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THE HEAT KERNEL OF A BALL IN $\mathbf{C}^n$

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Abstract

By introducing the horosphere coordinate of a unit ball B^n in $\mathbf{C}^n$ and an integral transformation formula of functions in such coordidates, the author constructs the heat kernel $H_{B^n}(z, w, t)$ of the heat equation associated to the Bergman metric of B^n. That is

$$H_{B^n}(z, w, t)=c_n\left(-\frac{1}{\pi}\right)^n \frac{e^{-n^2 t}}{\sqrt{t}}\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1} e^{-\frac{\sigma^2}{4t}}\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2 r(z,w)+\tau^2} d\tau,$$

where c_n is a well-defined constant and $r(z, w)$ is the geodesic destance of two points z and w of B^n and $t\in\mathbf{R}^+$. Since

$$H_{B^m\times B^n}=H_{B^m}\cdot H_{B^n},$$

then

$$G((z_1, z_2), (w_1, w_2))=-\int_0^{\infty}H_{B^m}(z_1, w_1, t)H_{B^n}(z_2, w_2, t)dt$$

is the Green function of the topological product space $B^m\times B^n$.

Let

$$B^n=\{z=(z^1, \cdots, z^n)\in\mathbf{C}^n \mid |z|^2:=|z^1|^2+\cdots+|z^n|^2<1\}$$

be the unit ball in $\mathbf{C}^n$ and

$$ds^2=\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar\beta}(z)dz^\alpha d\bar z^\beta, \text{ where } h_{\alpha\bar\beta}(z)=\frac{\delta_{\alpha\beta}}{1-|z|^2}+\frac{\bar z^\alpha z^\beta}{(1-|z|^2)^2}, \tag{1}$$

the invariant differential metric of B^n (that is, Bergman metric of B^n after multiplying the factor $(n+1)$ to the former). Now the corresponding Laplace-Beltrami operator is

$$\Delta=4\sum_{\alpha,\beta=1}^{n} h^{\bar\beta\alpha}(z)\frac{\partial^2}{\partial z^\alpha\partial\bar z^\beta}, \text{ where } h^{\bar\beta\alpha}(z)=(1-|z|^2)(\delta_{\alpha\beta}-z^{\bar\beta}z^\alpha), \tag{2}$$

The purpose of this paper is to construct the heat kernel $H(z, w, t)$ of B^n, that is, to find a function $H(z, w, t)$ with $(z, w, t)\in B^n\times B^n\times\mathbf{R}^+$ such that for any continuous and bounded function $\varphi(z)$ in B^n the integral

$$\Phi(w, t)=\int_{B^n} H(z, w, t)\varphi(z)\dot z, \tag{3}$$

where $\dot z$ is the invariant volume element of B^n, is the solution of the heat equation

$$\frac{\partial\Phi}{\partial t}=\Delta\Phi \tag{4}$$

and satisfies the initial condition

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$$\lim_{t\to 0}\Phi(w,\ t)=\varphi(w). \tag{5}$$

The main result is

Theorem. *The heat kernel of the heat equution (4) in B^n is*

$$H(z,\ w,\ t)=C_n\left(-\frac{1}{\pi}\right)^n\frac{1}{\sqrt{t}}e^{-n^2t}\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}e^{-\frac{\sigma^2}{4t}}\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2r(z,\,w)+\tau^2}d\tau, \tag{6}$$

where

$$C_n=\frac{(2n)!}{2^{3n-2}(n-1)!\,n!\sqrt{\pi}} \tag{7}$$

and $r(z,\ w)$ is the geodesic distance of the metric (1) between the points z and w of B. Moreover, it can be written into the form

$$H(z,\ w,\ t)=\frac{C_n}{2t^n\sqrt{t}}e^{-n^2t-r^2(z,\,w)/(4t)}\cdot\sum_{k=0}^{n-1}(-2t)^k\int_0^{\infty}\frac{f_{n,n-k}(\tau+r(z,\ w))\mathrm{sh}(\tau+r(z,\ w))}{[\mathrm{ch}^2(\tau+r(z,\ w))-\mathrm{ch}^2r(z,\ w)]^{1/2}}e^{-(\tau^2+2\tau r(z,\,w))/(4t)}d\tau, \tag{8}$$

where $f_{n,n-k}(\sigma)$ $(k=1,\ 2,\cdots n)$ are defined by the folliwing recurrent formulae $(n=1,\ 2,\ \cdots)$

$$\begin{cases} f_{1,1}(\sigma)=\dfrac{\sigma}{\mathrm{sh}\sigma}, \\ f_{n+1,n+1}(\sigma)=\dfrac{\sigma}{\mathrm{sh}\sigma}f_{n,n}(\sigma),\ f_{n+1,1}(\sigma)=\dfrac{1}{\mathrm{sh}\sigma}\dfrac{\partial}{\partial\sigma}f_{n,1}(\sigma) \\ f_{n+1,k+1}(\sigma)=\dfrac{\sigma}{\mathrm{sh}\sigma}f_{n,k}(\sigma)+\dfrac{1}{\mathrm{sh}\sigma}\dfrac{\partial}{\partial\sigma}f_{n,k+1}(\sigma). \\ \quad k=1,\ \cdots,\ n+1. \end{cases} \tag{9}$$

The theorem has been proved for $n=1$ (see [1]). The main points of the proof for $n\geqslant 2$ are the generalization of the horocycle coordinate used in [1] to the horosphere coordinate and that of the integral tansformation formulas (2) and (3) in [1] to higher dimensions.

§1. The Geodesic Coordinates and Horosphere Coordinates

It is known[2] that the geodesic distance with respect to the metric (1) between the points z and w of B^n is

$$r(z,\ w)=\frac{1}{2}\log\frac{1+[(z-w)(I-\bar{w}'z)^{-1}\overline{(z-w)'}/(1-w\bar{z}')]^{1/2}}{1-[(z-w)(I-\bar{w}'z)^{-1}\overline{(z-w)'}/(1-w\bar{z}')]^{1/2}}, \tag{1.1}$$

where we regard a point $z=(z^1,\ \cdots,\ z^n)$ in $\mathbb{C}^n$ at the same time as a $1\times n$ matrix and denote by A' the transposed matrix of a matrix A. It is easy to see

$$\mathrm{th}^2 r(z, w) = \frac{(z-w)(I-\bar{w}'z)^{-1}\overline{(z-w)'}}{1-w\bar{z}'} = 1 - \frac{(L-z\bar{z}')(1-w\bar{w}')}{|1-w\bar{z}'|^2}. \tag{1.2}$$

When $w=0$, denote

$$r = r(z, 0) = \frac{1}{2}\log\frac{1+\sqrt{z\bar{z}'}}{1-\sqrt{z\bar{z}'}}, \tag{1.3}$$

the geodesic distance from 0 to $z \in B^n$. Then

$$z = \sqrt{z\bar{z}'}\ \frac{z}{\sqrt{z\bar{z}'}} = \mathrm{th} r \cdot u,\ u\bar{u}' = 1, \tag{1.4}$$

where (r, u) is called the geodesic coordinrte of the point $z \in B^n$. Since the components of $u=(u^1, \cdots, u^n)$are not independent but depent on only $2n-1$ real parameters, we should, for the precise definition of geodesic coordinate, find out these $2n-1$ parameters. Let

$$u^j = \rho_j e^{i\theta_j}$$

with real θ_j and $\rho_j(\geqslant 0)$and $\rho_1^2 + \cdots + \rho_n^2 = 1$, or (when $n \geqslant 2$)

$$\rho_1 = \cos\varphi_1,\ \rho_2 = \sin\varphi_1\cos\varphi_2,\ \cdots,\ \rho_{n-1} = \sin\varphi_1\cdots\sin\varphi_{n-2}\cos\varphi_{n-1},\ \rho_n = \sin\varphi_1\cdots\sin\varphi_{n-1}, \tag{1.6}$$

where

$$0 \leqslant \varphi_j \leqslant \pi/2 \quad (j=1, \cdots, n-1).$$

Lemma 1. 1. *The invariant metric in the geodesic coordinate is*

$$ds^2 = dr^2 + \mathrm{sh}^2 r \sum_{j,k=1}^{n} [\rho_j^2\delta_{jk} + \mathrm{sh}^2 r \rho_j^2\rho_k^2] d\theta_j d\theta_k$$
$$+ \mathrm{sh}^2 r [d\varphi_1^2 + \sin^2\varphi_1 d\varphi_2^2 + \cdots + \sin^2\varphi_1\cdots\sin^2\varphi_{n-l} d\varphi_{n-1}^2]$$

and the inverse matrix$(g^{jk})$of the metric tensor is

$$(g^{jk})_{1\leqslant j,k\leqslant 2n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\mathrm{sh}^2 r}\left(\dfrac{\delta_{jk}}{\rho_j^2} - \mathrm{th}^2 r\right)_{1\leqslant j,k\leqslant n} & 0 \\ 0 & 0 & \Theta^{-1} \end{pmatrix},$$

where we set $x^1 = r,\ x^2 = \theta_1,\ \cdots,\ x^{n+1} = \theta_n,\ x^{n+2} = \varphi_1,\ \cdots,\ x^{2n} = \varphi_{n-1}$ *and* $\rho_1, \cdots, \rho_n$ *are defined by*(1.6)*and*

$$\Theta^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \sin^{-2}\varphi_1 & \cdots & 0 \\ & & \cdots\cdots\cdots & \\ 0 & 0 & \cdots & [\sin^2\varphi_1\cdots\sin^2\varphi_{n-2}]^{-2} \end{pmatrix}.$$

Hence

$$g := [\det(g^{jk})]^{-1} = \mathrm{sh}^{4n-2} r \,\mathrm{ch} r\, g_0,$$

where

$$g_0 = \rho_1^2\cdots\rho_n^2 \sin^{2(n-1)}\varphi_1\cdots\sin^2\varphi_{n-2} = \cos^2\varphi_1\cdots\cos^2\varphi_{n-1}\sin^{4(n-1)}\varphi_1\cdots\sin^2\varphi_{n-1} \tag{1.7}$$

and the Laplace-Beltrami operator becomes

$$\Delta=\frac{1}{\sqrt{g}}\sum_{j,k=1}^{2n}\frac{\partial}{\partial x^j}\left(\sqrt{g}\,g^{jk}\frac{\partial}{\partial x^k}\right)$$
$$=\frac{1}{\text{sh}^{2n-1}r\text{ch}r}\frac{\partial}{\partial r}\left(\text{sh}^{2n-1}r\text{ch}r\frac{\partial}{\partial r}\right)+\frac{1}{\text{sh}^{2r}}\sum_{j,k=1}^{n}\left(\frac{\delta_{jk}}{\rho_j\rho_k}-\text{th}^2r\right)\frac{\partial^2}{\partial\theta_j\partial\theta_k}$$
$$+\frac{1}{\sqrt{g_0}}\frac{1}{\text{sh}^2r}\sum_{j=1}^{n-1}\frac{\partial}{\partial\varphi_j}\left(\frac{\sqrt{g_0}}{\sin^2\varphi_1\cdots\sin^2\varphi_{j-1}}\frac{\partial}{\partial\varphi_j}\right). \tag{1.8}$$

Proof Differentiating (1.4) we have

$$dz=\frac{1}{\text{ch}^2r}dru+\text{th}r\,du=\left(\frac{dr}{\text{ch}^2r},\ \text{th}r\,du\right)\begin{pmatrix}u\\ I\end{pmatrix}.$$

According to (1),

$$ds^2=(1-z\bar{z}')^{-1}dz(I-\bar{z}'z)^{-1}d\bar{z}'$$
$$=\text{ch}^2r\left(\frac{dr}{\text{ch}^2r},\ \text{th}r\,du\right)\begin{pmatrix}u\\ I\end{pmatrix}(I-\text{th}^2r\bar{u}'u)^{-1}(\bar{u}',I)\overline{\left(\frac{dr}{\text{ch}^2r},\ \text{th}r\,du\right)}'$$
$$=\text{ch}^2r\left(\frac{dr}{\text{ch}^2r},\ \text{th}r\,du\right)\begin{pmatrix}\text{ch}^2r & \text{ch}^2ru\\ \text{ch}^2r\bar{u}' & (I-\text{th}^2r\bar{u}'u)^{-1}\end{pmatrix}\overline{\left(\frac{dr}{\text{ch}^2r},\ \text{th}r\,du\right)}'$$
$$=dr^2+\text{sh}x\text{ch}r\,dr(u\,d\bar{u}'+du\bar{u}')+\text{sh}^2r\,du(I-\text{th}^2r\bar{u}'u)^{-1}\overline{du}'.$$

Notice that $ud\bar{u}'+du\bar{u}'=0$ and

$$(I-\text{th}^2x\bar{u}'u)^{-1}=I+\text{sh}^2r\bar{u}'u.$$

We have

$$ds^2=dr^2+\text{sh}^2r(du\,\overline{du}'+\text{sh}^2r\,|du\,\bar{u}'|^2).$$

Since

$$du\,\overline{du}'=\sum_{j=1}^{n}d\rho_j^2+\sum_{j=1}^{r}\rho_j^2\,d\theta_j^2$$

and

$$|du\bar{u}'|^2=\left|\sum_{j=1}^{n}(\rho_j\,d\rho_j+i\rho_j^2\,d\theta_j)\right|^2=\left(\sum_{j=1}^{n}\rho_j^2\,d\theta_j\right)^2,$$

we obtain

$$ds^2=dr^2+\text{sh}^2r\sum_{j,k=1}^{n}(\rho_j^2\delta_{jk}+\text{sh}^2r\rho_j^2\rho_k^2)d\theta_j\,d\theta_k+\text{sh}^2r\sum_{j=1}^{n}d\rho_j^2.$$

Substituting (1.6) into the last term of the above expression, we obtain ds^2 in geodesic coordinate. Let $\rho=(\rho_1,\ \rho_2,\ \cdots,\ \rho_n)$ and

$$\Lambda=\begin{pmatrix}\rho_1 & & \\ & \ddots & \\ & & \rho_n\end{pmatrix}.$$

Then tho matrix

$$(\rho_j^2\delta_{jk}+\text{sh}^2r\rho_j^2\rho_k^2)_{1\le j,k\le n}=\Lambda^2+\text{sh}^2\Lambda\rho'\rho\Lambda=\Lambda(I+\text{sh}^2r\rho'\rho)\Lambda.$$

Hence

$$(\rho_j^2\delta_{jk}+\text{sh}^2r\rho_j^2\rho_k^2)^{-1}_{1\le j,k\le n}=\Lambda^{-1}(I+\text{sh}^2r\rho'\rho)^{-1}\Lambda^{-1}=\Lambda^{-1}(I-\text{th}^2r\rho'\rho)\Lambda^{-1}$$
$$=\Lambda^{-2}-\text{th}^2r\Lambda^{-1}\rho'\rho\Lambda^{-1}=\left(\frac{\delta_{jk}}{\rho_j^2}-\text{th}^2r\right)_{1\le j,k\le n}.$$

The lemma is proved.

Now let $\sigma\in\mathbf{R}$, $\tau\in\mathbf{R}$, $u_2=(u^2,\cdots u^n)\in\mathbf{C}^{n-1}$ and $z_1=z^1$, $z=(z^2,\cdots,z^n)$. Set

$$z_1=\frac{(1+2u_2\bar{u}_2')e^{2\sigma}-1-2i\tau e^{2\sigma}}{(1+2u_2\bar{u}_2')e^{2\sigma}+1-2i\tau e^{2\sigma}},\quad z_2=\frac{2\sqrt{2}\,u_2e^{\sigma}}{(1+2u_2\bar{u}_2')e^{2\sigma}+1-2i\tau e^{2\sigma}},\tag{1.9}$$

where $(\sigma, u_2, \tau)\in\mathbf{R}\times\mathbf{C}^{n-1}\times\mathbf{R}$ is called horosphere coordinate. In fact,

$$\begin{cases}1-z_1=\dfrac{e^{-\sigma}}{\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma}-i\tau e^{\sigma}},\\ 1-z\bar{z}'=\dfrac{1}{(\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma}}\end{cases}.\tag{1.10}$$

Hence

$$\frac{|1-z_1|^2}{1-z\bar{z}'}=e^{-2\sigma}\tag{1.11}$$

or

$$\left|z^1-\frac{1}{1+e^{-2\sigma}}\right|^2+\frac{e^{-2\sigma}}{1+e^{-2\sigma}}(|z^2|^2+\cdots+|z^n|^2)=\left(\frac{e^{-2\sigma}}{1+e^{-2\sigma}}\right)^2.\tag{1.12}$$

This shows that for a fixed σ, the locus of (1.9) is a elliptic sphere which lies inside B^n and is tangent to the boundary ∂B^n of B^n at the point $(1, 0, \cdots, 0)$. Moreover, by (1.10) we see that the inverse transform of (1.9) is

$$\begin{cases}\sigma=\dfrac{1}{2}\log[(1-z\bar{z}')/|1-z_1|^2],\\ u_2=\dfrac{1}{\sqrt{2}}z_2(1-z_1)^{-1}|1-z_1|(1-z\bar{z}')^{-\frac{1}{2}},\\ \tau=\dfrac{1}{2i}(z_1-\bar{z}_1)(1-z\bar{z}')^{-1}.\end{cases}\tag{1.13}$$

Denote

$$x=(x^1,\cdots,x^{2n})\in\mathbf{R}^{2n},\ x^1=\sigma,\ x_2=(x^2,\cdots,x^n),$$
$$y_2=(x^{n+1},\cdots,x^{2n-1}),\ x^{2n}=\tau \text{ such that } (x_2,y_2)=\left(\frac{u_2+\bar{u}_2}{2},\ \frac{u_2-\bar{u}_2}{2i}\right).\tag{1.14}$$

Then the metric ds^2 in (1) can be written into the horosphere coordinate such that

$$ds^2=\sum_{j,k=1}^{2n}g_{jk}(x)dx^jdx^k\tag{1.15}$$

which is a Riemann metric. We are to write out metric tensor g_{jk}.

Lemma 1.2. *Denote*

$$G=(g_{jk})_{1\le j,k\le 2n},$$

the matrix of the metric tensor g_{jk}. *Then*

$$G=AA',\tag{1.16}$$

where

$$A=\begin{pmatrix}1 & \sqrt{2}x_2 & \sqrt{2}y_2 & 2\tau\\ 0 & \sqrt{2}I^{(n-1)} & 0 & -2y_2'\\ 0 & 0 & \sqrt{2}I^{(n-1)} & 2x_2'\\ 0 & 0 & 0 & 1\end{pmatrix}.\tag{1.17}$$

Hence $g=\det G=2^{2n-2}$ *and*

$$G^{-1}=(g^{jk})_{1\le j,k\le 2n}=A'^{-1}A^{-1}$$

$$=\begin{pmatrix} 1 & -x_2 & -y_2 & -2\tau \\ -x_2' & \frac{1}{2}I+x_2'x_2 & x_2'y_2 & y_2'+2\tau x_2' \\ -y_2' & y_2'x_2 & \frac{1}{2}I+y_2'y_2 & -x_2'+2\tau y_2' \\ -2\tau & y_2+2\tau x_2 & -x_2+2\tau y_2 & 1+2x_2x_2'+2y_2y_2'+4\tau^2 \end{pmatrix} \tag{1.18}$$

and

$$\Delta=\frac{\partial^2}{\partial\sigma^2}+2n\frac{\partial}{\partial\sigma}-2\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\sigma}\right)-\sum_{j=2}^{2n-1}\frac{\partial}{\partial x^j}\left(x^j\frac{\partial}{\partial\sigma}\right)$$
$$+\sum_{j=2}^{2n}\sum_{k=1}^{n2}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial}{\partial x^k}\right). \tag{1.19}$$

Proof Denote

$$a=e^{2\sigma}+b\bar{b}'-2i\tau e^{2\sigma},\quad b=\sqrt{2}\,u_2e^{\sigma}. \tag{1.20}$$

Then (1.9) becomes

$$z_1=\frac{a-1}{a+1},\quad z_2=\frac{2b}{a+1}, \tag{1.21}$$

and we have

$$\frac{a+\bar{a}}{2}-b\bar{b}'=e^{2\sigma},\quad 1-z\bar{z}'=\frac{4e^{2\sigma}}{|a+1|^2}. \tag{1.22}$$

Then from (1.21)

$$dz=(dz_1,\ dz_2)=\left(\frac{2da}{(a+1)^2},\ 2\frac{(a+1)db-bda}{(a+1)^2}\right)$$
$$=\frac{2}{(a+1)^2}(da,\ db)\begin{pmatrix}1 & -b\\ 0 & (a+1)I\end{pmatrix}$$

and from (1)

$$ds^2=\frac{dz\,\overline{dz'}}{1-z\bar{z}'}+\frac{dz\bar{z}'z\,\overline{dz'}}{(1-z\bar{z}')^2},$$

where

$$\frac{dz\,\overline{dz'}}{1-z\bar{z}'}=\frac{e^{-2\sigma}}{|a+1|^2}(da,\ db)\begin{pmatrix}1+b\bar{b}' & -(\bar{a}+1)b\\ -(a+1)\bar{b}' & |a+1|^2I\end{pmatrix}\overline{(da,\ db)'}$$

and

$$\frac{dz\bar{z}'z\,\overline{dz'}}{(1-z\bar{z}')^2}$$
$$=\frac{e^{-4\sigma}}{4|a+1|^2}(da,\ db)\begin{pmatrix}\bar{a}-1-2b\bar{b}'\\ 2(a+1)\bar{b}'\end{pmatrix}(a-1-2b\bar{b}',\ 2(\bar{a}+1)b)\overline{(da,\ db)'}$$
$$=\frac{e^{-4\sigma}}{4|a+1|^2}(da,\ db)\begin{pmatrix}|a-1-2b\bar{b}'|^2 & 2(\bar{a}-1-2b\bar{b}')(\bar{a}+1)b\\ 2(a-1-2b\bar{b}')(a+1)\bar{b}' & 4|a+1|^2\bar{b}'b\end{pmatrix}\overline{(da,\ db)'}$$
$$=\frac{e^{-4\sigma}}{4|a+1|^2}(da,\ db)$$
$$\times\begin{pmatrix}|a+1|^2-4\left(\frac{a+\bar{a}}{2}-b\bar{b}'\right)(1+b\bar{b}') & -2|a+1|^2b+4\left(\frac{a+\bar{a}}{2}-b\bar{b}'\right)(\bar{a}+1)b\\ -2|a+1|^2\bar{b}'+4\left(\frac{a+\bar{a}}{2}-b\bar{b}'\right)(a+1)\bar{b}' & 4|a+1|^2\bar{b}'b\end{pmatrix}$$
$$\times\overline{(da,\ db)'}$$

$$=\frac{e^{-4\sigma}}{4|a+1|^2}(da,\ db)$$
$$\times\left[\begin{pmatrix}|a+1|^2 & -2|a+1|^2b\\ -2|a+1|^2\bar b' & 4|a+1|^2\bar b'b\end{pmatrix}+4e^{-2\sigma}\begin{pmatrix}-(1+b\bar b') & (\bar a+1)b\\ (a+1)\bar b' & 0\end{pmatrix}\right]\overline{(da,\ db)'}$$
$$=\frac{1}{4}e^{-4\sigma}(da,\ db)\left[\begin{pmatrix}1\\ -2\bar b'\end{pmatrix}(1,\ -2b)+\frac{e-e^{2\sigma}}{|a+1|^2}\begin{pmatrix}-(1+b\bar b') & (\bar a+1)b\\ (a+1)\bar b & 0\end{pmatrix}\right]\overline{(da,\ db)'}.$$

Hence

$$ds^2=\frac{1}{4}e^{-4\sigma}|da-2db\bar b'|^2+e^{-2\sigma}\,db\,\overline{db'}. \tag{1.23}$$

By (1.20) and (1.14) we have

$$db=\sqrt{2}\,e^{\sigma}[(x_2d\sigma+dx_2)+i(y_2d\sigma+dy_2)],$$
$$db\bar b'=2e^{2\sigma}[(x_2x_2'+y_2y_2')d\sigma+dx_2x_2'+dy_2y_2'+i(dy_2x_2'-dx_2y_2')],$$
$$da=2e^{2\sigma}[(1+2x_2x_2'+2y_2y_2')d\sigma+2(dx_2x_2'+dy_2y_2')-i(2\tau\,d\sigma+d\tau)],$$
$$da-2db\bar b'=2e^{2\sigma}[d\sigma-i(2\tau\,d\sigma+d\tau+2dy_2x_2'-2dx_2y_2')]$$

and

$$db\,\overline{db}'=2e^{2\sigma}[(x_2\,d\sigma+dx_2)(x_2\,d\sigma+dx_2)'+(y_2d\sigma+dy_2)(y_2d\sigma+dy_2)'].$$

Then (1.23) becomes

$$ds^2=d\sigma^2+2(x_2d\sigma+dx_2)(x_2d\sigma+dx_2)'+2(y_2d\sigma+dy_2)(y_2d\sigma+dy_2)'$$
$$+(2\tau d\sigma+d\tau+2dy_2x_2'-2dx_2y_2')^2$$
$$=(d\sigma,\ dx_2,\ dy_2,\ d\tau)AA'(d\sigma.\ dx_2,\ dy_2,\ d\tau)', \tag{1.24}$$

where A is defined in (1.17). Hence

$$A^{-1}=\begin{pmatrix}1 & -x_2 & -y_2 & -2\tau\\ 0 & \frac{1}{\sqrt{2}}I^{(n-1)} & 0 & \sqrt{2}\,y_2'\\ 0 & 0 & \frac{1}{\sqrt{2}}I^{(n-1)} & -\sqrt{2}\,x_2'\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{1.25}$$

and $G^{-1}=A'^{-1}A^{-1}$ can be obtained by direct calculation. This means that

$$\begin{cases}g^{11}=1,\ g^{1j}=g^{j1}=-x^j(j=2,\ \cdots,\ 2n-1),\ g^{1,2n}=g^{2n,1}=-2x^{2n},\\ \text{the other } g^{jk} \text{ are polynomials of } x^2\ \cdots,\ x^{2n} \text{ of degree 2.}\end{cases} \tag{1.26}$$

Hence

$$\Delta=\sum_{j,k=1}^{2n}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left(\sqrt{g}\,g^{jk}\frac{\partial}{\partial x^k}\right)$$
$$=\sum_{k=1}^{2n}\frac{\partial}{\partial x^1}\left(g^{1k}\frac{\partial}{\partial x^k}\right)+\sum_{j=2}^{2n}\sum_{k=1}^{n}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial}{\partial x^k}\right)$$
$$=\frac{\partial^2}{\partial\sigma^2}-\sum_{k=2}^{2n-1}x^k\frac{\partial^2}{\partial\sigma\partial x^k}-2\tau\frac{\partial^2}{\partial\sigma\partial\tau}+\sum_{j=2}^{2n}\sum_{k=1}^{2n}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial}{\partial x^k}\right)$$
$$=\frac{\partial^2}{\partial\sigma^2}+2n\frac{\partial}{\partial\sigma}-2\frac{\partial}{\partial\tau}\left(\tau\frac{\partial}{\partial\sigma}\right)-\sum_{k=2}^{2n-1}\frac{\partial}{\partial x^k}\left(x^k\frac{\partial}{\partial\sigma}\right)+\sum_{j=2}^{2n}\sum_{k=1}^{2n}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial}{\partial x^k}\right).$$

This proves the lemma.

§2. The Integral Transformations

At first we see that the coordinate transformation (1.9) can be written into

$$z_1=\frac{\operatorname{sh}\sigma+u_2\bar{u}_2'e^{\sigma}-i\tau e^{\sigma}}{\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma}-i\tau e^{\sigma}},\quad z_2=\frac{\sqrt{2}\,u_2}{\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma}-i\tau e^{\sigma}}. \tag{2.1}$$

Since $u_2=x_2+iy_2=(x^2,\ \cdots,\ x^n)+i(x^{n+1},\ \cdots x^{2n-1})$, we denote the Euclidean volume element by

$$\dot{u}_2=dx^2\cdots dx^{2n-1}. \tag{2.2}$$

Lemma 2.1. *Let $h(\xi)$ be an infinitely differentiable function of $\xi\in\mathbf{R}$ such that for any non-negative integer k and for any polynomial $p(\xi)$ of ξ*

$$\lim_{|\xi|\to\infty} p(\xi)\frac{\partial^k h(\xi)}{\partial\xi^k}=0. \tag{2.3}$$

Then the integral transformation

$$\begin{aligned} f(\sigma)&=e^{n\sigma}\int_{\mathbf{R}^{2n-1}} h((\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma})d\tau u_2\\ &=\int_{\mathbf{R}^{2n-1}} h((\operatorname{ch}\sigma+u_2\bar{u}_2')^2+\tau^2)d\tau\dot{u}_2 \end{aligned} \tag{2.4}$$

has an inverse transformation

$$h(\operatorname{ch}^2r)=\left(-\frac{1}{\pi}\right)^n\int_{-\infty}^{\infty}\left[\frac{1}{\operatorname{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}f(\sigma)\right]_{\operatorname{ch}^2\sigma=\operatorname{ch}^2r+\eta^2}d\eta, \tag{2.5}$$

that is

$$\left(-\frac{1}{\pi}\right)^n\int_{-\infty}^{\infty}\left[\frac{1}{\operatorname{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\int_{\mathbf{R}^{2n-1}}h((\operatorname{ch}\sigma+u_2\bar{u}_2')^2+\tau^2)d\tau\dot{u}_2\right]_{\operatorname{ch}^2\sigma=\operatorname{ch}^2r+\eta^2}d\eta,$$
$$=h(\operatorname{ch}^2r). \tag{2.6}$$

Proof Instroduce the polar coordinate $(\rho,\ \theta_1,\ \cdots,\ \theta_{2n-2})$ of $(x^2,\ \cdots,\ x^{2n-1})\in\mathbf{R}^{2n-1}$. Then

$$\dot{u}_2=\rho^{2n-3}\,d\rho\dot{\omega},$$

where $\dot{\omega}$ is the volume element of a unit sphere S^{2n-2}. Denote $\omega=2\pi^{n-1}/(n-2)!$ the volume of S^{2n-2}. Then

$$\begin{aligned} f(\sigma)&=\omega\int_{-\infty}^{\infty}d\tau\int_0^{\infty}h((\operatorname{ch}\sigma+\rho^2)^2+\tau^2)\rho^{2n-3}\,d\rho\\ &=\frac{\omega}{2}\int_{-\infty}^{\infty}d\tau\int_0^{\infty}h((\operatorname{ch}\sigma+\xi)^2+\tau^2)\xi^{n-2}\,d\xi. \end{aligned}$$

Hence

$$\begin{aligned} \left(\frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}f(\sigma)&=\frac{\omega}{2}\int_{-\infty}^{\infty}d\tau\int_0^{\infty}\xi^{n-2}\left(\frac{\partial}{\partial\operatorname{ch}\sigma}\right)^{n-1}h(\operatorname{ch}\sigma+\xi)^2+\tau^2)d\xi\\ &=\frac{\omega}{2}\int_{-\infty}^{\infty}d\tau\int_0^{\infty}\xi^{n-2}\left(\frac{\partial}{\partial\xi}\right)^{n-1}h((\operatorname{ch}\sigma+\xi)^2+\tau^2)d\xi\\ &=-\frac{\omega}{2}(n-2)\int_{-\infty}^{\infty}d\tau\int_0^{\infty}\xi^{n-2}\frac{\partial^{n-2}}{\partial\xi^{n-2}}h((\operatorname{ch}\sigma+\xi)^2+\tau^2)d\xi=\cdots\\ &=(-1)^{n-1}\frac{\omega}{2}(n-2)!\int_{-\infty}^{\infty}h(\operatorname{ch}^2\sigma+\tau^2)d\tau. \end{aligned}$$

Then

$$\frac{1}{\operatorname{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}f(\sigma)=(-\pi)^{n-1}\int_{-\infty}^{\infty}\frac{\partial}{\partial\operatorname{ch}^2\tau}f(\operatorname{ch}^2\sigma+\tau^2)d\tau$$

and

$$\int_{-\infty}^{+\infty}\left[\frac{1}{\operatorname{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}f(\sigma)\right]_{\operatorname{ch}\sigma=\operatorname{ch}^2r+\eta^2}d\eta$$

$$=(-\pi)^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\operatorname{ch}^2\sigma}h(\operatorname{ch}^2\sigma+\tau^2)\right]_{\operatorname{ch}^2\sigma=\operatorname{ch}^2r+\eta^2}d\tau\, d\eta$$

$$=(-\pi)^{n-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\xi}h(\xi)\right]_{\xi^2=\operatorname{ch}^2r+\eta^2+\tau^2}d\tau d\eta=2\pi(-\pi)^{n-1}\int_0^{\infty}\left[\frac{\partial h(\xi)}{\partial\xi}\right]_{\xi=\operatorname{ch}^2r+\rho^2}\rho d\rho$$

$$=\pi(-\pi)^{n-1}\int_0^{\infty}\left[\frac{\partial h(\xi)}{\partial\xi}\right]_{\xi^2=\operatorname{ch}^2r+\rho^2}d(\operatorname{ch}^2r+\rho^2)=\pi(-\pi)^{n-1}\int_0^{\infty}dh(\operatorname{ch}^2r+\rho^2)$$

$$=(-\pi)^n h(\operatorname{ch}^2r).$$

Thus the lemma is proved.

Lemma 2. 2. *Let $h(\xi)$ be a function of ξ satisfying the condition of Lemma 2.1. Then the integral*

$$f(\sigma)=e^{n\sigma}\int_{\mathbf{R}^{2n-1}}h((\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma})d\tau\dot{u}_2$$

satifies

$$\left(\frac{\partial^2}{\partial\sigma^2}-n^2\right)f(\sigma)=\int_{\mathbf{R}^{2n-1}}[\Delta h(\operatorname{ch}^2\xi)]_{\operatorname{ch}^2\xi=(\operatorname{ch}\sigma+u_2\bar{u}_2')^2+\tau^2}d\tau\dot{u}_2$$

where Δ is the Laplace-Beltrami operater.

Proof By(1.10)

$$h((\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma})=h\left(\frac{1}{1-z\bar{z}'}\right)=h(\operatorname{ch}^2\xi)$$

when $z=\operatorname{th}\xi u$. By Lemmas 1.1 and 1.2

$$[\Delta h(\operatorname{ch}^2\xi)]_{\operatorname{ch}^2\xi=(\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma}}$$

$$=\sum_{j,k=1}^{2n}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^j}\left[\sqrt{g}\,g^{jk}\frac{\partial}{\partial x^k}h((\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma})\right],$$

where $x=(x^1\ \cdots,\ x^{2n})$is the horosphere coordinate when σ, τ, u_2 are related to x by (1.14). By the assumption of $h(\xi)$,

$$\int_{\mathbf{R}}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial h}{\partial x^k}\right)dx^j=0,\ j,\ k=1,\ \cdots,\ 2n,$$

and by Lemma 1.2 again,

$$\int_{\mathbf{R}^{2n-1}}[\Delta h(\operatorname{ch}^2\xi)]_{\operatorname{ch}^2\xi=(\operatorname{ch}\sigma+u_2\bar{u}_2')^2+\tau^2}d\tau\dot{u}_2$$

$$=e^{n\sigma}\int_{\mathbf{R}^{2n-1}}[\Delta h(\operatorname{ch}^2\xi)]_{\operatorname{ch}^2\xi=(\operatorname{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma}}d\tau\dot{u}_2$$

$$=e^{n\sigma}\int_{\mathbf{R}^{2n-1}}\left[\frac{\partial^2h}{\partial\sigma^2}+2n\frac{\partial h}{\partial\sigma}-2\frac{\partial}{\partial\tau}\left(\tau\frac{\partial h}{\partial\sigma}\right)-\sum_{j=2}^{2n-1}\frac{\partial}{\partial x^j}\left(x^j\frac{\partial h}{\partial\sigma}\right)\right.$$

$$\left.+\sum_{j=2}^{2n}\sum_{k=1}^{2n}\frac{\partial}{\partial x^j}\left(g^{jk}\frac{\partial h}{\partial x^k}\right)\right]d\tau\, dx^2\cdots dx^{2n-1}$$

$$= e^{n\sigma}\int_{\mathbf{R}^{2n-1}}\left[\frac{\partial^2 h}{\partial\sigma^2}+2n\frac{\partial h}{\partial\sigma}\right]d\tau\, dx^2\cdots dx^{2n-1}$$
$$=\left(\frac{\partial^2}{\partial\sigma^2}-n^2\right)\int_{\mathbf{R}^{2n-1}} e^{n\sigma}h((\mathrm{ch}\sigma+u_2\bar{u}_2'e^{\sigma})^2+\tau^2e^{2\sigma})d\tau\dot{u}_2.$$

Lemma 2.2 is proved.

§ 3. The Proof of the Theorem

It is known[2] that any $T\in \mathrm{Aut}(B^n)$, the group of holomorphic automorphism of B^n, is of the form

$$T(z)=\frac{z-a}{1-z\bar{a}'}B(a)U,\ B(a)=\frac{1}{\sqrt{1-a\bar{a}'}}\left(I-\frac{1-\sqrt{1-a\bar{a}'}}{a\bar{a}'}\bar{a}'a\right), \tag{3.1}$$

wheae $a\in B^n$ and U is an $n\times n$ unitary matrix. Now we set

$$H(z,\ w,\ t)$$
$$=h(\mathrm{ch}^2 r(z,\ w),\ t)$$
$$=C_n\left(-\frac{1}{\pi}\right)^n\frac{1}{\sqrt{t}}e^{-n^2t}\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1} e^{-\sigma^2/(4t)}\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2 r(z,w)+\tau^2}d\tau, \tag{3.2}$$

where $r(z,\ w)$ is the geodesic distance defined by (1.1), which is invariant under $\mathrm{Act}(B^n)$ i. e., $r(T(w))=r(z,\ w)$. Hence

$$H(T(z),\ T(w),\ t)=H(z,\ w,\ t)\quad \text{for any } T\in\mathrm{Aut}(B^n). \tag{3.3}$$

For a fixed $w\in B^n$. we can choose T such that $T(w)=0$. Hence without loss of generality we can argue in what follows only for the special case $w=0$.

Let

$$f(\sigma,\ t)=\frac{1}{\sqrt{t}}e^{-n^2t-\sigma^2/(4t)} \tag{3.4}$$

which satisfies

$$\frac{\partial f}{\partial t}=\frac{\partial^2 f}{\partial\sigma^2}-n^2 f. \tag{3.5}$$

Then from(3.2), Lemma 2.2 and Lemma 2.1 for fixed t,

$$\frac{\partial H(z,\ 0,\ t)}{\partial t}$$
$$=C_n\left(-\frac{1}{\pi}\right)^n\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\frac{\partial f(\sigma,\ t)}{\partial t}\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2 r+\eta^2}d\eta$$
$$=C_n\left(-\frac{1}{\pi}\right)^n\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\left(\frac{\partial^2}{\partial\sigma^2}-n^2\right)f(\sigma,\ t)\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2 r+\eta^2}d\eta$$
$$=C_n\left(-\frac{1}{\pi}\right)^n\int_{-\infty}^{\infty}\left[\frac{1}{\mathrm{sh}2\sigma}\frac{\partial}{\partial\sigma}\left(\frac{1}{\mathrm{sh}\sigma}\frac{\partial}{\partial\sigma}\right)^{n-1}\right.$$
$$\left.\times\int_{\mathbf{R}^{2n-1}}(\Delta h(\mathrm{ch}^2\xi))_{\mathrm{ch}^2\xi=(\mathrm{ch}\,\sigma+u_2\bar{u}_2')^2+\tau^2}\,d\tau\,\dot{u}_2\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2 r+\eta^2}d\eta=\Delta h(\mathrm{ch}^2 r).$$

Since ds^2 is invariant under $\mathrm{Aut}(B^n)$, Δ is invariant under $\mathrm{Aut}(B^n)$, i. e., $\Delta_{T(z)}=\Delta_z$ for any $T\in\mathrm{Aut}(B^n)$. This shows that for $w\in B^n$,

$$\frac{\partial H(z, w, t)}{\partial t} = \Delta H(z, w, t). \tag{3.6}$$

For the proof that H is the heat kernel, it remains to prove that for any continuous and bounded function $\varphi(z)$ in B^n,

$$\lim_{t\to 0}\int_{B^n} H(z, w, t)\varphi(z)\dot{z} = \varphi(w), \tag{3.7}$$

where

$$\dot{z} = \left(\frac{i}{2}\right)^n \frac{dz^1\wedge d\bar{z}^1\wedge\cdots\wedge dz^n\wedge d\bar{z}^n}{(1-z\bar{z}')^{n+1}} \tag{3.8}$$

is the invariant volume element of B^n. At first we prove

Lemma 3.1. *Let* $X = \frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}$ *and* $A(\sigma) = \sigma/\operatorname{sh}\sigma$ *and* $f(\sigma, t)$ *is defined by* (3.4) *Then*

$$X^n f = \sum_{k=0}^{n-1}\left(-\frac{1}{2t}\right)^{n-k} f_{n,n-k}(\sigma) f, \tag{3.9}$$

where $f_{n,n-k}(\sigma)$ *are defined by the following recurrent formula* $(n=1,2,\cdots)$

$$\begin{cases} f_{11}(\sigma) = A(\sigma), \\ f_{n+1,n+1}(\sigma) = A(\sigma) f_{n,n}(\sigma),\ f_{n+1,1}(\sigma) = X f_{n,1}(\sigma), \\ f_{n+1,k+1}(\sigma) = A(\sigma) f_{n,k}(\sigma) + X f_{n,k}(\sigma),\ k=1, \cdots, n-1. \end{cases} \tag{3.10}$$

Hence $f_{n,k}(\sigma)$ *are polynomials of* $X^j A(\sigma)$ $(j=0, \cdots, n-1)$ *and are bounded functions of* $\sigma\in\mathbf{R}^+$.

Proof Since $Xf(\sigma, t) = \left(-\frac{1}{2t}\right) A(\sigma) f(\sigma, t)$, we have from (3.9) by inductive assumption that

$$\begin{aligned} X^{n+1} f &= \left[\sum_{k=0}^{n-1}\left(-\frac{1}{2t}\right)^{n-k+1} A f_{n,n-k} + \sum_{k=0}^{n-1}\left(-\frac{1}{2t}\right)^{n-k} X f_{n,n-k}\right] f \\ &= \left[\left(-\frac{1}{2t}\right)^{n+1} A f_{n,n} + \sum_{k=1}^{n-1}\left(-\frac{1}{2t}\right)^{n+1-k}(A f_{n,n-k} + X f_{n,n-k}) + \left(-\frac{1}{2t}\right) X f_{n,1}\right] f \\ &= \left[\sum_{k=0}^{n}\left(-\frac{1}{2t}\right)^{n+1-k} f_{n+1,n+1-k}\right] f. \end{aligned}$$

Dividing both sides by f and comparing the coefficients of $(-1/2t)^{n+1-k}$ we obtain (3.10). By this recurrent formula we know that $f_{n,k}(\sigma)$ are polynomials of $X^k A(\sigma)$ $(k=0, 1, \cdots, n-1)$. Moreover, from the series

$$\operatorname{sh}\sigma = \sum_{k=0}^{\infty}\frac{\sigma^{2h+1}}{(2h+1)!},$$

we know that $A(\sigma)$ is an even power series and bounded for small σ. Then $\frac{\partial}{\partial\sigma}A(\sigma)$ is an odd power series and $XA(\sigma) = \frac{1}{\operatorname{sh}\sigma}\frac{\partial}{\partial\sigma}A(\sigma)$ is again even and bounded for small σ. Hence all $X^k A(\sigma)$ $(k=0\ 1, \cdots)$ are even and bounded for small σ. On the otherhand $1/\operatorname{sh}\sigma = 2e^{-t\sigma}/(1-e^{-2\sigma})$, whence all $X^k A(\sigma)$ are bounded for large $\sigma \geqslant 0$. Hence all $f_{n,n-k}(\sigma)$ are bounded. The lemma is proved.

Lemma 3.2.

$$\lim_{t\to 0}\int_{B^n} H(z,\ w,\ t)\dot{z}=1.$$

Proof According to (3.2), (3.3) and Lemma 3.1,

$$H(z,\ 0,\ t)=C_n\left(-\frac{1}{\pi}\right)^n\int_0^\infty\left[\frac{1}{2\mathrm{ch}\sigma}X^n f(\sigma,t)\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2r+\tau^2}d\tau$$

$$=\frac{C_n}{t^n}\left(\frac{1}{2\pi}\right)^n\sum_{k=0}^{n-1}(-2t)^k\int_0^\infty\left[\frac{1}{2\mathrm{ch}\sigma}f_{n,n-k}(\sigma)\frac{e^{-n^2t}}{\sqrt{t}}e^{-\sigma^2/(4t)}\right]_{\mathrm{ch}^2\sigma=\mathrm{ch}^2r+\tau^2}d\tau$$

$$=\frac{C_n}{2t^n\sqrt{t}}\left(\frac{1}{2\pi}\right)^n\sum_{k=0}^{n-1}(-2t)^k\int_r^\infty f_{n,n-k}(\sigma)\frac{\mathrm{sh}(\sigma)e^{-n^2t}}{\sqrt{\mathrm{ch}^2\sigma-\mathrm{ch}^2r}}e^{-\frac{\sigma^2}{4t}}d\sigma$$

$$=\frac{C_n}{2t^n\sqrt{t}}\left(\frac{1}{2\pi}\right)^n e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k$$

$$\times\int_0^\infty f_{n,n-k}(\tau+r)\frac{\mathrm{sh}(\tau+r)}{\sqrt{\mathrm{ch}^2(\tau+r)-\mathrm{ch}^2r}}e^{-\frac{(\tau+r)^2}{4t}}d\tau. \tag{3.11}$$

Changing τ to $2\sqrt{t}\,\tau$ and using $\mathrm{ch}^2r=(\mathrm{ch}2r+1)/2$ we have

$$H(z,\ 0,\ t)=\frac{\sqrt{2}\,C_n}{t^n}\left(\frac{1}{2\pi}\right)^n e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k$$

$$\times\int_0^\infty f_{n,n-k}(2\sqrt{t}\,\tau+r)\frac{\mathrm{sh}(2\sqrt{t}\,\tau+r)}{\sqrt{\mathrm{ch}(2(2\sqrt{t}\,\tau+r))-\mathrm{ch}(2r)}}e^{-\frac{(2\sqrt{t}\tau+r)^2}{4t}}d\tau.$$

Since the volume element with respect to the geodesic coordinate is

$$\dot{z}=\mathrm{sh}^{2n-1}r\,\mathrm{ch}r\sqrt{g_0}\,d\theta_1\cdots d\theta_n\,d\varphi_1\cdots d\varphi_{n-1},$$

where g_0 is defined by (1.7), we have

$$\int_{B^n} H(z,\ 0,\ t)\dot{z}=\frac{\sqrt{2}\,C_nB}{(2t)^n\pi^n}e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k\int_0^\infty\int_0^\infty f_{n,n-k}(2\sqrt{t}\,\tau+r)\mathrm{sh}(2\sqrt{t}\,\tau+r),$$

$$\times[\mathrm{ch}(2(2\sqrt{t}\,\tau+r))-\mathrm{ch}(2r)]^{-1/2}$$

$$\times\exp\left[-\frac{(2\sqrt{t}\,\tau+r)^2}{4t}\right]\mathrm{sh}^{2n-1}r\,\mathrm{ch}r\,d\tau\,dr, \tag{3.12}$$

where

$$B=\int_0^{2\pi}\cdots\int_0^{2\pi}\int_0^{\pi/2}\cdots\int_0^{\pi/2}\sqrt{g_0}\,d\theta_1\cdots d\theta_n\,d\varphi_1\cdots d\varphi_{n-1}=(2\pi)^n\int_0^{\pi/2}\cdots\int_0^{\pi/2}\sqrt{g_0}\,d\varphi_1\cdots d\varphi_{n-1}$$

$$=2^{2n-1}\pi^n n!/(2n)!. \tag{3.13}$$

Let $r=2\sqrt{t}\,x$ in (3.11). Then

$$\int_{B^n} H(z,\ 0,\ t)\dot{z}=\frac{\sqrt{2}\,C_nB2^n}{(2\sqrt{t}\,)^{2n-1}\pi^n}e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k$$

$$\times\int_0^\infty\int_0^\infty\frac{f_{n,n-k}(2\sqrt{t}\,(\tau+x))\mathrm{sh}(2\sqrt{t}\,(\tau+x))}{\sqrt{\mathrm{ch}(4\sqrt{t}\,(\tau+x))-\mathrm{ch}(4\sqrt{t}\,x)}}$$

$$\cdot e^{-(\tau+x)^2}\mathrm{sh}^{2n-1}(2\sqrt{t}\,x)\mathrm{ch}(2\sqrt{t}\,x)d\tau\,dx$$

$$=\frac{2^nC_nB}{\pi^n}e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k\int_0^\infty\int_0^\infty f_{n,n-k}(2\sqrt{t}\,(\tau+x))\frac{\mathrm{sh}(2\sqrt{t}\,(\tau+x))}{2\sqrt{t}}$$

$$\cdot\left[\frac{\mathrm{ch}(4\sqrt{t}\,(\tau+x))-\mathrm{sh}(4\sqrt{t}\,x)}{8t}\right]^{-\frac{1}{2}}$$

$$\times e^{-(\tau+x)^2}\frac{\mathrm{sh}^{2n-1}(2\sqrt{t}(\tau+x))}{(2\sqrt{t})^{2n-1}}\,\mathrm{ch}\,(2\sqrt{t}\,x)d\tau\,dx. \tag{3.14}$$

Hence

$$\lim_{t\to 0}\int_{B^n} H(z,0,t)\dot{z}=\frac{2^nC_nB}{\pi^n}\int_0^\infty\int_0^\infty f_{n,n}(0)\frac{\tau+x}{\sqrt{\tau^2+2\tau x}}e^{-(\tau+x)^2}x^{2n-1}d\tau\,dx$$

$$=\frac{2^nC_nB}{\pi^n}\int_0^\infty x^{2n-1}e^{-x^2}dx\int_0^\infty e^{-(\tau^2+2\tau x)}d\sqrt{\tau^2+2\tau x}$$

$$=\frac{2^nC_nB}{\pi^n}\cdot\frac{\sqrt{\pi}}{2}\int_0^\infty x^{2n-1}e^{-x^2}dx=\frac{\sqrt{\pi}}{2}\left(\frac{2}{\pi}\right)^nC_nB(n-1)\,!=1$$

because

$$C_n=\frac{(2n)\,!}{2^{3n-2}(n-1)\,!\,n\,!\sqrt{\pi}}.$$

The lemma is proved.

Now we can prove (3.7). Without loss of generality we only prove the case $w=0$. Denote by B^n_δ the ball of centre 0 and radius $\delta>0$. Since

$$\int_{B^n}H(z,0,t)\dot{z}=\int_{B^n}H(z,0,t)\,[\varphi(z)-\varphi(0)]\,\dot{z}+\varphi(0)\int_{B^n}H(z,0,t)\dot{z}$$

and the last term tends to $\varphi(0)$ when $t\to 0$, it is sufficient to prove that the first integral of the right hand side tends to 0 when $t\to 0$. In fact, for any givan $\varepsilon>0$, we can choose small $\delta>0$ such that

$$\left|\int_{B^n_\delta}H(z,0,t)\,[\varphi(z)-\varphi(0)]\,\dot{z}\right|\leqslant\frac{\varepsilon}{4}\int_{B^n_\delta}H(z,0,t)\dot{z}<\frac{\varepsilon}{4}\int_{B^n}H(z,0,t)\dot{z}<\frac{\varepsilon}{2}$$

according to Lemma 3.2 for small t. By hypothesis, there is a positive number M such that $|\varphi(z)|\leqslant\frac{1}{2}M$ in B^n. Then, fixed δ, we have

$$\left|\int_{B^n}H(z,0,t)\,[\varphi(z)-\varphi(0)]\,\dot{z}\right|<\frac{\varepsilon}{2}+M\int_{B^n-B^n_\delta}H(z,0,t)\dot{z}. \tag{3.16}$$

Applying(3.12)and(3.14)and setting $\delta_0=\frac{1}{2}\log\frac{1+\delta}{1-\delta}$, we have

$$\int_{B^n-B^n_\delta}H(z,0,t)\dot{z}$$

$$=\frac{\sqrt{2}\,C_nB}{(2t)^n\pi^n}e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k\int_{\delta_0}^\infty\int_0^\infty\frac{f_{n,n-k}(2\sqrt{t}\,x+r)\mathrm{sh}(2\sqrt{t}\,\tau+x)}{\sqrt{\mathrm{ch}(2(2\sqrt{t}\,\tau+r))-\mathrm{ch}(2r)}}$$

$$\times e^{-\frac{(2\sqrt{t}\tau+r)^2}{4t}}\,\mathrm{sh}^{2n-1}r\,\mathrm{ch}\,r\,d\tau\,dr$$

$$=\frac{2^nC_nB}{\pi^n}e^{-n^2t}\sum_{k=0}^{n-1}(-2t)^k\int_{\frac{\delta_0}{2\sqrt{t}}}^\infty\left[\int\frac{f_{n,n-k}(2\sqrt{t}\,(\tau+x))\frac{\mathrm{sh}(2\sqrt{t}\,(\tau+x))}{2\sqrt{t}}}{\sqrt{\frac{\mathrm{ch}(4\sqrt{t}\,(\tau+x))-\mathrm{ch}(4\sqrt{t}\,x)}{8t}}}e^{-\tau^2-2\tau x}d\tau\right]$$

$$\times e^{-x^2}\frac{\mathrm{sh}^{2n-1}(2\sqrt{t}\,x)}{(2\sqrt{t})^{2n-1}}\mathrm{ch}(2\sqrt{t}\,x)dx.$$

The integrals inside the brackets have limits when t tends to 0 respectively, because $f_{n,n-k}$ are bounded. Hence there is a positive number M_1 such that, when t is sufficiently small,

$$\int_{B^n-B^n_\delta} H(z,\ 0,\ t)\dot{z} \leqslant M_1 \int_{\delta_0/2\sqrt{t}}^{\infty} e^{-x^2} \frac{\operatorname{sh}^{2n-1}(2\sqrt{t}\,x)}{(2\sqrt{t}\,)^{2n-1}} \operatorname{ch}(2\sqrt{t}\,x)dx.$$

Since $\operatorname{ch}2\sqrt{t}\,x < \operatorname{ch}2x$ and

$$\frac{\operatorname{sh}(2\sqrt{t}\,x)}{2\sqrt{t}} = x\sum_{k=0}^{\infty} \frac{(2\sqrt{t}\,x)^{2k}}{(2k+1)!} \leqslant x\sum_{k=0}^{\infty} \frac{(2x)^{2k}}{(2k+1)!} = \frac{1}{2}\operatorname{sh}(2x),$$

we have, for t small sufficiently,

$$\int_{B^n-B^n_\delta} H(z,\ 0,\ t)\dot{z} \leqslant \frac{M_1}{2^{2n-1}} \int_{\delta_0/2\sqrt{t}}^{\infty} e^{-x^2}\operatorname{sh}^{2n-1}(2x)\operatorname{ch}(2x)dx < \frac{\varepsilon}{2M}$$

because the last integral is covergent. Then the absolute value of the integral in (3.16) is less than ε. This proves (3.7) and the proof of the theorem is completed.

Let $z=(z_1.\ z_2)\in B^m\times B^n$ and

$$H_{B^m\times B^n}(z,\ w,\ t) = H_{B^m}(z_1,\ w_1,\ t)H_{B^n}(z_2,\ w_2,\ t),$$

where $H_{B^m}(z_1,\ w_1,\ t)$ is the heat kernal of B^m with respect to the invariant Beltrami-Laplace operator Δ_{B^m}. It is not hard to see that for any continuous and bounded function $\varphi(z)$ in $B^m\times B^n$,

$$\lim_{t\to 0}\int_{B^m\times B^n} H_{B^m\times B^n}(z,\ w,\ t)\varphi(z)\dot{z} = \varphi(w).$$

Hence $H_{B^m\times B^n}(z,\ w,t)$ is the heat kernel of $B^m\times B^n$ because

$$\frac{\partial}{\partial t} H_{B^m\times B^n}(z,\ w,t) = (\Delta_{B^m}+\Delta_{B^n})H_{B^m\times B^n}(z,\ w,\ t).$$

Moreovor, by similar argument as in [1],

$$G(z,\ w) = -\int_0^{\infty} H_{B^m\times B^n}(z,\ w,\ t)dt$$

is the Green function of $B^m\times B^n$ with respect to $\Delta_{B^m}+\Delta_{B^n}$.

References

[1] Lu Qikeng, The heat kernels of a unit disc and polydisc, Research Memorandum, Tntsitute of Mathematics, Academia Ssnica, Beijing, 3: 17 (1987).

[2] Lu Qikeng, The Classical manifolds and classical domains, Sciences and Tech. Press, Shanghai, 1961.