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ON σ-FINITE INTEGRALS ON C^*-ALGEBRAS**

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Abstract

This paper considers positive linear functionals f defined on a dense * —subalgebra which is an analogue of the continuous functions with compact supports, of a C_*-algebra L^1-spaces associated with f are studied. Also introduced is an analogue of the bounde Borel functions vanishing at infinity.

§ 0. Introduction

Let $C_0(X)$ be the C^*-algebra of continuous functions vanishing at infinity, w X is a locally compact Hausdorff space. To study Borel measures on X, one study the positive linear functionals on $C_{00}(X)$, the space of continuous func with compact supports. In non-commutative cases, G. K. Pedersen[3] introduc non-commutative analogue of $C_{00}(X)$ for non-unital C^*-algebra A, na the minimal dense ideal I generated by $\{a\in A^+ | b\in A^+, \ [a]\leqslant b\}$. A sp (unbounded) positive linear functional, called C^*-integral, defined on I has studied[3~6]. In this paper, we consider a dense *-subalgebra $C_{00}(A)$ of a σ-u C^*-algebra A, another analogue of $C_{00}(X)$. "σ-finite integrals" defined on C are considered. It is shown in section 1 that every "unitarily bounded" po linear functional defined on $C_{00}(A)$ can be extended to a C^*-integral. L^1 space σ-finite integrals will be studied in section 2. In section 3, we introduce an ana of the bounded Borel functions vanishing at infinity.

§ 1. σ-Finite Integrals and Radon-Nikodym Theorem

Let A be a σ-unital C^*-algebra. Then A has a strictly positive element a f_n be continuous functions such that $f_n(t)=1$ if $t>1/n$ and $f_n(t)=0$ if $0\leqslant t$ $n=1, 2, \cdots$, Let $e_n=f_n(a)$, e_n are (open) projections in A^{**}. Let $A_n=e_nA^{**}e_n\cap A$ define $C_{00}(A)=\bigcup_{n=1}^{\infty} A_n$. Notice that $C_{00}(A)$ depends on a.

Proposition 1.1. *$C_{00}(A)$ is a norm dense, hereditary *-subalgebra of A.*

We define a σ-finite integral on a σ-unital C^*-algebra A to be a positive l

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-ctional f defined on $C_{00}(A)$ for some strictly positive element and f has form

$$f=\sum_{i=1}^{\infty} f_i,$$

ere each f_i is a bounded linear functional on A.

Let f be a positive linear functional defined on $C_{00}(A)$. If f can be extended to ormal weight on A^{**}, then $f=\Sigma f_i$[1] where each f_i is a bounded linear functional A. For every n, since $f(e_n)<\infty$, there are only countably many i's such that $e_n) \neq 0$. Hence $f=\sum_{i=1}^{\infty} f_i$ on $C_{00}(A)$.

Lemma 1.2 [3, Theorem 3.1]. *Let f be a positive linear functional defined on $C_{00}(A)$,*

$$\rho(x)=\inf\{f(s)+t \mid s\in C_{00}(A_+),\ t\in R^+,\ s+t\geqslant x\}$$

$x\in A_+$. Then f is a σ-finite integral if and only if $\rho(x)=0$ implies $f(x)=0$ for $C_{00}(A)_+$.

A positive linear functional f on $C_{00}(A)$ is called unitarily bounded if for every A_n, $n=1, 2, \cdots$,

$$\operatorname{Sup}\{|f(u^*xu)| \mid u \text{ unitary in } A_m+\mathbf{C}e_m,\ m\geqslant n\}<\infty.$$

The idea used in the proof of the following theorem is taken from [3, Theorem 1].

Theorem 1.3. *Every unitarily bounded positive linear functional f defined on (A) is a σ-finite integral.*

We omit the proof.

Let $f=\sum_{i=1}^{\infty} f_n$ be a σ-finite integral on A, then f can be normally extended to *. We will use the same notation f for the normal extension.

Lemma 1.4. *Let f be a unitarily bounded positive linear functional defined on $_0(A)$. Suppose $x\in C_{00}(A)$, then*

$$\sup\{|f(y^*xz)| \mid y,\ z\in A^{**},\ \|y\|\leqslant 1 \text{ and } \|z\|\leqslant 1\}<\infty.$$

Corollary 1.5. *Every unitarily bounded positive linear functional defined on $_0(A)$ can be extended uniquely to a C^*-integral.*

Proof By Lemma 1.4 f (the normal extension) is finite on $AC_{00}(A)A$. Clearly $C_{00}(A)A$ is a norm dense ideal of A. Since $C_{00}(A)\subset\mathcal{I}$, the Pedersen's ideal, by [7, 6.1.], $\mathcal{I}=AC_{00}(A)A$. It follows from [3, Theorem 3.7] that $f|_{\mathcal{I}}$ is a C^*-integral. The uniqueness follows immediately.

The following example shows that not all σ-finite integrals are unitarily bounded.

Example. Let H be a separable Hilbert space with orthonormal basis $\{\xi_i\}$. Let A be the C^*-algebra of all compact operators on H. Let

$$e_n=\sum_{i=1}^{n}(\xi_i\otimes\xi_i).$$

Corresponding to tho approximate identity $\{e_n\}$, we have

$$C_{00}(A)=\bigcup_{n=1}^{\infty} e_nAe_n.$$

Define $f_i(x)=\langle x\xi_i|\xi_i\rangle$ for $x\in A$ and

$$f=\sum_{i=1}^{\infty} i\, f_i.$$

Then f is a σ-finite integral defined on $C_{00}(A)$, but not unitarily bounded. So the normal extension of f cannot be finite on I_+.

A version of the following Radon-Nikodym theorem was found by Sakai [8] the case that f is bounded. A similar version for faithful normal weight on v Neumann algebra was proved in [11].

Theorem 1.6. *Let A be a σ-unital C^*-algebra, f a unitarily bounded (but n not be bounded) positive linear functional defined on $C_{00}(A)$ for some strictly posi element of A. Let g be a self-adjoint linear functional defined on $C_{00}(A)$. If*

$$|g(x)|\leqslant f(|x|) \text{ for all } x\in C_{00}(A)_{s.a.}$$

then there is $h\in A^{**}_{s.a.}$ with $\|h\|\leqslant 1$ such that

$$g(x)=f(hx+xh)/2 \text{ for all } x\in C_{00}(A).$$

Proof Let $f'_n=f|_{A_n}$, $g_n=g_n|A_n$. We can extend f_n and g_n to $e_nA^{**}e_n$ normally. use the same notation for the normal extensions. Thus

$$|g_n(x)|\leqslant|g_n(|x|)|\leqslant f'_n(|x|) \text{ for all } x\in(e_nA^{**}e_n)_{s.a.}$$

A slight modification of the proof of [7, 5. 3. 2] shows that there exists $h_n\in(e_nA^{**}e$ such that $\|h_n\|\leqslant 1$ and

$$g_n(x)=f_n(h_nx+xh_n)/2=f(h_nx+xh_n)/2 \text{ for all } x\in(e_nA^{**}e_n)_{s.a.}.$$

We may assume that $h_n\to h$ weakly, with $h\in A^{**}_{s.a.}$ and $\|h_n-h\|\leqslant 1$. Fix m. Let

$$L=\sup\{f(ye_my)\,|\,y\in A^{**}_{s.a.},\ \|y\|\leqslant 1\}.$$

By Lemma 1.4, $L<\infty$. Let

$$f=\sum_{i=1}^{\infty} f_i,$$

where f_i is a bounded positive linear functional on A (by Theorem 1.3.). For e $\varepsilon>0$, there is an integer N such that

$$\sum_{i=N+1}^{\infty} f_i(e_m)<(\varepsilon/(L+1))^2/4.$$

For every $x\in(A_m)_{s.a.}$ and $\|x\|\leqslant 1$, there is m_0 such that

$$\left|\sum_{i=1}^{N} f((h_n-h)x+x(h_n-h))\right|<\varepsilon$$

whenever $n\geqslant m_0$, thus

$$\begin{aligned}
&|f((h_n-h)x+x(h_n-h))|\\
&<\varepsilon+\left|\sum_{i=N+1}^{\infty} f_i((h_n-h)x)\right|+\sum_{i=N+1}^{\infty}\left|\overline{f_i((h_n-h)x)}\right|\\
&<\varepsilon+2\Big[\sum_{i=N+1}^{\infty} f_i((h_n-h)e_m(h_n-h))\Big]^{1/2}\Big[\sum_{i=N+1}^{\infty} f_i(x^*x)\Big]^{1/2}
\end{aligned}$$

$$<\varepsilon+2f\left((h_n-h)\,e_m\,(h_n-h)\right)^{1/2}\left[\sum_{i=N+1}^{p} f_i(e_m)\right]^{1/2}$$

$$<2\varepsilon,\ \text{if } n\geqslant m_0.$$

nce $f(h_nx+xh_n)=f(h_mx+xh_m)$ for every x A_m and $n\geqslant m$, we conclude that

$$f(hx+xh)/2=f(h_nx+xh_n)/2 \text{ for } x\in A_m$$

ıd $n\geqslant m$. Thus $g(x)=f(hx+xh)/2$ for all $x\in C_{00}(A)$.

Let f be a σ-finite integral defined on $C_{00}(A)$ for some strictly positive element ɔf A. We extend f to A^{**} normally as before. Let π_f be the representation given ' f. Then we have

Proposition 1.7. $\ker f\supset\ker\pi_f$.

Since $\ker f\supset\ker\pi_f$, f deduces a σ-finite integral $\bar f$ on $\pi_f(A)$ by $f(x)=\bar f(\pi_f(x))$. ırthermore, we have $\xi_n\in H_f$ such that

$$f(x)=\bar f(\pi_f(x))=\sum_{n=1}^{\infty}\langle\pi_f(x)\xi_n|\xi_n\rangle.$$

we can extend $\bar f$ to $B(H_f)$ by

$$\bar f(y)=\sum_{n=1}^{\infty}\langle y\xi_n|\xi_n\rangle \text{ for } y\in B(H_f).$$

Now we conclude the section with the following Radon-Nikodym theorem, the oof of it is essentially the same as f is bounded.

Proposition 1.8. *Let f be a σ-finite integral defined on $C_{00}(A)$. If ϕ is a self-joint linear functional defind on $C_{00}(A)$ such that*

$$|\phi(x)|\leqslant f(|x|) \text{ for all } x\in C_{00}(Ae)_{s.a.},$$

m there exists $h\in\pi_f(A)_{s.a.}$, $\|h\|\leqslant1$ such that $\phi(x)=\bar f(h\pi_f(x))$ for all $x\in C_{00}(A)$.

§ 2. L^1 Spaces and Their Duals

Let A be a σ-unital C^*-algebra, $C_{00}(A)$ be *a* norm dense, hereditary ubalgebra defined by a strictly positive element a of A as in section 1. Let f be a finite integral defined on $C_{00}(A)$. For every

$$x\in\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.},$$

define

$\|_1=\sup\Big\{|\phi(x)|\ \big|\ |\phi(y)|\leqslant f(|y|)$ for all $y\in C_{00}(A)_{s.a.}$ and are real linear functionals fined on $\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}\Big\}$. Let

$$N_1=\Big\{x\in\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}\ \big|\ \|x\|_1=0\Big\},$$

$$L_0^1=\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}/N_1$$

and $(L^1)_{s.a.}$ be the completion of L_0^1 in the norm $\|\bar{x}\|$, where $\bar{x}$ is the image of x under the quotient mapping. Clearly, $C_{00}(A)_{s.a.}/N_1$ is dense in $(L^1)_{s.a.}$. We will denote L^1, the complexification of $(L^1)_{s.a.}$.

Proposition 2.1. *If f is unitarily bounded, then*

$$N_{s.a.}=\{h\in A^{**}_{s.a.}\,|\,f(hx+xh)=0 \text{ for all } x\in C_{00}(A)\}$$

is weakly closed.

We now define $(L_\infty)_{s.a.}=A^{**}_{s.a.}/N_{s.a.}$. The complexification of $(L_\infty)_{s.a.}$ will be denoted by L_∞. Let $x\in A^{**}_{s.a.}$, $\bar{x}$ be the image of the quotient mapping, we define $\|\cdot\|$ as the norm of $\bar{x}$ in L_∞.

Theorem 2.2. *Suppose that f is unitarily bounded. Then there is a one to linear contractive mapping from $(L^1)^*$ into L_∞.*

Proof Let $\phi\in((L^1)_{s.a.})^*$, $\|\phi\|\leqslant 1$. By the definition of $\|\cdot\|_1$, $|\phi(x)|\leqslant\|x\|$ $f(|x|)$ for all $x\in C_{00}(A)_{s.a.}$. By Theorem 1.6, there is $h\in A^{**}_{s.a.}$, $\|h\|\leqslant 1$ such that

$$\phi(x)=f(hx+xh)/2 \text{ for all } x\in C_{00}(A)_{s.a.}.$$

We define $\Phi_1(\phi)=\bar{h}$, where "$-$" is the quotient map from $A^{**}_{s.a.}$ onto L_∞. It is eas see that Φ_1 is well defined, one to one and linear. By the proof of Theorem 1.6, may assume that there are $h_k\in(e_kA^{**}e_k)_{s.a.}$, $\|h_k\|\leqslant 1$ such that

$$h_k\to h \text{ weakly and}$$

$$f(h_kx+xh_k)/2=\phi(x) \text{ for } x\in(e_kA^{**}e_k),$$

furthermore, we may assume that $\|h_k\|\leqslant\|h_k\|_\infty+1/k$.

Let $\lambda_k=\inf\{\|h_k\chi_\sigma(h_k)\|\,|\,f(1-\chi_\sigma(h_k))=0\}$, where χ_σ is the characteristic func of the set $\{t\colon |t|>\sigma\}$. Clearly, $1-\chi_\sigma(h_k)\in N_{s.a.}$, if $f(1-\chi_\sigma(h_k))=0$. Thus $\lambda_k\geqslant\|h$ For every $\varepsilon>0$, there exists a projection $p_k\in e_k A^{**}e_k$ such that $p_kh_k=h_kp_k$, $f(p_k)$ and $h_kp_k=p_kh_kp_k\geqslant(\lambda_k-\varepsilon)f(p_k)\geqslant(\lambda_k-\varepsilon)\|p_k\|_1$. Hence $\|\phi\|\geqslant\lambda_k-\varepsilon$, for every ε So we have $\|\phi\|\geqslant\lambda_k\geqslant\|h_k\|_\infty$. Since $h_k\to h$ weakly, $\|h\|\leqslant\underline{\lim}\|h_k\|$.

Since $\|h_k\|\leqslant\|h_k\|_\infty+1/k$, we conclude that $\|h\|\leqslant\underline{\lim}\|h_k\|_\infty\leqslant\|\phi\|$. Thus

$$\|\Phi_1(\phi)\|=\|h\|_\infty\leqslant\|\phi\|.$$

This completes the proof.

Theorem 2.3. *There is an isometric isomorphism between $\pi_f(A)$, the commu of $\pi_f(A)$ and $(L^1)^*$; consequently, L^1 is isometric isomorphic to $\pi_f(A)'_*$, the predu $\pi_f(A)$.*

Proof It is sufficient to show that $\pi_f(A)_{s.a.}$ is isometric isomorphic to $(L^1$ Let $h\in\pi_f(A)_{s.a.}$, we define $\Phi_2(h)(x)=f(h\pi_f(x))$ for all $x\in C_{00}(A)_{s.a.}$. By an computation, we have

$$|\Phi_2(h)(x)|\leqslant\|h\|\bar{f}(\pi_f(|x|)) \text{ for } x\in(A_n)_{s.a.}$$

and all n. Thus $|\Phi_2(h)(x)/\|h\||\leqslant f(|x|)$ for all $x\in C_{00}(A)_{s.a.}$. Since $\Phi_2(h)$ is a linear functional, by the definition of $\|\cdot\|_1$,

$$|\Phi_2(h)(x)|/\|h\|\leqslant\|x\|_1 \text{ for all } x\in C_{00}(A)_{s.a.}$$

Thus $\|\Phi_2(h)\| \leqslant \|h\|$.

Let ξ be the densely defined map from A to H_f. For fixed $h\in\pi_f(A)_{s.a.}$, let

$$x\in C_{00}(A)+ \text{with } \|x^2\|_1=1, \text{ i. e. } f(x^2)=1.$$

We have

$$\langle\xi_x|\xi_x\rangle=f(x^2)=1.$$

Then $|\Phi_2(h)(x^2)|=|\langle h\xi_x|\xi_x\rangle|$. Hence

$$\|\Phi_2(h)\|\geqslant\sup\{|h\xi_x|\xi_x\rangle|\,|\,\|\xi_x\|=1,\ x\in C_{00}(A)_+\}.$$

Let $y\in C_{00}(A)_{s.a.}$ with $\|\xi_y\|=1$ and $y=y_+-y_-$. We have $y_+y_-=y_-y_+=0$. Thus

$$\begin{aligned}\langle h\xi_y|\xi_y\rangle&=\langle h(\xi_{y+}-\xi_{y-})|(\xi_{y+}-\xi_{y-})\rangle\\&=\langle h\xi_{y+}|\xi_{y+}\rangle+\langle h\xi_{y-}|\xi_{y-}\rangle\\&\quad-\bar f(\pi_f(y_+)h\pi_f(y_-))-\bar f(\pi_f(y_-h\pi_f(y_+))\\&=\langle h\xi_{y+}|\xi_{y+}\rangle+\langle h\xi_{y-}|\xi_{y-}\rangle\\&=(\|\xi_{y+}\|^2+\|\xi_{y-}\|^2)\sup\{|\langle h\xi_x|\xi_x\rangle|\,|\,\|\xi_x\|=1,\ x\in C_{00}(A)_+\}\\&=\sup\{|\langle h\xi_x|\xi_x\rangle|\,\|\xi_x\|=1,\ x\in C_{00}(A)_+\}\leqslant\Phi_2(h).\end{aligned}$$

Hence $\|h\|=\|\Phi_2(h)\|$.

Let $\phi\in((L^1)_{s.a.})^*$, $\|\phi\|\leqslant1$. By the definition of $\|\cdot\|_1$,

$$|\phi(x)|\leqslant f(|x|) \text{ for all } x\in C_{00}(A)_{s.a.}$$

By Proposition 1.8, there is $h\in\pi_f(A)_{s.a.}$ such that $\|h\|\leqslant1$ and

$$\phi(x)=\bar f(h\pi_f(x)) \text{ for all } x\in C_{00}(A).$$

Hence Φ_2 is also onto.

Let L be the set of those elements $\bar h\in L_\infty$ such that $f(hx+xh)/2\leqslant Hf(|x|)$ for $x\in C_{00}(A)_{s.a.}$, where H does not depend on x. Clearly L is the image of Φ_1, hence is a linear space.

Corollary 2.4. *If f is unitarily bounded, then there is a one to one linear contractive mapping from $\pi_f(A)$ onto L.*

We now again assume that f is unitarily bounded. Let $h\in A^{**}_{s.a.}$, an argument used in section 1 shows that $f(hx+xh)$ is defined on every

$$x\in\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}.$$

For every

$$x\in\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.},$$

define $\|x\|^1=\sup\{f(hx+xh)/2\,|\,h\in A^{**}_{s.a.},\ \|h\|\leqslant1\}$. Clearly, $\|\cdot\|^1$ is a semi-norm.

$$N^1=\Big\{x\in\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}\,\|x\|^1=0\Big\}.$$

$$\bar L^1=\Big(\bigcup_{n=1}^{\infty}e_nA^{**}e_n\Big)_{s.a.}/N^1$$

and $(\bar L^1)_{s.a.}$ be the completion of $\bar L^1$. We will use the notation $\bar L^1$ for the complexication of $(\bar L^1)_{s.a.}$.

Proposition 2.5. *Let*

$$x\in\Big(\bigcup_{n=1}^{\infty} e_nA^{**}e_n\Big)_{s.a.},$$

then there are $x_n\in C_{00}(A)_{s.a.}$ *such that*

$$\|x_n-x\|^1\to 0.$$

Theorem 2.6. *If f is unitarily bounded, then there is an isomorphism* Φ_3 *from* L_∞ *onto* $(\bar{L}^1)^*$. *Moreover,* $\|\Phi_3\|\leqslant 1$ *and* $\|\Phi_3^{-1}\|<\infty$.

Proof Let $h\in A^{**}_{s.a.}$, $\|h\|\leqslant 1$. We define $\Phi_3(h)(x)=f(hx+xf)/2$ for $x\in C_{00}(A$ $\Phi_3(h)$ uniquely determines a (norm less than one) real linear functional on $(L^1)_s$ Thus Φ_3 is a linear map from $A^{**}_{s.a.}$ to $[(L^1)_{s.a.}]^*$ such that $\|\Phi_3\|\leqslant 1$.

Suppose that $\phi\in[(L^1)_{s.a.}]^*$ such that $\|\phi\|\leqslant 1$. Then $|\phi(x)|=\sup\{|f(hx+xh)|$ $h\in A^{**}_{s.a.}\|h\|\leqslant 1\}$ for all $x\in C_{00}(a)_{s.a.}$.

Fixed n, the set $S=\{f(h\cdot+\cdot h)/2\,|\,h\in A^{**}_{s.a.},\ \|h\|\leqslant 1\}$ is a convex compact sub of $[(e_nA^{**}e_n)_*]_{s.a.}$

If $x\in(e_nA^{**}e_n)$ and $\|x\|\leqslant 1$, we have

$$|f(hx)|\leqslant f(he_nh)^{1/2}f(x^*x)^{1/2}\leqslant L^{1/2}f(e_n)^{1/2},$$

where L is the same as in Proposition 2.5. Thus

$$|f(hx+xh)/2|\leqslant L^{1/2}f(e_n)^{1/2}$$

for all $x\in(e_nA^{**}e_n)$ and $\|x\|\leqslant 1$. Hence $f(hx+xh)/2$ is a bounded linear functio on $(e_nA^{**}e_n)$. Since it is also normal, we conclude that $S\subset[(e_nA^{**}e_n)_*]_{s.a.}$. Supp that $h_n\in A^{**}_{s.a.}$, $h_n\to h$ weakly, we may also assume that $\|h_n-h\|\leqslant 1$. Then for every $e_nA^{**}e_n$, as in the proof of Theorem 1.6,

$$f((h_n-h)x+x(h_n-h))\to 0.$$

Since the unit ball of $A^{**}_{s.a.}$ is convex and weakly compact, we conclude that S convex and compact.

Let $\phi|_n=\phi|_{e_nA^{**}e_n}$. We have

$$|\phi_n(x)|\leqslant L^{1/2}f(e_n)^{1/2} \text{ for all } x\in(e_nA^{**}e_n)$$

and $\|x\|\leqslant 1$. So ϕ_n is a bounded linear functional on $e_nA^{**}e_n$. Suppose that

$$x_\alpha,\ x\in(e_nA^{**}e_n)_{s.a.}$$

such that $x_\alpha\nearrow x$. We may also assume that $\|x_\alpha-x\|\leqslant 1$. For every $h\in A^{**}_{s.a.}$, $\|h\|$ since $x-x_\alpha\geqslant 0$,

$$\begin{aligned}
&|f(h(x-x_\alpha))|\\
&\leqslant f(h(x-x_\alpha)h)^{1/2}f((x-x_\alpha))^{1/2}\\
&=f(h(x-x_\alpha)^{1/2}e_n(x-x_\alpha)^{1/2}h)f((x-x_\alpha))^{1/2}\\
&\leqslant L^{1/2}f((x-x_\alpha))^{1/2}.
\end{aligned}$$

Hence

$$|\phi_n((x-x_\alpha))|\leqslant L^{1/2}f((x-x_\alpha))^{1/2}\to 0,$$

since f is normal. We conclude that

$$\phi_n \in [(e_n A^{**} e_n)_*]_{s.a.}$$

Now, if $\phi_n \notin S$, then by the Hahn-Banach theorem there is an element x in $(e_n A^{**} e_n)_*]^*_{s.a.}$ $(=(e_n A^{**} e_n)_{s.a.})$ and a real number t such that $\phi_n(x) > t \geq S(a)$. Since $= -S$, we conclude that

$$\phi_n(x) > \sup\{|f(hx+xh)|/2 | h \in {}^{**}_{s.a.}\ \|h\| \leq 1\},$$

contradiction. Thus there is $h_n \in A^{**}_{s.a.}$, and $\|h_n\| \leq 1$ such that

$$\phi(x) = \phi_n(x) = f(h_n x + x h_n)/2$$

r all

$$x \in (e_n A^{**} e_n).$$

y the proof of Theorem 1.6, there is $h \in A^{**}_{s.a.}$ and $\|h\| \leq 1$ such that.

$$\phi(x) = f(hx+xh)/2$$

r all $x \in C_{00}(A)$.

Hence Φ_3 is onto.

By the definition of N^1, we have ker $\Phi_3 = N^1$. If we use the same notation Φ_3 for ie composition of Φ_3 and the quotient map from $A^{**}_{s.a.}$ onto $(L_\infty)_{s.a.}$, we have that Φ_3 one to one and onto. By open mapping theorem, $\|\Phi_3^{-1}\| < \infty$. We complete our proof.

The following corollary is a stronger version of Theorem 1.6.

Corollary 2.7. *Let f be a unitary bounded (may not be bounded) positive linear nctional defined on $C_{00}(A)$, g a selfadjoint linear functional defined on $C_{00}(A)$. If $(x)| \leq \sup\{|f(hx+xh)|/2|\ h \in A^{**}_{s.a.},\ \|h\| \leq 1\}$ for all $x \in C_{00}(A)_{s.a.}$ then there is $h \in {}^{**}_{s.a.}$ such that $\|h\| \leq 1$ and*

$$g(x) = f(hx+xh)/2 \text{ for all } x \in C_{00}(A).$$

ie proof is the key part of the proof of Theorem 2.6.

Remark. If f is a trace, then $L^1 = \bar{L}^1$. Moreover $\Phi_1^{-1} = \Phi_3$ will be an isometric omorphism.

We notice that both the definitions of L^1 and $\bar{L}^1$ depend on $C_{00}(A)$. If f is iitarily bounded, we can extend $\|\cdot\|_1$ and $\|\cdot\|^1$ to the minimal dense ideal I as llowing. If $x \in I_{s.a.}$,

$$\|x\|_1 = \sup\{|f(hx+xh)|/2 | \bar{h} \in \Phi_1((L^1)^*_{s.a.}),\ \|h\| \leq 1\},$$

$$\|x\|^1 = \sup\{|f(hx+xh)1/2 | h \in A^{**}_{s.a.},\ \|h\| \leq 1\}.$$

ie following theorem shows that both L^1 and $\bar{L}^1$ contain I no matter how different $_{00}(A)$ may be.

Theorem 2.8. *For every element $x \in I$, there are $x_n \in C_{00}(A)$ such that*

$$\|x_n - x\|_1 \to 0 \text{ and } \|x_n - x\|^1 \to 0.$$

Remark. L^1 spaces for von Neumann algebras associated with a faithful ormal weight have been studied (e. g. [2], and [10]). The faithfulness of the normal weights plays an important rule in [2] and [10]. Since abelian von Neumann algebras are special cases of $C(X)$, it is probably more natural to consider integrals initially defined on C^*-algebras. Notice also that even if a σ-finite integral f is faithful on $C_{00}(A)_+$, it may not be faithful on $\pi_f(A)^{\overline{w}}$.

§ 3. The C^*-Algebras $M_0(A)$, $B_0(A)$ and Other Non-Commutative Measure Theorems

Definition 3.1. *Let M be a von Neumann algebra, f a weight on M. Suppose x_n, $x \in M$. We say $\{x_n\}$ f-converges to x, if for every $\varepsilon > 0$, $\sigma > 0$ and a projection $p \in$ there are projections $q_n \in M$, $q_n \leqslant p$ and an integer N such that*

$$\|(x_n - x)q_n\| < \sigma \text{ and } f(p - q_n) < \varepsilon$$

for all $n \geqslant N$.

Let $\chi_\sigma(t)$ be the characteristic function of the set $\{t: |t| \geqslant \sigma\}$.

Proposition 3.2. *Suppose that f is bounded. If x_n f-converges to x, then for eve $\sigma > 0$,*

$$f(\chi_\sigma(|x_n - x|)) \to 0 \text{ as } n \to \infty.$$

Proposition 3.3. *Suppose that x_n f-converges to x and f is normal. Then th exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and projections $p_i \in M$ such that $p_i \leqslant p_{i+1}$, $f(1-p_i)$ - and $\|(x_n - x)p_i\| \to 0$ as $k \to \infty$ for each i.*

Proof By the definition of f-convergence, we have a subsequence $\{x_k^{(1)}\} \subset \{$ and projections $q_k^{(1)} \in M$ subh that $q_{k+1}^{(1)} \leqslant q_k^{(1)}$, $f(1 - q_1^{(1)}) < 1/2$, $f(q_k^{(1)} - q_{k+1}^{(1)}) < (1/2)$ and $\|(x_k^{(1)} - x)q_k^{(1)}\| < 1/k$. Let $q^{(1)} = S - \lim q_k^{(1)}$. Since f is normal, $f(1 - q^{(1)}) <$ Moreover $\|(x_k^{(1)} - x)q^{(1)}\| \to 0$.

We also have a subsequence $\{x_k^{(2)}\} \subset \{x_k^{(1)}\}$, a projection $q^{(2)} \in M$ such t $f(1 - q^{(1)} - q^{(2)}) < 1/2$, $q^{(2)} \leqslant 1 - q^{(1)}$ and $\|x_k^{(2)} - xq^{(2)}\| \to 0$. By induction, there projections

$$q^{(n)} \in M,\ q^{(n)} \leqslant 1 - \sum_{i=1}^{n-1} q^{(i)},$$

$$f\left(1 - \sum_{i=1}^{n} q^{(i)}\right) < 1/2^n \text{ and } \{x_k^{(n)}\} \subset \{x_k^{(n-1)}\}$$

such that $\|(x_k^{(n)} - x)q^{(n)}\| \to 0$. Take

$$x_{n_1} = x_1^{(1)},\ x_{n_2} = x_2^{(2)},\ \cdots,\ p_i = \sum_{n=1}^{i} q^{(n)}.$$

We have $f(1 - p_i) \to 0$ and $\|(x_{n_k} - x)p_i\| \to 0$ for every i.

Corollary 3.4. *Suppose that f is normal, x_n f-converges to x and $\{x_n\}$ is bound Then there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and a projection $p \in M$ such that $x_{n_k} p \to$ strongly and $f(1-p) = 0$.*

Corollary 3.5. *Let M be a von Neumann algebra, f a faithful normal weight M. Suppose x_n, $x \in M$ and $\|x_n\|$ is bounded such that x_n f-converges to x. Then x_n converges to x strongly.*

The following is a non-commutative version of Lebesque's Dominant nvergence Theorem

Theorem 3.6. *Let M be a von Neumann algebra, f a normal weight on M. ppose that x_n, $x\in M$, $y\in M_+$ such that $f(y)<+\infty$ and $|x_n-x|\leqslant y$ for all n. Then*

$$f(|x_n-x|)\to 0 \text{ as } n\to\infty,$$

one of the following holds

(1) *x_n converges to x strongly,*

(2) *$|x_n-x|\to 0$ weakly,*

(3) *x_n f-converges to x,*

(4) *x_n converges to x in the sence of Proposition 3.2.*

Remark. If a weight has the convergence property stated in Theorem 3.5, then $x\in M_+$, $x_n\nearrow x$ and $f(x)<\infty$ imply $f(x_n)\nearrow f(x)$.

Let X be a locally compact Hausdorff space. We say a function f defined on X vanishing at infinity if, for every $\varepsilon>0$, there is a compact subset C of X such it $|f(t)|<\varepsilon$ if $t\notin C$. Let A be a σ-unital C^*-algebra (without unit), $\{e_n\}$ be the ments defined by a strictly positive element a as in the section 1. Let $M_0(A)$ be norm closure of $\bigcup_{n=1}^{\infty} e_nA^{**}e_n$, $B_0(A)$ be the norm closure of $\bigcup_{n=1}^{\infty} e_nBe_n$, where B is enveloping Borel *-algebra of A. Notice that since A is σ-unital, $1\in B$. viously, $B_0(A)$ is an analogue of the bounded Borel functions vanishing at inity. Corollary 3.9 and Theorem 3.11 will convince us of that. Unlike L^1 which pend on $C_{00}(A)$, $M_0(A)$ and $B_0(A)$ do not depend on the choices of $C_{00}(A)$.

Theorem 3.7. *$M_0(A)$ (respectively $B_0(A)$) is the smallest hereditary C^*-algebra of A^{**} (respectively B) containing A.*

Proof Let $x\in B_+$, $y\in B_0(A)$ such that $x\leqslant y$. We may assume that $\|y\|\leqslant 1$ and re is an element $z\in B$ such that $z^*z=x$. By [7, 1. 4. 5], there is $u\in B$, $\|u\|\leqslant 1$ and $\alpha<1/2$ such that $z=uy^\alpha$. Therefore $x=y^\alpha u^*uy^\alpha$. There are $y_n\in \bigcup_{n=1}^{\infty} e_nBe_n$ such that $\|y_n-y\alpha\|\to 0$, as $n\to\infty$. Clearly $y_n\, u^*uy_n\in B_0(A)$. Hence $x=y^\alpha u^*uy^\alpha$ is in $A_0(A)$.

Corollary 3.8. $M_0(A)$ and $B_0(A)$ do not depend on $\{e_n\}$.

Corollary 3.9. *Let f be a bounded Borel function on R such that $\lim_{t\to 0} f(t)=0$. en $f(x)\in B_0(A)$ (respectively $M_0(A)$) for all $x\in B_0(A)_{s.a.}$ (respectively $M_0(A)_{s.a.}$).*

Theorem 3.10. *Let A be a σ*-unital C^*-algebsa, *$x\in M_0(A)$. For every σ-finite egral f defined on some $C_{00}(A)$, every $\varepsilon>0$, $\sigma>0$ and a projection*

$$p\in \mathbb{C}1+\bigcup_{n=1}^{\infty} e_nA^{**}e_n,$$

there are projection

$$p_0 \in \mathbf{C}1 + \bigcup_{n=1}^{\infty} e_n A^{**} e_n,$$

$p_0 \leqslant p$ and $y \in C_{00}(A)$ such that

$$\|(x-y)p_0\| < \sigma, \ \|y\| \leqslant \|xp\| \text{ and } f(p-p_0) < \varepsilon.$$

Moreover, there exist a projection $q_0 \leqslant p$ in A^{**} and a sequence $y_n \in C_{00}(A)$ such that

$$\|(y_n - x)q_0\| \to 0 \text{ as } n \to \infty, \ f(p-q_0) < \varepsilon.$$

Proof For every $\sigma > 0$, there is

$$y \in \bigcup_{n=1}^{\infty} e_n A^{**} e_n$$

such that $\|y\| \leqslant \|xp\|$ and

$$\|(x-y)p\| < \sigma/2.$$

Suppose $y \in e_m A^{**} e_m$. Since

$$p \in \mathbf{C}1 + \bigcup_{n=1}^{\infty} e_n A^{**} e_n,$$

$p = 1 - p_1$, where

$$p_1 \in \bigcup_{n=1}^{\infty} e_n A^{**} e_n.$$

We may assume that $p_1 \in e_m A^{**} e_m$. Let $p_2 = e_m - p_1$. So $f(p_2) < \infty$. Since $C_{00}(A$ norm dense in A, by [9, Corollary 4.14], there exist $p \in e_m A^{**} e_m$, $p \leqslant p_2$ and $z \in C_{00}$ such that

$$\|(y-z)p'\| < \sigma/2, \ \|z\| \leqslant \|yp_2\| \text{ and } f(p_2 - p') < \varepsilon.$$

Let $p_0 = (1 - e_m) + p$, then $f(p - p_0) = f(p_2 - p') < \varepsilon$. We have

$$\|(x-z)p_0\| \leqslant \|(x-y)p_0\| + \|(y-z)p_0\| < \sigma/2 + \|(y-z)p\| < \sigma,$$

so we complete the first part of our proof.

By the first part and induction, we can find a decreasing sequence $\{q_n$ projections in

$$\mathbf{C}1 + \bigcup_{n=1}^{\infty} e_n A^{**} e_n$$

and a sequence $\{y_n\}$ in $C_{00}(A)$ with the properties

$$\|(x-y_n)q_n\| < 1/n \text{ and } f(q_n - p_n) < (1/2)^n \varepsilon,$$

$n = 1, 2, \cdots$.

Let $q_0 = s - \lim q_n$. Then we get

$$\|(x-y_n)q_n\| < 1/n \text{ and } f(p-q_0) < \varepsilon.$$

We complete our proof.

Theorem 3.11. (Generalized Lusin's Theorem) *Let A be σ-unital C^*-alg Take an arbitary σ-finite integral f defined on some $C_{00}(A)$, projection*

$$p \in \mathbf{C}1 + \bigcup_{n=1}^{\infty} e_n A^{**} e_n$$

$\varepsilon > 0$ and $\sigma > 0$. Then for every $x \in M_0(A)$, there exist a projection $p_0 \leqslant p$ in A^{**} and $y \in A$ such that

$$xp_0=yp_0,\ f(p-p_0)<\varepsilon \text{ and } \|y\|\leqslant(1+\sigma)\|xp_0\|.$$

Proof Notice that for every

$$z\in\mathbf{C}1+\bigcup_{n=1}^{\infty} e_nA^{**}e_n,$$

, $xz\in M_0(A)$. Now we can use Theorem 3.10 to prove Theorem 3.11 as in [9, heorem 4.10].

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