right)\right]^{-3/2}.$$

or example, we verify condition (d). For $x\geqslant x_0$, i. e. $x_1\geqslant 1$, $x_2\geqslant \varepsilon_0$, and $t, s\in[0, 1]$, e have

$$H(t, s, x_1, x_2)>2\ln 2,$$

$$H_2(t, s, x_1, x_2)\geqslant\frac{\sqrt[3]{\varepsilon_0}\,\mathrm{arc\,tg}^2(1+\varepsilon_0^2)}{1+\mathrm{arc\,tg}^2(1+\varepsilon_0^2)}>\frac{\sqrt[3]{\varepsilon_0}\,\mathrm{arc\,tg}^2 1}{1+\mathrm{arc\,tg}^2 1}=2\varepsilon_0\ln 2.$$

onsequently, (d) is satisfied for $x_0=(1, \varepsilon_0)$ and $k(t, s)\equiv 2\ln 2(0\leqslant t, s\leqslant 1)$. Hence, ur conclusion follows from Theorem 1.

Theorem 2. *Let P be a cone in the real Banach space E. suppose that conditions* c) *and* (n) *of Theorem* 1 *are satisfied. Moreover, assume that*

(d′) *there exist $x_0\in P\setminus\{\theta\}$ and $k\in C[I\times I, R^1]$ such that*

$$H(t, s, x)\geqslant k(t, s)x_0 \text{ for } x\geqslant x_0 \tag{29}$$

nd

$$\int_I k(t, s)ds\geqslant 1 \text{ for } t\in I. \tag{30}$$

hen, equation (1) *has at least one solution $x^*(t)$ in $C[I, E]$ such that $x^*(t)\geqslant x_0$ for all* $\in I$.

Proof As in the proof of Theorem 1, (22) holds with $R_0>\|x_0\|$. Choosing R such at (23) is satisfied and letting $D=\{x\in C[I, E]\mid \|x\|_C\leqslant R \text{ and } x(t)\geqslant x_0 \text{ for all } t\in I\}$, e see clearly that D is a bounded closed convex set in $C[I,E]$ and $D\neq\varnothing$ since $\bar{x}\in D$, here $\bar{x}(t)\equiv x_0$ for $t\in I$. Similar to the proof of (26), we can get $A(D)\subset D$, where A defined by (7), which is a strict set contraction from $C[I, E]$ into $C[I, E]$ by emma 2. Hence, by Sadovskii's fixed point theorem (see [4]), A has a fixed point x^* n D.

Remark 2. Obviously, condition (d′) is weaker than condition (d) and in heorem 2 P may be any cone which is not necessary to be normal and solid.

Example 2. Consider the infinite system of nonlinear integral equations

$$x_n(t)=\int_0^1 H_n(t, s, x_1(s), x_2(s), \cdots)ds \quad (n=1, 2, 3, \cdots), \tag{31}$$

here

$$H_n(t, s, x_1, x_2, \cdots)$$
$$=\frac{1}{n}(2-ts)\sqrt[3]{1+x_{n+1}+x_{2n}}-\frac{1}{n}t^2 s\sin(t+s-x_n)$$
$$(n=1, 2, 3, \cdots).$$

Conclusion. *Infinite system* (31) *has at least one continuous solution* $\{x_1^*(t), x_2^*(t), \cdots, x_n^*(t), \cdots\}$ *such that $x_n^*(t)\to 0$ as $n\to\infty$ and $x_1^*(t)\geqslant 1$, $x_n^*(t)\geqslant 0$ $(n=2, 3, \cdots)$ for all $t\in[0, 1]$.*

Proof Let $I=[0, 1]$, $E=c_0=\{x=(x_1, x_2, \cdots, x_n, \cdots)\,|\,x_n\to 0\}$ with norm $\|x\|=\sup_n |x_n|$, $P=\{x=(x_1, x_2, \cdots, x_n, \cdots)\in c_0\,|\,x_n\geqslant 0,\ n=1, 2, 3, \cdots\}$ and $H(t, s, x)=(H_1(t, s, x_1, x_2, \cdots), \cdots, H_n(t, s, x_1, x_2, \cdots), \cdots)$, where $x=(x_1, x_2, \cdots, x_n, \cdots)$. It is not difficult to showthat all conditions of Theorem 2 are satisfied with $x_0=(1, 0, 0, 0, \cdots)$. For example, (17) follows from the inequalities

$$|H_n(t, s, x_1, x_2, \cdots)|\leqslant\frac{1}{n}(2\sqrt[3]{1+2\|x\|}+1),\ (n=1, 2, 3, \cdots). \tag{32}$$

And, moreover, by virtue of (32) we can easily prove that the set $H(t, s, B)$ relatively compact in $E=c_0$ for any bounded $B\subset E=c_0$, and so (15) is satisfie Finally, for $x\geqslant x_0$, i. e. $x_1\geqslant 1$, $x_n\geqslant 0 (n=2, 3, \cdots)$, and $t, s\in[0, 1]$, we have

$$H_1(t, s, x_1, x_2, \cdots)\geqslant 2-ts-t^2s,$$
$$H_n(t, s, x_1, x_2, \cdots)\geqslant 0 \quad (n=2, 3, \cdots).$$

Since

$$\int_0^1 (2-ts-t^2s)\,ds=2-\frac{1}{2}(t+t^2)\geqslant 1 \text{ for all } t\in[0, 1],$$

we see that condition (d′) is satisfied for $x_0=(1, 0, 0, 0, \cdots)$ and $k(t, s)=2-ts-t^2$ Consequently, our conclusion follows from Theorem 2.

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