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SYMMETRIES AND THE CALCULATIONS OF DEGREE**

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Abstract

This paper considers the calculations of Leray–Schauder degree of equivariant compact operators under any compact Lie group actions. The main results include two parts. One is a local Leray–Schauder index formula on regular zero orbits. The other is a generalized Borsuk theorem.

§0. Introduction

In this paper we consider the calculations of Leray–Schauder degree of quivariant compact operators under any compact Lie group actions. The main esults include two parts. One is a local Leray–Schauder index formula on regular ero orbits. The other is a generalized Borsuk theorem.

The zero points of equivariant operators, which appear as orbits here, are enerally not isolated. Hence, some useful and efficient results about the local index f degree cannot be used again. Naturally the locally calculating problem of the ndex for zero orbits should be investigated. We study this problem in § 1, and give formula of the local index for regular zero orbits, which is related to the topology Euler characteristic) of the orbits.

On the other hand, it is well known that the classical Borsuk–Ulam theorem as played an important role for dealing with symmetric nonlinear problems. Based n this theorem Lusternik–Schnirelman category theory and the related notion of enus were founded, which have been used to treat even functionals and to obtain any stationary points for the variational problems. In finite dimensional case the orsuk–Ulam theorem states that if Ω is a symmetric bounded open neighbourhood f the origin in R^n, and f is an odd continuous map of $\partial\Omega$ into R^k, $k<n$, then f must anish somewhere. This is an immediate corollary of the following Borsuk's theorem: Ω is as above and f is a continuous odd map of $\partial\Omega$ into $R^n\backslash\{\theta\}$, then deg (f, Ω, θ)

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is odd. This shows that the appearance of symmetry can provide some quantitative information for the calculation of degree.

In recent years, much work was devoted to the generalizations of the above theorems. In [1] and [8] Benci, Fadell and Rabinowitz, to develop an S^1 index theory, gave an S^1-version of the Borsuk–Ulam theorem. The initial proof of it given in [1] employed some theory in algebraic topology, such as Chern class. Afterwards, in [3] Nirenberg gave an elementary analytic proof for a slightly generalized form by means of transversality lemma. In addition, there is a lot of work concerned wi the generalization of the Borsuk theorem. For instance, in [11] for finite groups torus groups, in [15] for Z_p groups (p prime integer); and when $\mathrm{Fix}_G \neq \{\theta\}$, in [13 [9] for S^1 group, in [7] for finite groups or torus groups. Other work is referred the references of the above papers. However, observing all these researches on t global generalizations of the Borsuk theorem we find that the transformation grou are restricted to the finite groups, S^1 or torus groups which all are commutati groups except finite groups, and that there is no result for other group actions su as $SO(n)$, $O(n)$, S^3, which should be more complicated and appear more natural in applications than finite groups, S^1 or T^n.

In § 2, we consider the global calculation of Leray–Schauder degree under t action of a compact Lie group, and we obtain a generalization of the Borsuk theor which we call a generalized Borsuk theorem (cf. § 2 Theorem 2.1). This resu dividesthe degree calculation into two parts. One part is the degree of the m restricted to the fixed point space of the group action, and the other part is a line combination of Euler characteristics of the orbits. Obviously, the former correspor to the part on which there is no influence of the group action, and the latter sho clearly how the symmetries influence the global degree. We also discuss some versi of the generalized Borsuk theorem for certain concrete groups. Moreover, we point that our theorems imply all previous results about Z_2, S^1, T^n and finite groups.

This work was motivated by the studies of equivariant Morse theory for isola critical orbits in [19] by means of which some slightly further results about degree calculation of equivariant potential operators were obtained in [20].

Throughout this note, the following notations are used. G always denote compact Lie group. A G-Hilbert space means that there is an isometric lin representation of G on X. For fixed x, $G(x)=\{gx \mid g\in G\}$ is called a G-orbit, whic a compact submanifold of X. The normal bundle of $G(x)$ in X is denoted $\nu G(x)$. The closed subgroup of G defined by $G_x=\{g\in G \mid gx=x\}$ is called the isotr group of x. If H is a closed subgroup of G, G/H denotes the left coset space of H G. $\mathrm{Fix}_G=\{x\in X \mid gx=x,\ \forall g\in G\}$ is called fixed point space. The concepts of tube and slice are often used. We refer these concepts and other terminology on the compact ie transformation groups to [2].

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§1. Local Result–An Index Formula of Regular Zero Orbits

Let X be a Banach space and let $f=I+K$: $X\to X$ be continuous, where K is a mpact operator. Assume that x_0 is an isolated zero point of f, then the index of f ith respect to x_0, ind (f, x_0), is well defined. And from the Leray–Schauder index rmula (cf. [4]), if $(-1)\notin\sigma(DK(x_0))$, then

$$\operatorname{ind}(f,\ x_0)=(-1)^{\rho(DK(x_0))},\tag{1.1}$$

here $\rho(DK(x_0))$ equals to the sum of the algebraic multiplicities of all eigenvalues $DK(x_0)$ which are less than -1.

The zero points of a G–equivariant operator are generally not isolated. Therefore e should consider the index of isolated zero orbits.

Let X be a Hilbert space*) and $T(G)$ be a smooth isometric linear representation a compact Lie group G. When X' is also a G–space on which the representation G is $T'(G)$, $f\in C(X,\ X')$ is calleds G–equivariant if

$$f(T_gx)=T'_gf(x),\quad \forall g\in G,\ x\in X.\tag{1.2}$$

Set

$$C^k_G(X,\ X)=\{f\in C^k(X,\ X)\,|\,f \text{ is } G\text{–equivariant}\},$$
$$F_G(X,\ X)=\{f\in C^2_G(X,\ X)\,|\,f-I \text{ is a compact operator}\},$$
$$R_G(X,\ X)=\{f\in F_G(X,\ X)\,|\,\text{the zero orbits of } f \text{ all are regular}\}.\tag{1.3}$$

Now, assume $f\in C_G(X,\ X)$ and that N is an isolated zero orbit of f. Assume at $f-I$ is compact. Suppose that O is an isoated neighbourhood of N, i. e. O does t contain other zero points of f except N. The index of f with respect to N is fined as follows:

$$\operatorname{ind}\ (f,\ N)=\deg(f,\ O,\ \theta).\tag{1.4}$$

It is easy to see (1.4) is independent of the choise of O.

Definition 1.1. *Suppose that $f\in C^1_G(X,\ X)$ and $N=G(x_0)$ is a zero orbit of f. is called a regular zero orbit of f if*

$$Df(x_0):\ X/T_{x_0}N\to \operatorname{Ran} Df(x_0)\tag{1.5}$$

is an isomorphism.

*) Our results are also true for Banach space X, the proof is referrd eto [22].

One can easily prove that the regular zero orbits are isolated. Below, we shall calculate the index of a regular zero orbit. Let N be a regular zero orbit of f with orbit type (H). By orbit type (H) we mean $(H)=\{K \mid K$ is a subgroup of G, K and H are conjugate$\}$. And we say N has orbit type (H) if for $x\in N$, $(x_x)=(H)$, (cf. [2]). Moreover, if we denote the left coset space of H in G by G/H, then there is a diffeomorphism between N and G/H. Our main conclusion is as follows.

Theorem 1.1. *Let $f\in F_G(X, X)$. Suppose that N is a regular zero orbit of f with orbit type (H), then there is a nonnegative integer ρ_0 such that*

$$\operatorname{ind}(f, N)=(-1)^{\rho_0}\chi(N)=(-1)^{\rho_0}\chi(G/H), \tag{1.6}$$

where $\chi(N)$ is the Euler characteristic of N.

By means of G-equivariant tubular neighbourhood theorem (cf. [2]), denoting the tangent bundle of N by TN and the normal bundle by νN, then there is a orthogonal decomposition at each $x\in N$:

$$X=T_xN\oplus\nu_xN, \tag{1.7}$$

$\pi:\nu N\to N$ is G-equivariant projection. Moreover, there is a diffeomorphism i:

$$i:\ \nu N(\varepsilon)\to O_\varepsilon(N),$$
$$i\ (x,\ v)=x+v,\ x\in N,\ v\in\pi^{-1}(x),$$

where

$$\nu V(\varepsilon)=\{(x,\ v)\mid \|v\|<\varepsilon\},$$
$$O_\varepsilon(N)=\{x\mid \operatorname{dist}(x,\ N)<\varepsilon\}.$$

For any $x_0\in N$, $S_{x_0}=\nu_{x_0}N(\varepsilon)=\{(x_0, v)\mid \|v\|<\varepsilon\}$ is a slice at x_0 and

$$\nu N(\varepsilon)=G(S_{x_0})$$

is a G-invariant tubular neighbourhood.

In the following we shall consider the problem on this neighbourhood.

We always write $T_gx=gx$ for the simplicity if there is no obscurity.

Lemma 1.1. *For any $g\in G$, $x\in N$ we always have*

$$Df(gx)=gDf(x)g^{-1}. \tag{1.8}$$

Moreover, g: Ker $Df(x)\to$ Ker $Df(gx)$ *and* g: Ran $Df(x)\to$ Ran $Df(gx)$ *a[re] isomorphisms.*

Proof Differentiate the equality $f(gx)=gf(x)$, the result follows immediatel[y].

Now, if N is a regular zero orbit, we fix a point $x_0\in N$. Without loss of generali[ty] assume $G_{x_0}=H$ and write $Df(x_0)=A\in\mathscr{L}(X,\ X)$, where $A=I+K$ and K is a line[ar] compact operator. Therefore, there is an orthogonal decomposition of X:

$$X=\operatorname{Ker}A\oplus Z_1=Z_2\oplus\operatorname{Ran}A, \tag{1.}$$

where $Z_1=\nu_{x_0}N$, $\dim Z_2=\dim\operatorname{Ker}A=\dim N<+\infty$.

From (1.8) and $hx_0=x_0\ \forall h\in H$, we have $Df(x_0)h=Df(hx_0)h=hDf(x_0)$. A[nd] then we have

Lemma 1.2. *$T(H)$ and A are commutative. Therefore,* Ker$A$, Ran$A$, Z_1 *and*

$_2$ *in* (1.9) *all are* $T(H)$*-invariant subspaces of* X.

Proposition 1.1. *Assume that* A *is given as above. Then for any* $\varepsilon>0$ *there is a* *near compact operator* $\Theta\in\mathscr{L}(X, X)$ *satisfying the following conditions:*

(i) $\|\Theta\|_{\mathscr{L}(X,X)}<\varepsilon$;

(ii) $T(H)$ *and* Θ *are commutative;*

(iii) $[\mathrm{Ker}(I+\Theta)A]\cap[\mathrm{Ran}(I+\Theta)A]=\{\theta\}$. (1.10)

Its proof is rather technical and we give it in § 3. However, we now use this roposition to complete the proof of Theorem 1.1.

1° On $\nu N(\varepsilon)$,

$$f(x)=f(\pi(x))+Df(\pi(x))(x-\pi(x))+o(\|x-\pi(x)\|) \text{ as } \|x-\pi(x)\|\to 0.$$

ince N is a regular zero orbit and N is compact, there is $\delta>0$ independent of $x\in N$. t. for $x\in\nu N(\varepsilon)$, $\|Df(\pi(x))(x-\pi(x))\|\geqslant\delta\ \|x-\pi(x)\|$. Let $\tilde f(x)=Df(\pi(x))(x-$ $(x))$, then for $\varepsilon>0$ small enough we have

$$\deg(f, \nu N(\varepsilon), \theta)=\deg(\tilde f, \nu N(\varepsilon), \theta). \tag{1.11}$$

2° Let $x_0\in N$ be the point fixed as above. By Lemma 1.1,

$$Df(\pi(gx))=Df(g\pi(x))=gDf(\pi(x))g^{-1},$$

. e. for $x\in\nu_{x_0}N(\varepsilon)$, $Df(\pi(gx))=gAg^{-1}$. Define a map F: $N\cong G/H\to\mathscr{L}(X, X)$ by

$$F(gH)=g(I+\Theta)g^{-1}, \tag{1.12}$$

vhere Θ is given in Proposition 1.1. From the property (ii) of Θ one can easily heck that F is well defined on N. Let π': $G\to G/H$ be the projectio n of Lie group onto its homogeneous space G/H, it is easy to see that $F\circ\pi'(g)=g(I+\Theta)g^{-1}$ is a mooth map from G into $\mathscr{L}(X, X)$, so F: $G/H\to\mathscr{L}(X, X)$ is also smooth. Now let

$$E(x)=F(\pi(x))\tilde f(x)=F(\pi(x))\circ Df(\pi(x))(x-\pi(x)). \tag{1.3}$$

Note that

$$\begin{aligned}&\sup_{x\in\nu N(\varepsilon)}\|E(x)-\tilde f(x)\|\\ &\leqslant\sup_{x\in\nu N(\varepsilon)}\|F(\pi(x))-I\|\,\|Df(\pi(x))(x-\pi(x))\|\\ &\leqslant\sup_{x\in G,x\in\nu_{x_0}N(\varepsilon)}\|g\Theta g^{-1}\|\,\|Df(gx)\|\,\|g(x-\pi(x)\|\leqslant\varepsilon\|\Theta\|\,\|A\|.\end{aligned}$$

So if $\|\Theta\|$ is small enough, it suffices to consider deg $(E, \nu N(\varepsilon), \theta)$. In addition, we leclare that for any $x\in\nu N(\varepsilon)$,

$$\mathrm{Ker}[F(\pi(x))\circ Df(\pi(x))]\cap\mathrm{Ran}[F(\pi(x))\circ Df(\pi(x))]=\{\theta\}. \tag{1.14}$$

In fact, by the definition of F for $x\in\nu_{x_0}N$,

$$F(\pi(gx))\circ Df(\pi(gx))=g(I+\Theta)Ag^{-1},\ \forall g\in G.$$

Nevertheless

$$\mathrm{Ker}[g(I+\Theta)Ag^{-1}]=g\,\mathrm{Ker}[(I+\Theta)A]$$

and

$$\mathrm{Ran}[g(I+\Theta)Ag^{-1}]=g\,\mathrm{Ran}[(I+\Theta)A].$$

And the (1.14) follows from

$$g\,\mathrm{Ker}[(I+\Theta)A]\cap g\,\mathrm{Ran}[(I+\Theta)A]$$
$$=g\{\mathrm{Ker}[(I+\Theta)A]\cap\mathrm{Ran}[(I+\Theta)A]\}$$
$$=\{\theta\}\ (\text{by Proposition. 1.1 (iii)}).$$

3° From (1.14) one can see that

$$P(\pi(x))F(\pi(x))Df(\pi)x))P(\pi)x))$$

is a linear isomorphism of $\nu_{\pi(x)}N$ onto itself, where $P(\pi(x))$ is the orthogonal projection onto the normal space $\nu_{\pi(x)}N$ at $\pi(x)$. Define a homotopy map

$$J:\ [0,\ 1]x\nu N(\varepsilon)\to X \text{ by}$$
$$J_t(x)=(1-t)E(x)+t(P(\pi(x))E(x)+V(\pi(x))),$$

where V is a smooth vector field on N, i. e. a smooth section of TN. By Sard's lemma we may assume that the zero points of V are nondegenerate, that is, if $V(x)=0$, $DV(x)$ is a linear isomorphism (cf. [12]). So V has only finite zero points, say $\{x_1, \cdots, x_m\}$, $m<+\infty$.

We verify $J_t(x)\neq\theta$ for any $(t,\ x)\in[0,\ 1]\times\partial(\nu N(\varepsilon))$, $t=0$ it is true. If

$$(t,\ x)\in(0,\ 1]\times\partial(\nu N(\varepsilon)),\ J(t,\ x)=\theta,$$

then $P(\pi(x))E(x)=\theta$ and $tP^{\perp}(\pi(x))E(x)+(1-t)V(\pi(x))=\theta$, where $P^{\perp}$ is the projection onto the tangent space $T_{\pi(x)}N$ at $\pi(x)$. But from the above observation, $P(\pi(x))E(x)=\theta$ implies $x=\pi(x)\in N$, this is a contradiction.

Now, $J_1(x)=P(\pi(x))E(x)+V(\pi(x))$, the two terms are orthogonal sum, and the zero set of J_1 is precisely $\{x_1,\ \cdots,\ x_m\}$. Thus

$$\deg(E,\ \nu N(\varepsilon),\ \theta)=\deg(J_1,\ \nu N(\varepsilon),\ \theta)=\sum_{j=1}^{m}\mathrm{ind}\ (J_1,\ x_j).\tag{1.15}$$

4° Take $Z\in X$, it is easily verified that

$$DJ_1(x_j)(Z)=\tilde{D}(\mathrm{PFDf})(\pi(x_j))\circ D\pi(x_j)(Z)(x_j-\pi(x_j))+DV\circ D\pi(x_j)(Z)$$
$$+P(\pi(x_j))F(\pi(x_j))Df(\pi(x_j))(Z-D\pi(x_j)Z)$$
$$=P(\pi(x_j))F(\pi(x_j))Df(\pi(x_j))P(\pi(x_j))(Z)$$
$$+DV(\pi(x_j))P^{\perp}(\pi(x_j))(Z),$$

where $\tilde{D}$ denotes the differentiation along the tangent space of N. And we have used $x_j-\pi(x_j)=0$ and $D\pi(x_j)=P^{+}(\pi(x_j))$. By (1.14) and the assumption on V, we see that $\{x_1,\ \cdots,\ x_m\}$ are nondegenerate zero points of J_1. By (1.1) and (1.15),

$$\mathrm{ind}\ (f,\ N)=\sum_{j=1}^{m}(-1)^{\rho}(DJ_1(x_j)).\tag{1.16}$$

By the expression of $DJ_1(x_j)$ we have

$$\rho(DJ_1(x_j))=\rho(D(PE(x_j))+\rho(D(V\circ\pi)(x_j)).$$

Let $x_j=g_jx_0$, then

$$D(PE(x_j)=P(x_j)F(x_j)Df(x_j)P(x_j)$$
$$=g_jP(x_0)(I+\Theta)AP(x_0)g_j^{-1}.$$

Thus,

$$\rho(D(PE)(x_j))=\rho(P(x_0)(I+\Theta)AP(x_0)).$$

Set $C=P(x_0)(I+\Theta)AP(x_0)$: $\nu_{x_0}N\to\nu_{x_0}N$, then C is an isomorphism and $C-I$ is compact operator (by Proposition 1.1(i)). Let $\rho(C)=\rho_0(\nu_{x_0}N$ as a closed subspace f X), then

$$\text{ind}\ (f,N)=\sum_{j=1}^{m}(-1)^{\rho(D(PE)(x_j))}\cdot(-1)^{\rho(D(V\circ\pi)(x_j))}$$
$$=(-1)^{\rho_0}\sum_{j=1}^{m}(-1)^{\rho(DV(x_j))}=(-1)^{\rho_0}\chi(N).$$

The last equality is due to the Poincaré-Hopf theorem (cf. [12]).

Remark 1.1. For a given group we may use the index formula to give much nore information. For example, if $G=T^n=S^1x\cdots xS^1$ (n times) and N is a nontrivial regular zero orbit of f, then ind $(f,N)=0$.

Remark 1.2. Some results related to our work can be found in [14], [18], in which the index for general zero manifolds was discussed. For potential operators we obtained some slightly better results in [20].

Remark 1.3. The result (1.6) might be true for a continuous action $T(G)$. In [6] Dancer obtained a result which implies the zero orbit of a smooth map should be smooth even if the action is not smooth.

§ 2. Global Result-Generalized Borsuk Theorem

We begin to consider the global calculation of degree for equivariant operatoas, and our main result is the fo llowing one which we call generalized Borsuk theorem.

Theorem 2.1 *Let X be a Hilbert space*) and $T(G)$ be a smoothly isometric representation of compact Lie group G on X. $\Omega\subset X$ is a G-invariant bounded open set. Denote the orbit types of $T(G)$ in Ω by (G_i), $i=1,2,\cdots,k$, where $G_0=G$. Assume that f: $\Omega\to X$ is a continuous G-equivariant map and that $f-Id$ is compact. If $\theta\notin f(\partial\Omega)$, we have*

$$\deg(f,\Omega,\theta)=\deg(f|_{\text{Fix}_G\cap\Omega},\ \text{Fix}_G\cap\Omega,\theta)+\sum_{i=1}^{k}\alpha_i\chi(G/G_i),\tag{2.1}$$

where $\{\alpha_i\}_{i=1}^{k}$ is a group of integers depending on f.

Remark 2.1. It was proved that the orbit types of $T(G)$ are finite when X is a finitely dimentional Euclidean space (cf. [2]). When X is infinitely dimensional, while there may be an infinite number of isotropy groups G_i(i. e. an infinite number of orbit types) it can be shown that $\chi(G/G_i)$ can only take a finite number of values (cf. [5]).

Before proving it we point out that Theorem 2.1 implies the classical Borsuk

*) We have proved this theorem in [22] for X being a Banach space and $T(G)$ being a continuous isometric representation.

theorem (cf. § 0) and a series of previous generalizations to S^1, T^n and finite groups. For the simplicity, we introduce a nonnegative integer $\nu_{T(G)}(\Omega)$.

Definition 2.1. *Let $T(G)$ be an isometric representation of G on X. $\Omega\subset X$ is an invariant set. If for any $x\in\Omega\backslash\mathrm{Fix}_G$, $\chi(G(x))=0$, define $\nu_{T(G)}(\Omega)=0$. Otherwise, define $\nu_{T(G)}(\Omega)=$ the greatest common divisor of $\{|\chi(G(x))|\,|x\in\Omega\backslash\mathrm{Fix}_G\}$. $\nu_{T(G)}$ is short for $\nu_{T(G)}(X)$.*

Corollary 2.1. *Let the assumptions in Theorem 2.1 hold, then*

$$\deg(f,\ \Omega,\ \theta)=\deg(f|_{\mathrm{Fix}_G\cap\Omega},\ \mathrm{Fix}_G\cap\Omega,\ \theta)+\alpha\cdot\nu_{T(G)}(\Omega). \tag{2.2}$$

In particular, if $\mathrm{Fix}_G=\{\theta\}$ and $\theta\in\Omega$, then

$$\deg(f,\ \Omega,\ \theta)=1+\alpha\cdot\nu_{T(G)}(\Omega), \tag{2.3}$$

where α is an integer depending on f.

Remark 2.2. In fact, ν is a kind of measure for the uniformity of the topolog of orbits. Below we can see (2.2) and (2.3) include all previous work in this field Take $G=Z_2$ and then $\nu_{T(G)}=2$. (2.3) is just the famous Borsuk theorem (cf. § 0) Take $G=S^1$ and note that all nontrivial orbitt of S^1 action are homeomorphic to S^1 then $\nu_{T(G)}=0$. And (2.3) is just tho S^1-Borsuk theorem (cf. [1], [8], etc.). Take $G=T$ and similarly $\nu_{T(G)}=0$ (cf. [11]). Take $G=Z_p$, then $\nu_{T(G)}=p$ provided p is prime. Tak G a finite group and denote the order of G by $|G|$, then $\nu_{T(G)}=$the G. C. D. of al divisors of $|G|$ which are less than $|G|$ (cf. [15], [11], etc.). When $\mathrm{Fix}_G\neq\theta$ corresponding to the above various group actions, (2.2) was discussed in [7], [9] [13]. Further references can be found in these papers.

Corollary 2.2. *Let the assumptions in Theorem 2.1 hold. In addition, assum that there are only two orbit tyyes $G_0=G$, $G_1=\{e\}$. Then*

$$\deg(f,\ \Omega,\ \theta)=\deg(f|_{\mathrm{Fix}_G},\ \mathrm{Fix}_G\cap\Omega,\ \theta)+\alpha\cdot\chi(G). \tag{2.4}$$

In particular, if $\dim G\geqslant 1$, then

$$\deg(f,\ \Omega,\ \theta)=\deg(f|_{\mathrm{Fix}_G},\ \mathrm{Fix}_G\cap\Omega,\ \theta). \tag{2.5}$$

Example 2.1. Let X be a Hilbert G-space and $\dim G\leqslant 3$. Assume that th number of connected components of G is 2^m for a certain $m\in N$. Assume that $\Omega\subset X$ is an invariant set, and that all orbits of the G-action in Ω are orientable. Ther $\nu_{T(G)}(\Omega)$ is even.

In fact, since $\dim G\leqslant 3$ for every nontrivial orbit N $\dim N\leqslant 3$. If $\dim N=1$ o 3 by virtue of Poincaré's duality theorem (cf. [10]) $\chi(N)=0$. If $\dim N=2$, i follows from Corollary (26.11) in [10] that $\chi(N)$ is even. If $\dim N=0$, by th assumption N contains 2^t points for a certain $1\leqslant t\leqslant m$, so $\chi(N)$ is even.

Corollary 2.3. *Let the assumptions in Theorem 2.1 hold. In addition, assum that G and Ω satisfy the conditions in the above example. If $f|_{\mathrm{Fix}_G}=id$, then $\deg(f,\ \Omega,\ \theta)$ is odd.*

Remark 2.3. The orientable condition in Example 2.1 is essential. For ample, $G=So(3)$ and $H=O(2)$, then $G/H=P^2$ is nonorientable and $\chi(P^2)=1$.

Now, a generalized form of the classical Borsuk–Ulam theorem (cf. § 0) can be ven as follows.

Theorem 2.2. *Let $T(G)$ be a linear representation of compact Lie group G on* R^n. *Assume that $R^k\subset R^n$ is a $T(G)$-invariant subspace $K<n$ and that $\Omega\subset R^n$ is a $T(G)$-invariant bounded open set $\theta\in\Omega$ with $\nu_{T(G)}(\Omega)\neq 1$. If $f\colon \partial\Omega\to R^k$ is a $T(G)$-equivariant continuous map satisfying $f|_{\mathrm{Fix}_G}=id$, then f must vanish somewhere.*

Proof Firstly, we extend f to $\tilde f\colon \Omega\to R^k\subset R^n$ with $\tilde f|_{\partial\Omega}=f$, $\tilde f$ being $T(G)$-equivariant. Since $\tilde f|_{\mathrm{Fix}_G\cap\partial\Omega}=\mathrm{id}$ deg $(\tilde f|_{\mathrm{Fix}_G}, \mathrm{Fix}_G\cap\Omega, \theta)=1$. If $\theta\notin\tilde f(\partial\Omega)=f(\partial\Omega)$ by Corollary 2.1, $\deg(\tilde f, \Omega, \theta)\neq 0$. On the other hand, from $k<n$, we may choose $y\in R^n\backslash R^k$ and $\|y\|$ small s. t. deg $(\tilde f, \Omega, \theta)=\deg(\tilde f, \Omega, y)=0$, a contradiction.

Remark 2.4. Further discussions can be made for $\{\alpha_i\}_{i=1}^k$ in Theorem 2.1. To shorten the paper we do not consider them here.

In order to prove Theorem 2.1 we firstly give a density theorem which shows a kind of weak equivariant transversary property and is a generalization of Sard's lemma (or Sard–Smale's lemma) in the category of equivariant maps. It mry be useful in some other situations. Let X be a complete G-Hilbert space with an isometric representation $T(G)$. Let $B\subset X$ be a G-invariant set, define

$R_G(X, X; B)=\{f\in F_G(X, X)\mid$ the zero orbits of f in B are regular$\}$.

Theorem 2.3. *$R_G(X, X)$ is dense in $F_G(X, X)$, where $R_G(X, X)=R_G(X, X; X)$.*

Before proving Theorem 2.3, we use it to give the proof of Theorem 2.1.

Lemma 2.1. *Let $f\in F_G(X, X)$ and Ω be a bounded G-invariant open set and that $\Omega\cap\mathrm{Fix}_G=\phi$. If $\theta\notin f(\partial\Omega)$, then*

$$\deg(f, \Omega, \theta)=\sum_{i=1}^k\beta_i\chi(G/G_i), \tag{2.6}$$

where $\{\beta_i\}$ is a group of integers.

Proof By the homotopy invariance of degree and Theorem 2.3 we may assume $f\in R_G(X, X)$. Then it is easy to see f has only finite zero orbits which all are regular. Since $\Omega\cap\mathrm{Fix}_G=\emptyset$, every zero orbit must be of form G/G_i, $1\leqslant i\leqslant k$. By virtue of Theorem 1.1 the result follows.

Lemma 2.2. *Let $f\in F_G(X, X)$. Assume $x_0\in\mathrm{Fix}_G$ is an isolated zero points of f, then*

$$\mathrm{ind}(f, x_0)=\mathrm{ind}(f, x_0)=\sum_{i=1}^k\beta_i\cdot\chi(G/G_i) \tag{2.7}$$

where $\{\beta_i\}_{i=1}^k$ is a group of integers.

Proof Let P be the projection onto Fix_G and $Q=I-P$. For $\varepsilon_1>0$, $\varepsilon_2>0$, set

$$A(\varepsilon_1, \varepsilon_2)=\{x\in X\mid \|Qx\|\leqslant\varepsilon_1,\ \|Px-x_0\|\leqslant\varepsilon_2\}.$$

Since $f(Px)\in \mathrm{Fix}_G$, x_0 is also an isolated zero point of $f|_{\mathrm{Fix}_G}$ and (2.7) makes sense. Let $h(x)=Qx+f(Px)$ and define

$$\tilde{f}(x)=(1-\eta(\|Qx\|^2))h(x)+\eta(\|Qx\|^2)f(x), \tag{2.8}$$

where η is a nondecreasing smooth function such that $\eta(t)=0$ if $0\leqslant t\leqslant\delta_1$, $\eta(t)=1$ if $\delta_2\leqslant t\leqslant\varepsilon_1$; $0<\delta_1<\delta_2<\varepsilon_1$ are determined in the sequel. Obviously, $\tilde{f}$ is equivariant and $\tilde{f}(x)=f(x)$ if $\|Qx\|\geqslant\delta_2$. On the other hond, if δ_2 is small enough,

$$\begin{aligned}\|\tilde{f}(x)-f(x)\|&=(1-\eta(\|Qx\|^2))\,\|h(x)-f(x)\|\\&=\|Qx\|+\|Df(Px)\|\cdot\|Qx\|+o(\|Qx\|)\\&=o(1)\text{ as }\delta_2\to 0.\end{aligned}$$

So, if δ_2 is small enough, $\deg(f, A, \theta)=\deg(\tilde{f}, A, \theta)$. In $A(\delta_1, \varepsilon_2)$, $\tilde{f}(x)=h(x)$ and on $\partial(A(\delta_1, \varepsilon_2))$, $h\neq\theta$. In fact if $x\in\partial(A(\delta_1, \varepsilon_2))$, $h(x)=Qx+f(Px)=\theta$, the $Qx=\theta$, $f(Px)=\theta$ i. e. $x=Px\Rightarrow x=x_0$, a contradiction. So

$$\begin{aligned}&\deg(\tilde{f}, A(\varepsilon_1, \varepsilon_2), \theta)\\&=\deg(\tilde{f}, A(\delta_1, \varepsilon_2), \theta)+\deg(\tilde{f}, A(\varepsilon_1, \varepsilon_2)\backslash A(\delta_1, \varepsilon_2), \theta).\end{aligned} \tag{2.9}$$

Obviously, $A(\varepsilon_1, \varepsilon_2)\backslash A(\delta_1, \varepsilon_2)$ is G-invariant and

$$[A(\varepsilon_1, \varepsilon_2)\backslash A(\delta_1, \varepsilon_2)]\cap \mathrm{Fix}_G=\emptyset.$$

Then it follows from Lemma 2.1 that

$$\deg(\tilde{f}, A(\varepsilon_1, \varepsilon_2)\backslash A(\delta_1, \varepsilon_2), \theta)=\sum_{i=1}^{k}\beta_i\chi(G/G_i). \tag{2.10}$$

Since $h(x)=Qx+f(Px)$, we can apply the product formula of degree (cf. [4] to obtain

$$\begin{aligned}&\deg(h, A(\delta_1, \varepsilon_2), \theta)\\&=\deg(h|_{\mathrm{Fix}_G}, \mathrm{Fix}_G\cap A, \theta)\cdot\deg(h|_{(\mathrm{Fix}_G^\perp)}, \mathrm{Fix}_G^\perp\cap A, \theta)\\&=\deg(f|_{\mathrm{Fix}_G}, \mathrm{Fix}_G\cap A, \theta)=\mathrm{ind}(f|_{\mathrm{Fix}_G}, x_0).\end{aligned}$$

Therefore, the result follows immediately.

Proof of Theorem 2.1 We firstly use a smooth map to approximate f, and the use the invariant Haar measure on Lie group G to average the approximated ma and obtain a map $\tilde{f}\in F_G(X, X)$. Again using Theorem 2.3 we can assume $\tilde{f}\in R_G(X, X)$. And it has only finite zero orbits which all are regular. Denote the nontrivial zero orbits by $N_1, \cdots, N_m$ and the isolated zero points in Fix_G by $x_1, \cdots x_l$. Therefore, by the homotopy invariance of degree and Theorem 1.1 and Lemma 2.2,

$$\begin{aligned}&\deg(f, \Omega, \theta)\\&=\deg(\tilde{f}, \Omega, \theta)=\sum_{j=1}^{m}\mathrm{ind}(\tilde{f}, N_j)+\sum_{j=1}^{l}\mathrm{ind}(\tilde{f}, x_j)\\&=\sum_{j=1}^{m}(-1)^{\sigma_j}\chi(N_j)+\sum_{j=1}^{l}(\mathrm{ind}(\tilde{f}|_{\mathrm{Fix}_G}, x_j)+\sum_{i=1}^{k}\beta_i^j\chi(G/G_i)\\&=\deg\tilde{f}|_{\mathrm{Fix}_G}, \Omega\cap\mathrm{Fix}_G, \theta)+\sum_{i=1}^{k}\alpha_i\chi(G/G_i)\end{aligned}$$

$$=\deg(f|_{\mathrm{Fix}_G}, \Omega\cap \mathrm{Fix}_G, \theta)+\sum_{i=1}^{k}\alpha_i\chi(G/G_i).$$

Now we prove Theorem 2.3. We need the following lemmas.

Lemma 2.3. *$F_G(X, X)$ is a closed subspace of $C^2(X, X)$. And then $F_G(X, X)$ is a second category complete metric space.*

Lemma 2.4. *Let $A=I+K$, K being a compact operator, then there is an $\varepsilon>0$ such that for any $B=I+K'$ satisfying K' a linear compact operator and $\|B-A\|_{\mathscr{L}(X)}<\varepsilon$*

$$\dim \operatorname{Ker} B\leqslant \dim \operatorname{Ker} A. \tag{2.11}$$

Lemma 2.5. *Assume $x\in X$ and $N=G(x)$ is a G-orbit. Assume that S_x is a slice at x and that $GS_x=B$ is a tube of N. Then for any $y\in B$,*

$$\dim G(y)\geqslant \dim N. \tag{2.12}$$

Lemma 2.6. *Assume that $B\subset X$ is a bounded closed G-invariant subset and $f\in R_G(X, X; B)$. Then f has at most finite zero orbits in B.*

The proofs of the above lemmas are usual arguments. We do not give them here to shorten the paper and the reader can refer to [21].

Lemma 2.7. *Assume that $B\subset X$ is a G-invariant bounded closed set, then $R_G(X, X; B)$ is an open subset of $F_G(X, X)$.*

Proof Let $f\in R_G(X, X; B)$. It suffices to prove that there is an $\varepsilon>0$ such that for any $F\in F_G(X, X)$, if $\|F-f\|_{F_G}<\varepsilon$, then $F\in R_G(X, X; B)$.

By Lemma 2.6, f has finite zero orbits in B, say $N_1, \cdots, N_k$. Take $x_i\in N_i$. Let S_{x_i} be an open slice at x_i, the radius of which is small enough such that for any $y\in S_{x_i}$, $\dim\operatorname{Ker} Df(y)\leqslant\dim\operatorname{Ker} Df(x_i)=\dim N_i$(by Lemma 2.4). Thus, if $\|F-f\|<\varepsilon'$ is small, $\|DF(y)-Df(x_i)\|\leqslant\varepsilon'+\|Df(y)-Df(x_i)\|$ is also small. So by Lemma 2.4 again, for any $y\in S_{x_i}$, if $F(y)=0$,

$$\begin{aligned}\dim \operatorname{Ker} DF(y)&\leqslant\dim \operatorname{Ker} Df(x_i)=\dim N_i\\ &\leqslant\dim G(y)\leqslant\dim \operatorname{Ker} DF(y)\ \text{(by Lemma 2.5)}.\end{aligned} \tag{2.13}$$

This shows that $G(y)$ is a regular zero orbit.

Now, we can take $\varepsilon>0$ small enough such that (2.13) holds and that F has no zero orbits in

$$B\Big\backslash\bigcup_{i=1}^{k}G(S_{x_i}).$$

That is to say $F\in R_G(X, X; B)$.

Now, we need a Palais's theorem (cf. [16]).

Lemma 2.8 (Palais). *Let P be a statement valued function defined for all compact Lie groups. If whenever G is a compact Lie group the truth of $P(H)$ for all $H<G$ implies the truth of $P(G)$, then $P(G)$ is true for all compact Lie groups. Hence in a proof that $P(G)$ is valid for all compact Lie groups G it suffices to prove $P(G)$ for an arbitrary compact Lie group G under the assumption that $P(H)$ is valid whenever $H<G$. Here $H<G$ means H is a closed subgroup G and $H\neq G$.*

Lemma 2.9. *Let $x\in\backslash\mathrm{Fix}_G$, then there is a closed slice S_x at x and the corresponding tubular neighbourhood $B_x=G(S_x)$ such that $R_G(X, X, B_x)$ is dense in $F_G(X, X)$.*

Proof Let $G_x=H$. From $x\in X\backslash\mathrm{Fix}_G$, $H<G$. For any given $f\in F_G(X, X)$, $f\in F_H(X, X)$. By Lemma 2.8, we assume that $R_H(X, X; B_x)$ is dense in $F_H(X, X)$. Then for any $\varepsilon'>0$ (determined below) there is an $f_1\in R_H(X, X; B_x)$, $\|f_1-f\|<\varepsilon'$. Using a similar construotson as partition of unity without loss of generality we may assume $f_1=f$ on $X\backslash B_x$. Let $f_1'=f_1|_{S_x}$, then $f_1'\in F_H(S_x, X)$ for S_x is H invariant. From $f_1\in R_H(X, X; B_x)$ we have $f_1'\in R_H(S_x, X)$. Define a map $\tilde{f}$: X-X by

$$\tilde{f}(z)=\begin{cases} gf_1'(y) \text{ if } z\in B_x,\ z=gy,\ y\in S_x, \\ f(z) \text{ if } z\in X\backslash B_x. \end{cases}$$

We shall prove that $\tilde{f}$ is well defined and satisfies $\tilde{f}\in R_G(X, X; B_x)$ and $\|H f\|_{F_G}<\varepsilon'$ provided ε' is small enough. Firstly, in B_x if $g_1y_1=g_2y_2$ then $g_1^{-1}g_2y_2=y$ Thus $\exists\, h\in H$ $g_2=g_1h$. Then $\tilde{f}(g_2y_2)=g_2f_1'(y_2)=g_1hf_1'(y_2)=g_1f_1'(hy_2)=g_1f_1'(y_1)$ $\tilde{f}(g_1y_1)$, and on ∂S_x $f_1'=f$. So $\tilde{f}$ is well defined and G-equivariant. For t] smoothness, it suffices to verify the smoothness of $\tilde{f}$ along the tangent space of orbi This is guaranteed by the definition of $\tilde{f}$ and the smoothness of G-action. Th($\tilde{f}\in F_G(X, X)$.

Note that all orbits in B_x intersect S_x. Assume $y\in S_x$ and $\tilde{f}(y)=0$. Then X $T_yG(y)\oplus\nu_yG(y)$. One can easily verify that $\nu_yG(y)$ is the same as $\nu_yH(y)$ in T_y (where $T_yS_x=T_yH(y)\oplus\nu_yH(y)$). Since $f_1'\in R_H(S_x, X)$, $\tilde{f}\in R_G(X, X; B_x)$ follow Now, it is evident that

$$\|\tilde{f}-f\|_{C^0}=\sup_{g\in G, y\in S_x}\|gf_1'(y)-gf(y)\|\leqslant\|f_1-f\|_{C^0}<\varepsilon'.$$

Again by Lemma 1.1, $D\tilde{f}(gy)=gD\tilde{f}(y)g^{-1}$. So it suffices to see the points (S_x. In this situation, the differentiation is divided into two parts: one part is parall to the tangent space of S_x at y and the other part is vertical to the tangent space S_x. The former acts on f_1' which is an approximation of f. The latter part can be se as follows. Without loss of generality, assume $y=x$. Locally, f can be written as

$$f(gx)=F([g])f(x),$$

where $F([g])$: $G/H\to\mathscr{L}(X, X)$ is defined by $F([g])x=gx$. Then,

$$D_{T_xG(x)}f(x)Y=D_{T_xG(x)}F([e])Y\cdot f(x),\ \forall Y\in T_xG(x).$$

Similarly,

$$D_{T_xG(x)}\tilde{f}(x)Y=D_{T_xG(x)}F([e])Y\cdot f_1'(x),\ \forall Y\in T_xG(x).$$

Without loss of generality, assume there is an $M>0$, $\|D_{T_yG(y)}F\|\leqslant M$ for $y\in S_x$ (if not, we can reduce the radius of S_x). Now, we have

$$\sup_{y\in S_x}\|D_{T_yG(y)}\tilde{f}(y)-D_{T_yG(y)}f(y)\|_{\mathscr{L}(x)}$$
$$\leqslant\sup_{y\in S_x}\sup_{\|Y\|=1}\|D_{T_yG(y)}F([e])Y(f(y)-f_1'(y))\|$$
$$\leqslant M\cdot\sup_{y\in S_x}\|f(y)-f_1'(y)\|\leqslant M\cdot\|f-f_1\|_{\mathscr{L}^0}.$$

herefore,

$$\|D\tilde{f}-Df\|_{C^0}\leqslant M\|f-f_1\|_{C^1}+\|f-f_1\|_{C^1}\leqslant M\varepsilon^1.$$

n a similar way, one can prove $\|D^2\tilde{f}-D^2f\|_{C^0}\leqslant M_1\varepsilon'$, where M_1 is a certain nstant. Then, if we choose ε' small enough, we have

$$\|\tilde{f}-f\|_{F_G(X,X)}\leqslant\varepsilon',$$

e. $R_G(X, X; B_x)$ is dense in $F_G(X, X)$.

Remark 2.5. When $G=\{e\}$, it is a consequence of Sard–Smale theorem (cf. [7]) that $R_G(X, X)$ is dense in $F_G(X, X)$.

Lemma 2.10. *Let $x\in\mathrm{Fix}_G$, then there is a closed ball neighbourhood B_x of x in* ix_G *such that $R_G(X, X'B_x)$ is dense in $F_G(X, X)$.*

Proof Since f is equivariant, $f_1=f|_{\mathrm{Fix}_G}$: $\mathrm{Fix}_G\to\mathrm{Fix}_G$. By Sard–Smale's lemma, e may take a point $a\in\mathrm{Fix}_G$ with $\|a\|$ small enough such that if $y\in\mathrm{Fix}_G$, $f_1(y)+a$ $=0$, then $Df_1(y)$: $\mathrm{Fix}_G\to\mathrm{Fix}_G$ is an isomorphism.

Moreover, $f+a$ is also equivariant. Thus, without loss of generality assume f ossesses this property. Then the zero points on Fix_G are isolated with respect to ix_G. Assume x is such a point. For simplicity, assume $x=\theta$. Take

$$B_x=\overline{B(\theta, \delta)}\cap\mathrm{Fix}_G$$

ich that f has no zero point except θ, $B(\theta, \delta)=\{x\in X\,|\,\|x\|<\delta\}$.

Since $f=I+K$ and K is compact, we have

$$Df(\theta)\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}I+A, & B\\0, & I+C\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix},$$

here $(X, Y)\in\mathrm{Fix}_G\oplus(\mathrm{Fix}_G)^\perp$, and $I+A$: $\mathrm{Fix}_G\to\mathrm{Fix}_G$ is an isomorphism. $Df(\theta)$ is ommutative with G, and so do A and C. In addition, A and C ar ecompact. Let $\eta\in$ $C^\infty(R_+, R_+)$ be a nonincreasing function such that $\eta(t)=1$ if $0\leqslant t\leqslant\delta$ and $\eta(t)=0$ f $t\geqslant2\delta$. Define

$$\tilde{f}(x)=f(x)+\varepsilon\eta(\|Qx\|^2)CQx$$

here Q: $X\to(\mathrm{Fix}_G)^\perp$ is the orthogonal projection.

It is easy to check that $\tilde{f}\in F_G(X, X)$ and $\tilde{f}|_{\mathrm{Fix}_G}=f|_{\mathrm{Fix}_G}$. Furthermore we eclare that θ is a regular zero point of $\tilde{f}$. In fact, $D\tilde{f}(\theta)=Df(\theta)+\varepsilon CQ$. Because -1 is isolated in $\sigma(C)$, we may choose $\varepsilon>0$ small enough so that

$$-\frac{1}{1+\varepsilon}\notin\sigma(C),$$

i. e. $I+(1+\varepsilon)C$: $(\mathrm{Fix}_G)^\perp\to(\mathrm{Fix}_G)^\perp$ is an isomorphism. Thus, $\tilde{f}\in R_G(X, X; B_x)$.

Proof of Theorem 2.3 For each $x\in X\setminus\mathrm{Fix}_G$, choose a tube B_x' such that Lemma

2.9 holds. And for each $x\in \mathrm{Fix}_G$ choose a ball B''_x in Fix_G such that Lemma 2.10 holds. Obviously, we may choose countable B'_x, B''_x so that they cover X. Rewrite them as B_1, B_2, $\cdots$. By virtue of Lemmas 2.7, 2.9, 2.10, $R_G(X, X; B_i)$ is an open dense subset of $F_G(X, X)$ for $i=1, 2, \cdots$. Since $F_G(X, X)$ is second categary, by applying Baire's theorem (cf. [24]) we see that $\bigcap_{i=1}^{\infty} R_G(X, X; B_i)$ is dense in $F_G(X, X)$. Moreover, it is evident that

$$\bigcap_{i=1}^{\infty} R_G(X, X; B_i)=R_G\Big(X, X; \bigcup_{i=1}^{n} B_i\Big)=R_G(X, X).$$

The proof is completed.

In a similar way, we can give another approximation theorem.

Theorem 2.4. *Suppose that X and G are given as above. In addition, assu $\dim G\geqslant 1$. Let $x\in X$ be a point with $F_x=\{e\}$, i. e. $N=G(x)$ is a free G-orbit. Assu $f\in F_G(X, X)$, and that N is aregular zero orbit of f. Then there are slice S_x at x a tube $B_x=G(S_x)$ of N so that for any given $\varepsilon>0$ there exists an $\tilde{f}\in F_G(B_x,$] satisfying* (i) $\|\tilde{f}-f\|_{U_x}<\varepsilon$ *and* (ii) $\tilde{f}(y)\neq 0$, $\forall y\in B_x$.

Remark 2.6. This theorem shows that the zero orbit of an equivariant m can be romoved by perturbations.

§3. The Proof of Proposition 1.1

In order to complete the proof, we shall construct the operator Θ step by step this section. And we continue to employ the notations in § 1.

Firstly, in (1.9) we decomposite the space X further. In the following, if X' a subspace of X, $P_{X'}$ denotes the orthogonal projetion onto X'. Let

$$Y_1=\mathrm{Ker}\ A\cap \mathrm{Ran}\ A,\quad M_1=Z_1\cap \mathrm{Ran}\ A,$$

$$\widetilde{M}_2=\text{the orthogonal complement of } Y_1\oplus M_1 \text{ in Ran } A,$$

$$P_{\mathrm{Ker}A}\widetilde{M}_2=Y_2,\quad P_{Z_1}\widetilde{M}_2=M_2.$$

We have the following orthognal splitting:

$$X=\underbrace{Y_0\oplus Y_2\oplus Y_1}_{\mathrm{Ker}A\oplus}\oplus\underbrace{M_1\oplus M_2\oplus M_0}_{Z_1,}\qquad (3.$$

where Y_0, M_0 are the orthogonal complements of $Y_2\oplus Y_1$ in Ker A and $M_1\oplus M_2$ Z_1 respectively.

Lmma 3.1. $\dim Y_2=\dim M_2=\dim \widetilde{M}_2<+\infty$.

Proof If $y\in \widetilde{M}_2$ and $P_{\mathrm{Ker}A}\,y=0$, then $y\in Z_1\cap \mathrm{Ran}\,A$, i. e. $y\in M_1$. Then by t definition of $\widetilde{M}_2$, $y=0$. Therefore, $P_{\mathrm{Ker}A}$: $\widetilde{M}_2\to Y_2$ is an isomorphism. It follows tl $\dim \widetilde{M}_2=\dim Y_2\leqslant \dim \mathrm{Ker}\,A<+\infty$. The other equality is similar.

Lemma 3. 2. *Let $Y\subset Z_1$ be a $T(H)$-invariant subspace, then AY is also a $T(H)$-invariant subspace. Moreover, if we denote $A=P_{AY}AP_Y$, $h_Y=P_YhP_Y$ and $h_{AY}=$*

$_{AY}hP_{AY}$ *for* $h\in H$, *then the following formula holds:*

$$\widetilde{A}h_Y=h_{AY}\widetilde{A},\quad \forall h\in H. \tag{3.2}$$

Proof This can be derived from the fact that A is commutative with $T(H)$ (cf. emma 1.2). We omit it here.

Since $Z_2=Y_0\oplus M_0\oplus\widetilde{M}_2^\perp$ (where $\widetilde{M}_2^\perp$ is the orthoganal complement of $\widetilde{M}_2$ in $_2\oplus M_2$) comparing this with (3.1) we have $\dim Y_1=\dim M_0$. Assume $\dim Y_1=n$. If $=0$, one may verify that $\Theta=0$ satisfies the requirements. In $n>0$, denote $L=$ $^{-1}Y_1\subset Z_1$ (A^{-1} is well defined on Ran A) and then $\dim L=n$ and $\dim M_0=n$. Let

$$M_0\cap L=X_0,\quad M_0\cap L^\perp=X_1,$$

$X_2=$ the orthogonal complement of $X_0\oplus X_1$ in M_0,

$$P_LX_2=\widetilde{X}_0,\quad P_{L^\perp}X_2=\widetilde{X}_1.$$

In a similar way with the proof of Lemma 3.1 we have $\dim\widetilde{X}_0=\dim\widetilde{X}_1=$ m X_2. And we have

$$Z_1=\underbrace{X_4\oplus X_1\oplus\widetilde{X}_1}_{L^\perp}\oplus\underbrace{\widetilde{X}_0\oplus X_0\oplus X_3}_{L} \tag{3.3}$$

Obviously, every space in (3.3) is $T(H)$-invariant. Let $Q=P_JAP_{\tilde{x}_0\oplus x_0}$: $\widetilde{X}_0\oplus$ $_0\to J\subset Y_1$, where $J=A(\widetilde{X}_0\oplus X_0)$. From Lemma 3.2,

$$Qh_{\tilde{x}_0\oplus x_0}=h_JQ\quad \forall h\in T(H). \tag{3.4}$$

et $S=P_EAP_{\tilde{x}_0}$: $\tilde{x}_0\to E\subset J\subset Y_1$, where $E=A\tilde{x}_0$. Similarly,

$$Sh_{\tilde{X}_0}=h_ES,\quad \forall h\in T(H). \tag{3.5}$$

oreover, $Q_{\tilde{X}_0}=S$.

Note that X_2 is an m-subspace of the $2m$-space $\widetilde{X}_1\oplus\widetilde{X}_0$ (wheredim $X_2=m$). ince $P_{x_1}x_2$ and $P_{\tilde{x}_0}x_2$ are isomorphic to X_2, there is a nonsingular linear ansformation B: $\widetilde{X}_1\to\widetilde{X}_0$ such that if $(w,z)\in\widetilde{X}_1\oplus\widetilde{X}_0$ then $(w,z)\in X_2\Leftrightarrow z=Bw$. y the $T(H)$-invariance of X_2, we can find

$$h_{\tilde{x}_0}B=Bh_{\tilde{x}_1},\quad \forall h\in T(H). \tag{3.6}$$

Now, we give a part of definition of Θ as follows:

$$\Theta_1x=\begin{cases} P_{\tilde{x}_0\oplus x_0}\Theta_1P_Jx=\varepsilon Q^{-1}x, & x\in J,\\ P_{\tilde{x}_1}\Theta_1P_Ex=\varepsilon B^{-1}S^{-1}x, & x\in E,\\ 0, & \text{other cases.}\end{cases}$$

By (3.3) Θ_1 is well defined. Since Θ_1 is defined on a finitely dimensional space, $_1$ is a compact operator. In addition, by the definition Θ_1 is commutative with (H). Unfortunately, at this time we can not guarantee Θ_1 satisfies the third roperty in Proposition 1.1, i. e. (1.10). Nevertheless, calculating directly we btain

$$P_{M_0}\Theta_1A:\ \tilde{x}_0\oplus x_0\to x_2\oplus x_0\subset M_0.$$

Hence, the orthogonal complement of $(I+\Theta_1)AZ_1$ in Z_1 is exactly X_1. Then

$$\operatorname{Ker}(A+\Theta_1A)\cap\operatorname{Ran}(A+\Theta_1A)=(A+\Theta_1A)X_3=AX_3\subset Y_1.$$

Up to now, if we take $(I+\Theta_1)A$ as the original A and continue on with the construction of Θ, we may assume that in (3.3) $\tilde{X}_1=\tilde{X}_0=X_0=\{\theta\}$, i. e. $L=X_3$, $M_0=X_1$. Moreover, we have $\dim L=\dim M_0=\dim Y_1=n_1$, where $Y_1=AL$. If $n_1=0$, we have arrived at the end. If $n_1>0$, we take $A+\Theta_1 A$ as the original one to go on to work. From $AM_0\cap \mathrm{Ker} A=\{\theta\}$, $P_{Z_1}AM_0=E_1$ is also an n_1 dimensional space. Let

$$F_0=E_1\cap L,\quad F_1=E_1\cap L^{\perp},$$

$$F_2=\text{the orthogonal complement of } (F_0\oplus F_1) \text{ in } E_1,$$

$$\tilde{F}_0=P_LF_2,\quad \tilde{F}_1=P_{L^\perp}F_2.$$

We have

$$Z_1=\underbrace{\tilde{Z}_1\oplus F_1\oplus \tilde{F}_1\oplus M_0}_{L^\perp}\oplus\underbrace{\tilde{F}_0\oplus F_0\oplus \tilde{L}}_{L.} \tag{3.7}$$

If $\dim \tilde{F}_0\oplus F_0>0$, by the same method as above we can construct a map $\Theta_2\in \mathscr{L}(X, X)$ compact commutative with $T(H)$ such that $\Theta_2 A$: $\tilde{F}_0\oplus F_0\to F_2\oplus F_0\subset I$ And then the orthogonal complement of $P_{Z_1}((I+\Theta_2)A)Z_1$ in Z_1, whose dimension : the same as the dimension of F_1, is exactly the orthogonal complement ($A^{-1}(F_2\oplus F_0)$ in M_0. If dim $F_2\oplus F_0>0$, we can reduce the dimension of Y_1 to n_1 $\dim F_2\oplus F_0$. If n_1-$\dim F_2\oplus F_0>0$, we an repeat the above proceduro to consider E_2 AF_1. If $\dim P_LE_2>0$, we can construct a map Θ_3 so that the dimension of Y_1 reduced to n_1-$\dim F_2\oplus F_0$-$\dim P_LE_2$. We declare that by the above procedure the: is an integer $j_0>0$ s.t. if we take

$$(I+\Theta_{j_0})(I+\Theta_{j_0-1})\cdots(I+\Theta_1)A$$

as the previous one then we have $\dim M_0=0$. In other words dim $Y_1=0$. Therefor when we write $I+\Theta=(I+\Theta_{j_0})\cdots(I+\Theta_1)$, the above facts show precisely that satisfies all properties of Proposition 1.1.

If the above declaration does not hold, we deduce a contradiction as follows. first, there is a series of spaces E_j, $j=0, 1, \cdots$, such that $E_{j+1}=AE_j$ and $P_LAE_j=\{\theta$ where $E_0=M_0$, $E_1=F_1$. Let $E=\bigoplus_{j=0}^{\infty} E_j$, then $E\subset L^\perp$ is a subspace of Z_1 and A: E E. We have $Z_1=\tilde{E}\oplus E\oplus L$ where $\tilde{E}$ is the corresponding orthogonal complemer Now $\tilde{A}=P_{\tilde{E}\oplus L}AP_{\tilde{E}\oplus L}$: $\tilde{E}\oplus L\to\tilde{E}\oplus L$ is a surjection and $\mathrm{Ker}\,\tilde{A}=L$. However, $\tilde{A}$ also a Fredholm operator with index 0 and this contradicts $\dim L=\dim M_0>0$. Tl shows that the above procedure can be completed with finite speps. So far, the pr is finished.

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