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GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS TO THE TYPICAL FREE BOUNDARY PROBLEM FOR GERENAL QUASILINEAR HYPERBOLIC SYSTEMS AND ITS APPLICATIONS

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Abstract

In this paper the authors prove the existence and uniqueness of global classical solutions to the typical free boundary problem for general quasilinear hyperbolic systems.

As an application, a unique global discontinuous solution only containing n shocks on $t\geqslant 0$ is obtained for a class of generalized Riemann problem for the quasilinear hyperbolic system of n conservation laws.

§1. Introduction

Under certain decay hypotheses we proved in [1] the existence and uniqueness of global classical solutions to the typical free boundary problem on an angular domain

$$D=\{(t,x)\,|\,t\geqslant 0,\ x_1(t)\leqslant x\leqslant x_2(t)\}$$
$$(x_1(0)=x_2(0)=0;\quad x_1(t)<x_2(t),\ \forall t>0) \tag{1.1}$$

for the first order reducible quasilinear hyperbolic system

$$\begin{cases}\dfrac{\partial r}{\partial t}+\lambda(r,s)\dfrac{\partial r}{\partial x}=0\\[2mm] \dfrac{\partial s}{\partial t}+\mu(r,s)\dfrac{\partial s}{\partial x}=0\end{cases}\qquad (\lambda(r,s)<\mu(r,s)), \tag{1.2}$$

and this result was used in [2] to discuss the global perturbatien of the Riemann problem for the system of one-dimensional isentropic flow and construct a global discontinuous solution only containing two shocks in a class of piecewise continuous and piecewise smooth functions.

In this paper we shall generalize the previous result to the typical free boundary problem on the angular domain (1.1) for the following general quasilinear hyperbolic system

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$$\sum_{j=1}^{n} \zeta_{lj}(u)\left(\frac{\partial u_j}{\partial t}+\lambda_l(u)\ \frac{\partial u_j}{\partial x}\right)=\mu_l(u) \quad (l=1,\ \cdots,\ n), \tag{1.3}$$

where $u=(u_1,\ \cdots,\ u_n)^T$ denotes the unknown vector function, $\zeta_{lj}(u)$, $\lambda_l(u)$, $\mu_l(u)$ $(l,\ j=1,\ \cdots,\ n)$ are suitably smooth functions of u and

$$\det(\zeta_{lj})\neq 0. \tag{1.4}$$

The boundary conditions are as follows:

on the free boundary $x=x_2(t)$,

$$g_r(a_r(t,\ x),\ u)=0\ (r=1,\ \cdots,\ m), \tag{1.5}$$

$$\frac{dx_2}{dt}=F_2(b_2(t,\ x),\ u),\ x_2(0)=0; \tag{1.6}$$

on the free boundary $x=x_1(t)$,

$$g_s(a_s(t,\ x),\ u)=0 \quad (s=m+1,\ \cdots,\ n), \tag{1.7}$$

$$\frac{dx_1}{dt}=F_1(b_1(t,\ x),\ u),\ x_1(0)=0. \tag{1.8}$$

We shall prove in § 4 that under certain reasonable hypotheses problem (1.3)—(1.8) admits a unique global classical solution on the angular domain (1.1). Then this result will be generalized in § 5 to the typical free boundary problem on a fan-shaped domain. Finally in § 6 we shall use the result of § 5 to consider the discontinuous initial value problem for the quasilinear hyperbolic system of conservation laws

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial f(u)}{\partial x}=0, & (1.9)\\ t=0:\ u=\begin{cases} u_0^-(x), x\leqslant 0,\\ u_0^+(x), x\geqslant 0,\end{cases} & (1.10)\end{cases}$$

where $u=(u_1,\ \cdots,\ u_n)^T$, $f(u)=(f_1(u),\ \cdots,\ f_n(u))^T$ is a suitably smooth function of u, $u_0^-(x)$ and $u_0^+(x)$ are given smooth functions on $x\leqslant 0$ and on $x\geqslant 0$ respectively with

$$u_0^-(0)\neq u_0^+(0). \tag{1.11}$$

Problem (1.9)—(1.10) may be regarded as a perturbation of the corresponding Riemann problem (1.9) and

$$t=0:\ u=\begin{cases} u_-,\ x\leqslant 0,\\ u_+,\ x\geqslant 0,\end{cases} \tag{1.12}$$

in which

$$u_\pm=u_0^\pm(0). \tag{1.13}$$

Suppose that $|u_+-u_-|$ is sufficiently small and the solution to Riemann problem (1.9), (1.12) is composed of constant states and n typical shocks. We shall prove in § 6 that problem (1.9)—(1.10) admits a unique global discontinuous solution only containing n shocks on $t\geqslant 0$ in a class of piecewise continuous and piecewise smooth functions and this solution is a global perturbation of the solution to the corresponding Riemann problem.

§ 2. An Uniform A Priori Estimate

In this section we consider the following typical boundary value problem on the angular domain (1.1) for the quasilinear hyperbolic system of diagonal form

$$\frac{\partial u_l}{\partial t}+\lambda_l(u)\frac{\partial u_l}{\partial x}=\sum_{j,k=1}^{n} a_{ljk}(u)u_j u_k \quad (l=1,\cdots,n), \tag{2.1}$$

$$x=x_2(t): u_r=\sum_{q=m+1}^{n}\theta_{rq}(t)u_q+\sum_{j,k=1}^{n} g_{rjk}(t,u)u_j u_k+g_r(t,u)b_r(t) \quad (r=1,\cdots,m), \tag{2.2}$$

$$x=x_1(t): u_s=\sum_{p=1}^{m}\theta_{sp}(t)u_p+\sum_{j,k=1}^{n} g_{sjk}(t,u)u_j u_k+g_s(t,u)b_s(t) \quad (s=m+1,\cdots,n). \tag{2.3}$$

Setting $F_i(t)=x_i'(t)$ $(i=1,2)$, we suppose that this problem satisfies the following conditions:

(H1) λ_l, a_{ljk}, θ_{lj}, g_{ljk}, g_l, b_l, F_i are all suitably smooth functions; moreover, g_{ljk} and g_l are bounded if u is bounded and $t\geqslant 0$.

(H2) Boundary conditions (2.2)—(2.3) possess a unique solution $u=u^0$.

(H3) There are no characteristic curves entering the domain (1.1) from the origin, i. e.

$$\lambda_r(u^0)<F_1(0)<F_2(0)<\lambda_s(u^0) \quad (r=1,\cdots,m;\ s=m+1,\cdots,n), \tag{2.4}$$

then

$$0<\sigma_r\triangleq\frac{F_1(0)-\lambda_r(u^0)}{F_2(0)-\lambda_r(u^0)},\ \sigma_s\triangleq\frac{\lambda_s(u^0)-F_2(0)}{\lambda_s(u^0)-F_1(0)}<1 \quad (r=1,\cdots,m;\ s=m+1,\cdots,n). \tag{2.5}$$

(H4) Let

$$\Theta=(\theta_{lj}(0)), \tag{2.6}$$

$$\Theta_{-1}=\mathrm{diag}(\sigma_1^{-1},\cdots,\sigma_n^{-1})\Theta, \tag{2.7}$$

we have

$$\|\Theta_{-1}\|_{\min}<1, \tag{2.8}$$

where the minimal characterizing number $\|A\|_{\min}$ of a $n\times n$ matrix $A=(a_{ij})$ is defined as

$$\|A\|_{\min}\triangleq\inf_{\substack{\gamma_i\neq 0\\(i=1,\cdots,n)}}\max_{i=1,\cdots,n}\sum_{j=1}^{n}\left|\frac{\gamma_i}{\gamma_j}a_{ij}\right| \quad (cf.\ [3]).$$

Lemma 2.1. *Under assumptions* (H1)—(H4), *there exist positive constants* ε_0 *and* ε $(0<\varepsilon\leqslant\varepsilon_0)$ *so small that if*

$$|\theta_{lj}(t)-\theta_{lj}(0)|\leqslant\varepsilon_0 \quad (l,j=1,\cdots,r),\ \forall t\geqslant 0, \tag{2.9}$$

$$|F_i(t)-F_i(0)|\leqslant\varepsilon_0 \quad (i=1,2),\ \forall t\geqslant 0, \tag{2.10}$$

$$|b_l(t)|\leqslant\frac{\varepsilon}{1+t}, \quad (l=1,\cdots,n),\ \forall t\geqslant 0, \tag{2.11}$$

then on the existence domain of the classical solution $u=u(t,x)$ *to problem* (2.1)—(2.3), *the following uniform a priori estimate holds:*

$$|u(t, x)| \leqslant \frac{K\varepsilon}{1+t}, \quad \forall t \geqslant 0, \tag{2.12}$$

where K is a positive constant independent of t.

Proof Noting (2.8), we may suppose that (cf. [3])

$$\sum_{p=1}^{m} \sum_{s=m+1}^{n} |\theta_{rs}(0)\theta_{sp}(0)| \sigma_r^{-1}\sigma_s^{-1} < 1 \quad (r=1, \cdots, m), \tag{2.13}$$

$$\sum_{q=m+1}^{n} \sum_{r=1}^{m} |\theta_{sr}(0)\theta_{rq}(0)| \sigma_r^{-1}\sigma_s^{-1} < 1 \quad (s=m+1, \cdots, n). \tag{2.14}$$

By (2.11), it is easy to see that for the value u^0 of the solution at the origin, uniquely determined by (2.2)—(2.3), we have

$$|u^0| \leqslant A_0\varepsilon, \tag{2.15}$$

where A_0 is a positive constant. Hence, we can take $\varepsilon > 0$ so small that

$$|u(t, x)| \leqslant \varepsilon_0, \tag{2.16}$$

provided that $t \geqslant 0$ is suitably small, where ε_0 is a small positive constant to be determined later on.

In order to prove (2.12), we first suppose that (2.16) holds on the whole existence domain of the classical solution. Let $\xi = f_l(\tau; t, x)$ be the l-th characteristic curve passing through the point (t, x) on the existence domain of the classical solution:

$$\begin{cases} \dfrac{df_l(\tau; t, x)}{d\tau} = \lambda_l(u(\tau, f_l(\tau; t, x))), \quad (\tau \leqslant t), \\ \tau = t: f_l(\tau; t, x) = x. \end{cases} \tag{2.17}$$

By (2.4), (2.10) and (2.16), for any $l=1, \cdots, n$ there exists a unique $\tau_l(t, x) \leqslant t$ such that

$$\begin{aligned} &f_r(\tau_r(t, x); t, x) = x_2(\tau_r(t, x)) \quad (r=1, \cdots, m), \\ &f_s(\tau_s(t, x); t, x) = x_1(\tau_s(t, x)) \quad (s=m+1, \cdots, n), \end{aligned} \tag{2.18}$$

provided that $\varepsilon_0 > 0$ is suitably small. Let

$$\begin{aligned} &\tau_{rs}(t, x) = \tau_s(\tau_r(t, x), x_2(\tau_r(t, x))), \\ &\tau_{sr}(t, x) = \tau_r(\tau_s(t, x), x_1(\tau_s(t, x))), \\ &(r=1, \cdots, m;\ s=m+1, \cdots, n). \end{aligned} \tag{2.19}$$

By (2.10) and (2.16)—(2.18) we have

$$\begin{aligned} \frac{(F_1(0)-\varepsilon_0)t-(F_2(0)+\varepsilon_0)\tau_r(t, x)}{t-\tau_r(t, x)} &\leqslant \frac{x_1(t)-x_2(\tau_r(t, x))}{t-\tau_r(t, x)} \\ &\leqslant \frac{x-x_2(\tau_r(t, x))}{t-\tau_r(t, x)} \\ &= \frac{f_r(t; t, x)-f_r(\tau_r(t, x); t, x)}{t-\tau_r(t, x)} \\ &\leqslant \lambda_r(u^0) + A_1\varepsilon_0, \end{aligned}$$

here and hereafter, $A_i (i=1, 2, \cdots)$ will denote positive constants independent of (t, x). Thus, noting (2.5), when $\varepsilon_0 > 0$ is suitably small, we have

$$t\leqslant(\sigma_r^{-1}+A_2\varepsilon_0)\tau_r(t,\ x)\quad (r=1,\ \cdots,\ m). \tag{2.20}$$

In a similar way we get

$$\tau_r(t,\ x)\leqslant(\sigma_s^{-1}+A_3\varepsilon_0)\tau_{rs}(t,\ x), \tag{2.21}$$

$$\tau_{rs}(t,\ x)\leqslant(\sigma_s+A_4\varepsilon_0)\tau_r(t,\ x), \quad \begin{pmatrix} r=1,\ \cdots,\ m; \\ s=m+1,\ \cdots,\ n \end{pmatrix}. \tag{2.22}$$

Observing (2.5) and (2.13), we can choose constants $\alpha>1$ and $\beta>1$ such that if $\varepsilon_0>0$ is suitably small, then

$$\begin{cases}(\sigma_r^{-1}+A_2\varepsilon_0)(\sigma_s^{-1}+A_3\varepsilon_0)\leqslant\alpha, \\ (\sigma_s+A_4\varepsilon_0)\beta<1,\end{cases} \quad \begin{pmatrix} r=1,\ \cdots,\ m; \\ s=m+1,\ \cdots,\ n \end{pmatrix} \tag{2.23}$$

and

$$\alpha\sum_{p=1}^{m}\sum_{s=m+1}^{n}|\theta_{rs}(0)\theta_{sp}(0)|<1. \tag{2.24}$$

Let

$$\begin{cases} u_l(t)=\max\limits_{x_1(t)\leqslant x\leqslant x_2(t)}|u_l(t,\ x)| \quad (l=1,\ \cdots,\ n), \\ u(t)=\max\limits_{l=1,\cdots,n}u_l(t), \\ V(t)=tu(t). \end{cases} \tag{2.25}$$

We want to prove that there exists a positive constant K_0 such that

$$V(t)\leqslant K_0\varepsilon \tag{2.26}$$

on the existence domain of the classical solution, provided that $\varepsilon_0>0$ and $\varepsilon>0$ are suitably small.

By the definition of $V(t)$, (2.26) obviously holds provided that $t\geqslant0$ is suitably small. Suppose that (2.26) holds for $0\leqslant t\leqslant T$, we shall prove that it still holds for $T\leqslant t\leqslant\beta T$, provided that the classical solution exists on $0\leqslant t\leqslant\beta T$.

Using boundary conditions (2.2) and (2.3), for any $(t,\ x)$ in the domain (1.1) with $T\leqslant t\leqslant\beta T$, we can integrate system (2.1) along the r-th characteristic curve from $(t,\ x)$ to $(\tau_r(t,\ x),\ x_2(\tau_r(t,\ x)))$ and then along the s-th characteristic curve from $(\tau_r(t,\ x),\ x_2(\tau_r(t,\ x)))$ to $(\tau_{rs}(t,\ x),\ x_1(\tau_{rs}(t,\ x)))$. Noting (2.9), (2.11), (2.16), (H1) and that $\tau_{rs}\leqslant\tau_r$, this procedure gives

$$\begin{aligned}|u_r(t,\ x)|\leqslant&\sum_{s=m+1}^{n}(|\theta_{sr}(0)|+A_5\varepsilon_0)|u_s(\tau_r(t,\ x),\ x_2(\tau_r(t,\ x)))| \\ &+\frac{A_5\varepsilon}{1+\tau_r(t,\ x)}+A_5\int_{\tau_r(t,x)}^{t}u^2(\tau)d\tau \\ \leqslant&\sum_{p=1}^{m}\sum_{s=m+1}^{n}(|\theta_{rs}(0)\theta_{sp}(0)|+A_6\varepsilon_0)|u_p(\tau_{rs}(t,x),\ x_1(\tau_{rs}(t,\ x)))| \\ &+\sum_{s=m+1}^{n}\frac{A_6\varepsilon}{1+\tau_{rs}(t,\ x)}+A_6\sum_{s=m+1}^{n}\int_{\tau_{rs}(t,x)}^{t}u^2(\tau)d\tau,\ (r=1,\ \cdots,\ m).\end{aligned} \tag{2.27}$$

When $T\leqslant t\leqslant\beta T$, it follows from (2.20)—(2.23) that

$$\frac{1}{\alpha}T\leqslant\frac{1}{\alpha}t\leqslant\tau_{rs}(t,\ x)<T. \tag{2.28}$$

By (2.24) we can choose ε_0 so small that

$$\alpha\sum_{p=1}^{m}\sum_{s=m+1}^{n}(|\theta_{rs}(0)\theta_{sp}(0)|+A_6\varepsilon_0)\leqslant\theta_0<1. \tag{2.29}$$

Noting (2.28)—(2.29) and the fact that (2.26) holds on $0\leqslant t\leqslant T$, it comes from (2.27) that

$$t|u_r(t, x)|\leqslant\theta_0K_0\varepsilon+A_7\varepsilon+A_7t\int_{T/\alpha}^{t}u^2(\tau)d\tau,\ (r=1, \cdots, m). \tag{2.30}$$

Similar estimates hold for $u_s(t, x)\,(s=m+1, \cdots, n)$. Then it is easy to get

$$V(t)\leqslant\theta_0K_0\varepsilon+A_8\varepsilon+A_8\int_{T/\alpha}^{t}\frac{V^2(\tau)}{\tau}d\tau,\ T\leqslant t\leqslant\beta T. \tag{2.31}$$

Suppose that $W(t)$ satisfies

$$W(t)=\theta_0K_0\varepsilon+A_8\varepsilon+A_8\int_{T/\alpha}^{t}\frac{W^2(\tau)}{\tau}d\tau,\quad T\leqslant t\leqslant\beta T. \tag{2.32}$$

Then we have

$$V(t)\leqslant W(t)=\frac{\theta_0K_0\varepsilon+A_8\varepsilon}{1-(\theta_0K_0\varepsilon+A_8\varepsilon)A_8\ln(\alpha t/T)}. \tag{2.33}$$

Hence there exists a positive constant K_0 only depending on A_8 and θ_0 but indenpendent of T, such that

$$V(t)\leqslant K_0\varepsilon,\quad T\leqslant t\leqslant\beta T, \tag{2.34}$$

provided that $\varepsilon>0$ is suitably small. Repeating the same argument, we get (2.26) on the whole existence domain of the classical solution. Noting that $u(0, 0)=u^0$ satisfies (2.15), from (2.26) we obtain immediately (2.12), which implies that the previous hypothesis (2.16) is reasonable provided that $\varepsilon>0$ is suitably small. This finishes the proof of Lemma 2.1.

Remark 2.2. The hypothesis that $u(t, x)$ is a C^1 solution is not necessary in the proof of Lemma 2.1. In fact, Lemma 2.1 is still valid provided that $u(t, x)$ is continuous, the charaeteristic curve exists and the integration along the characteristic curve makes sense.

§3. Typical Boundary Value Problems With Fixed Boundaries

We now consider the following typical boundary value problem on the angular domain (1.1) for the quasilinear hyperbolic system (1.3):

$$x=x_2(t):\ u_r=G_r(a_r(t), u_{m+1}, \cdots, u_n),\ (r=1, \cdots, m), \tag{3.1}$$

$$x=x_1(t):\ u_s=G_s(a_s(t), u_1, \cdots, u_m),\ (s=m+1, \cdots, n). \tag{3.2}$$

Still setting $F_i(t)=x_i'(t)\,(i=1, 2)$, we suppose that

(H1) ζ_{ij}, λ_l, G_l, a_l, F_i are all C^1 functions, μ_l are C^2 functions and

$$\mu_l(0)=0, \frac{\partial \mu_l}{\partial u_j}(0)=0, \ (l, j=1, \cdots, n). \tag{3.3}$$

(H2) Boundary conditions (3.1)—(3.2) possess a unique solution $u=u^0$ at the origin. Without loss of generality, we may suppose $u^0=0$; then

$$G_l(a_l(0), 0)=0 \quad (l=1, \cdots, n). \tag{3.4}$$

(H3) (1.4) and (2.4) (in which $u^0=0$) hold and

$$\zeta_{lj}(0)=\delta_{lj}=\begin{cases} 1, & l=j, \\ 0, & l\neq j. \end{cases} \tag{3.5}$$

(H4) Let

$$\Theta=\begin{pmatrix} 0 & \frac{\partial G_r}{\partial u_q}(a_r(0), 0) \\ \frac{\partial G_s}{\partial u_p}(a_s(0), 0) & 0 \end{pmatrix}\begin{pmatrix} r, p=1, \cdots, m; \\ s, q=m+1, \cdots, n \end{pmatrix}, \tag{3.6}$$

$$\Theta_{-1}=\operatorname{diag}(\sigma_1^{-1}, \cdots, \sigma_n^{-1})\Theta, \tag{3.7}$$

where $\sigma_l (l=1, \cdots, n)$ are defined by (2.5), we have

$$\|\Theta_{-1}\|_{\min}<1. \tag{3.8}$$

Theorem 3.1. *Under assumptions* (H1)—(H4), *there exist positive constants* ε_0 *and* ε $(0<\varepsilon\leqslant\varepsilon_0)$ *so small that if*

$$|a_l(t)-a_l(0)|, \ |a_l'(t)|\leqslant\frac{\varepsilon}{1+t}, \quad (l=1, \cdots, n), \forall t\geqslant 0 \tag{3.9}$$

and (2.10) *holds, then the typical boundary value problem* (1.3), (3.1)—(3.2) *admits a unique global* C^1 *solution* $u=u(t, x)$ *on the angular domain* D (*see* (1.1)); *moreover,*

$$|u(t, x)|\leqslant\frac{K\varepsilon}{1+t}, \ \forall (t, x)\in D, \tag{3.10}$$

$$\left|\frac{\partial u}{\partial x}(t, x)\right|, \ \left|\frac{\partial u}{\partial t}(t, x)\right|\leqslant\frac{K\varepsilon}{1+t}, \ \forall (t, x)\in D, \tag{3.11}$$

where K *is a positive constant independent of* t.

Proof By the local existence theorem of classical solutions (cf. [3]), in order to prove the global existence of C^1 solutions, it is only necessary to get some uniform a priori estimates for the solution itself and its first order derivatives, therefore it suffices to prove (3.10)—(3.11) on the existence domain of the classical solution.

Let

$$\begin{cases} v=(v_1, \cdots, v_n)^T=\zeta(u)u, \\ w=(w_1, \cdots, w_n)^T=\zeta(u)\frac{\partial u}{\partial x}, \end{cases} \tag{3.12}$$

where $\zeta(u)=(\zeta_{lj}(u))$ is the $n\times n$ mrtrix and $\frac{\partial u}{\partial x}=\left(\frac{\partial u_1}{\partial x}, \cdots, \frac{\partial u_n}{\partial x}\right)^T$. Similarly to the proof of Lemma 2.1, we may first suppose that

$$|u(t, x)|\leqslant\varepsilon_0 \tag{3.13}$$

holds on the existence domain of the classical solution. Hence, noting (3.5), (3.2) can be written as

$$\begin{cases} u_l = v_l + \sum_{j,k=1}^{n} b_{ljk}(v) v_j v_k, \ (l=1, \cdots, n), \\ \dfrac{\partial u_l}{\partial x} = w_l + \sum_{j,k=1}^{n} \bar{b}_{ljk}(v) v_j w_k, \ (l=1, \cdots, n), \end{cases} \tag{3.14}$$

provided that $\varepsilon_0 > 0$ is suitably small.

By system (1.3) it is easy to see that v and w satisfy the following system

$$\begin{cases} \dfrac{\partial v_l}{\partial t} + \lambda_l(v) \dfrac{\partial v_l}{\partial x} = \sum_{j,k=1}^{n} c_{ljk}(v) v_j v_k + \sum_{j,k=1}^{n} d_{ljk}(v) v_j w_k, \ (l=1, \cdots, n), \\ \dfrac{\partial w_l}{\partial t} + \lambda_l(v) \dfrac{\partial w_l}{\partial x} = \sum_{j,k=1}^{n} \bar{c}_{ljk}(v) w_j w_k + \sum_{j,k=1}^{n} \bar{d}_{ljk}(v) v_j w_k, \ (l=1, \cdots, n). \end{cases} \tag{3.15}$$

Moreover, boundary condition (3.1) can be rewritten as

$$x = x_2(t): u_r = \sum_{q=m+1}^{n} \frac{\partial G_r}{\partial u_q}(a_r(t), 0) u_q + \sum_{j,k=1}^{n} G_{rjk}(a_r(t), u) u_j u_k + G_r(a_r(t), 0), \ (r=1, \cdots, m); \tag{3.16}$$

then, noting (3.13) and (3.4) we get

$$x = x_2(t): \ v_r = \sum_{q=m+1}^{n} \frac{\partial G_r}{\partial u_q}(a_r(t), 0) v_q + \sum_{j,k=1}^{n} g_{rjk}(a_r(t), v) v_j v_k + g_r(a_r(t))(a_r(t) - a_r(0)), \ (r=1, \cdots, m). \tag{3.17}$$

Similarly, from (3.2) we get

$$x = x_1(t): \ v_s = \sum_{p=1}^{m} \frac{\partial G_s}{\partial u_p}(a_s(t), 0) v_p + \sum_{j,k=1}^{n} g_{sjk}(a_s(t), v) v_j v_k + g_s(a_s(t))(a_s(t) - a_s(0)), \ (s=m+1, \cdots, n). \tag{3.18}$$

Furthermore, differentiating (3.16) with respect to t and using system (1.3), (3.5) and (3.14), we get the boundary condition on $x = x_2(t)$ for w as follows:

$$x = x_2(t): \ w_r = \sum_{q=m+1}^{n} \frac{\partial G_r}{\partial u_q}(a_r(t), 0) \frac{F_2(t) - \lambda_q(0)}{F_2(t) - \lambda_r(0)} w_q + \sum_{j,k=1}^{n} \bar{g}_{rjk}(a_r(t), F_2(t), v) v_j v_k + \sum_{j,k=1}^{n} \bar{\bar{g}}_{rjk}(a_r(t), F_2(t), v) v_j w_k + \bar{g}_r(a_r(t), F_2(t), v) a_r'(t) \quad (r=1, \cdots, m). \tag{3.19}$$

Similarly we have

$$x = x_1(t): \ w_s = \sum_{p=1}^{m} \frac{\partial G_s}{\partial u_p}(a_s(t), 0) \frac{F_1(t) - \lambda_p(0)}{F_1(t) - \lambda_s(0)} w_p + \sum_{j,k=1}^{n} \bar{g}_{sjk}(a_s(t), F_1(t), v) v_j v_k + \sum_{j,k=1}^{n} \bar{\bar{g}}_{sjk}(a_s(t), F_1(t), v) v_j w_k + \bar{g}_s(a_s(t), F_1(t), v) a_s'(t), \quad (s=m+1, \cdots, n). \tag{3.20}$$

According to the properties of the minimal characterizing number (cf. [3]), the minimal characterizing number of the matrix

$$\begin{pmatrix} 0 & \dfrac{\partial G_r}{\partial u_q}(a_r(0), 0) \dfrac{F_2(0) - \lambda_q(0)}{F_2(0) - \lambda_r(0)} \\ \dfrac{\partial G_s}{\partial u_p}(a_s(0), 0) \dfrac{F_1(0) - \lambda_p(0)}{F_1(0) - \lambda_s(0)} & 0 \end{pmatrix}$$

is equal to the minimal characterizing mumber of the matrix

$$\begin{pmatrix} 0 & \frac{\partial G_r}{\partial u_q}(a_r(0),\ 0)\frac{F_1(0)-\lambda_r(0)}{F_2(0)-\lambda_r(0)} \\ \frac{\partial G_s}{\partial u_p}(a_s(0),\ 0)\frac{F_2(0)-\lambda_s(0)}{F_1(0)-\lambda_s(0)} & 0 \end{pmatrix}.$$

Thus, noticing (2.5) it is easy to verify that all assumptions in Lemma 2.1 are satisfied for problem (3.15), (3.17)—(3.20); then in the existence domain of the classical solution to the original typical boundary value problem we have

$$|v(t,\ x)|,\ |w(t,\ x)| \leqslant \frac{K_0 \varepsilon}{1+t}, \tag{3.21}$$

where K_0 is a positive constant independent of t, provided that ε_0 and ε are suitably small. Hence, estimates (3.10)—(3.11) come from (3.14) and system (1.3); then the previous hypothesis (3.13) is actually reasonable provided that ε is suitably small. The proof of Theorem 3.1 is complete.

For the purpose in § 6, we consider the following initial value problem with the initial data given on $x \leqslant 0$:

$$\begin{cases} \sum_{j=1}^{n} \zeta_{lj}(u)\left(\frac{\partial u_j}{\partial t} + \lambda_l(u)\ \frac{\partial u_j}{\partial x}\right) = 0,\ (l=1,\ \cdots,\ n), & (3.22) \\ t=0:\ u = u^0(x),\ (x \leqslant 0). & (3.23) \end{cases}$$

As a consequence of Theorem 3.1, we have

Corollary 3.2. *There exists a positive number ε so small that if $u^0(x) \in C^1$ and*

$$\begin{cases} |u^0(x) - u^0(0)| \leqslant \frac{\varepsilon}{1+|x|},\ \forall x \leqslant 0, \\ |u^{0\prime}(x)| \leqslant \frac{\varepsilon}{1+|x|},\ \forall x \leqslant 0, \end{cases} \tag{3.24}$$

then on the domain

$$\hat{D} = \{(t,\ x)\ |\ t \geqslant 0,\ x \leqslant \xi t\}, \tag{3.25}$$

where ξ satisfies

$$\xi < \min_{i=1,\cdots,n} \{\lambda_i(u^0(0)\}, \tag{3.26}$$

the initial value problem (3.22)—(3.23) *admits a unique global C^1 solution $u = u(t,\ x)$ with*

$$\begin{cases} |u(t,\ x) - u(0,\ 0)| \leqslant \frac{K\varepsilon}{1+t},\ \forall (t,\ x) \in \hat{D}, \\ \left|\frac{\partial u}{\partial x}(t,\ x)\right|,\ \left|\frac{\partial u}{\partial t}(t,\ x)\right| \leqslant \frac{K\varepsilon}{1+t},\ \forall (t,\ x) \in \hat{D}, \end{cases} \tag{3.27}$$

where K is a positive constant independent of t.

Proof Taking the transformation of independent variables

$$\bar{t} = t - \alpha x,\ \bar{x} = x, \tag{3.28}$$

where $\alpha > 0$ is so small that

$$1 - \alpha \cdot \max_{i=1,\cdots,n} \{\lambda_i(u^0(0))\} > 0, \tag{3.29}$$

the domain $\hat{D}$ is reduced to the angular domain

$$\bar{D}=\left\{(\bar{t},\ \bar{x})\,|\,\bar{t}\geqslant 0,\ -\frac{1}{\alpha}\bar{t}\leqslant\bar{x}\leqslant\frac{\xi}{1-\alpha\xi}\bar{t}\right\} \tag{3.30}$$

and the original initial value problem on $\hat{D}$ to the following boundary value problem on $\bar{D}$

$$\begin{cases}\sum_{j=1}^{n}\zeta_{lj}(u)\left(\frac{\partial u_j}{\partial\bar{t}}+\bar{\lambda}_l(u)\frac{\partial u_j}{\partial\bar{x}}\right)=0,\ (l=1,\ \cdots,\ n),\\ \bar{x}=-\frac{1}{\alpha}\bar{t}:\ u=u^0(\bar{x})=u^0\left(-\frac{1}{\alpha}\bar{t}\right),\end{cases} \tag{3.31}$$

where

$$\bar{\lambda}_l(u)=\frac{\lambda_l(u)}{1-\alpha\lambda_l(u)},\ (l=1,\ \cdots,\ n). \tag{3.32}$$

Problem (3.13) can be regarded as a special case of problem (1.3), (3.1)—(3.2) in which $m=0$ and Θ_{-1} is the null matrix, hence by Theorem 3.1 problem (3.31) admits a uniqne global C^1 solution $u=\bar{u}(\bar{t},\ \bar{x})$ on $\bar{D}$ with

$$\begin{cases}|\bar{u}(\bar{t},\ \bar{x})-\bar{u}(0,\ 0)|\leqslant\frac{K_0\varepsilon}{1+\bar{t}},\ \forall(\bar{t},\ \bar{x})\in\bar{D},\\ \left|\frac{\partial\bar{u}}{\partial\bar{x}}(\bar{t},\ \bar{x})\right|,\ \left|\frac{\partial\bar{u}}{\partial\bar{t}}(\bar{t},\ \bar{x})\right|\leqslant\frac{K_0\varepsilon}{1+\bar{t}},\ \forall(\bar{t},\ \bar{x})\in\bar{D},\end{cases} \tag{3.33}$$

provided that $\varepsilon>0$ is suitably small. Noticing (3.28)—(3.29), we get the global C^1 solution $u=u(t,\ x)=\bar{u}(\bar{t},\ \bar{x})$ on $\hat{D}$ and (3.26) holds. This ends the proof of Corollary 3.2.

§ 4. Typical Free Boundary Problems on an Angular Domain

Theorem 3.1 will be generalized to the typical free boundary problem in this section. For the free boundary problem (1.3)—(1.8) on the angular domain (1.1), we give the following assumptions:

(H1) There exists a unique state u^0 such that

$$g_l(a_l(0,\ 0),\ u^0)=0,\ (l=1,\ \cdots,\ n). \tag{4.1}$$

Without loss of generality, we may suppose that $u^0=0$; then

$$g_l(a_l(0,\ 0),\ 0)=0,\ (l=1,\ \cdots,\ n). \tag{4.2}$$

Moreover, (3.3) and (3.5) hold.

(H2) In a neighborhood of $u=u^0=0$, boundary conditions (1.5) and (1.7) can be rewritten as

$$x=x_2(t):\ u_r=G_r(a_r(t,\ x),\ u_{m+1},\ \cdots,u_n),\ (r=1,\ \cdots,\ m), \tag{4.3}$$

$$x=x_1(t):\ u_s=G_s(a_s(t,\ x),\ u_1,\ \cdots,\ u_m),\ (s=m+1,\ \cdots,\ n). \tag{4.4}$$

(H3) There are no characteristic curves entering the domain D from the origin, i. e.

$$\lambda_r(0)<F_1(b_1(0,\ 0),\ 0)<F_2(b_2(0,\ 0),\ 0)<\lambda_s(0),\ (r=1,\ \cdots,\ m;\ s=m+1,\ \cdots,\ n),\tag{4.5}$$

then

$$0<\sigma_r \triangleq \frac{F_1(b_1(0,\ 0),\ 0)-\lambda_r(0)}{F_2(b_2(0,\ 0),\ 0)-\lambda_r(0)},\ \sigma_s \triangleq \frac{\lambda_s(0)-F_2(b_2(0,\ 0),\ 0)}{\lambda_s(0)-F_1(b_1(0,\ 0),\ 0)}<1$$

$$(r=1,\ \cdots,\ m;\ \ s=m+1,\ \cdots,\ n.)\tag{4.6}$$

(H4) Let

$$\Theta=\begin{pmatrix} 0 & \frac{\partial G_r}{\partial u_q}(a_r(0,\ 0),\ 0) \\ \frac{\partial G_s}{\partial u_p}(a_s(0,\ 0),\ 0) & 0 \end{pmatrix},\ \begin{pmatrix} r,\ p=1,\ \cdots,\ m; \\ s,\ q=m+1,\ \cdots,\ n, \end{pmatrix},\tag{4.7}$$

$$\Theta_{-1}=\mathrm{diag}(\sigma_1^{-1},\ \cdots,\ \sigma_n^{-1})\Theta;\tag{4.8}$$

we have

$$\|\Theta_{-1}\|_{\min}<1.\tag{4.9}$$

Theorem 4.1. *Under assumptions* (H1)—(H4), *suppose that* ξ_{lj}, λ_l, G_l, a_l, F_i, b_i *are* C^1 *functions*, μ_l *are* C^2 *functions* ($l,\ j=1,\ \cdots,\ n;\ i=1,\ 2$) *and* (3.3) *holds. Then there exist positive constants* ε_0 *and* $\varepsilon\ (0<\varepsilon\leqslant\varepsilon_0)$ *so small that if on* $x=x_2(t)$

$$\begin{cases} |a_r(t,\ x)-a_r(0,\ 0)|\leqslant\frac{\varepsilon}{1+t},\ (r=1,\ \cdots,\ m),\ \forall t\geqslant 0, \\ \left|\frac{\partial a_r}{\partial x}(t,\ x)\right|,\ \left|\frac{\partial a_r}{\partial t}(t,\ x)\right|\leqslant\frac{\varepsilon}{1+t},\quad (r=1,\ \cdots,\ m),\forall t\geqslant 0, \end{cases}\tag{4.10}$$

$$|b_2(t,\ x)-b_2(0,\ 0)|\leqslant\varepsilon_0,\ \forall t\geqslant 0\tag{4.11}$$

and on $x=x_1(t)$

$$\begin{cases} |a_s(t,\ x)-a_s(0,\ 0)|\leqslant\frac{\varepsilon}{1+t},\ (s=m+1,\ \cdots,\ n),\ \forall t\geqslant 0, \\ \left|\frac{\partial a_s}{\partial x}(t,\ x)\right|,\ \left|\frac{\partial a_s}{\partial t}(t,\ x)\right|\leqslant\frac{\varepsilon}{1+t},\quad (s=m+1,\ \cdots,\ n),\forall t\geqslant 0, \end{cases}\tag{4.12}$$

$$|b_1(t,\ x)-b_1(0,\ 0)|\leqslant\varepsilon_0,\ \forall t\geqslant 0,\tag{4.13}$$

then the typical free boundary problem (1.3)—(1.8) *admits a unique global classical solution on the angular domain* D *(see* (1.1)): $u=u(t,\ x)\in C^1$, $x_1(t)$, $x_2(t)\in C^2$. *Moreover, we have* (3.10)—(3.11) *and*

$$|x_1'(t)-x_1'(0)|,\ |x_2'(t)-x_2'(0)|\leqslant K_0\varepsilon_0,\ \forall t\geqslant 0,\tag{4.14}$$

where K_0 *is a positive constant independent of* t.

Proof. According to the local existence theorem of classical solutions (see [3]), under aassumptions (H1)—(H4) the typical free bonndary problem (1.3)—(1.8) admits a unique classical solution $u=u(t,\ x)\in C^1$ and $x_1(t),\ x_2(t)\in C^2$ on a local domain

$$D(\delta)=\{(t,\ x)\ |0\leqslant t\leqslant\delta,\ x_1(t)\leqslant x\leqslant x_2(t)\}.\tag{4.15}$$

In order to get the global existence of classical solutions, it still suffices to prove the uniform estimates (3.10)—(3.11) on the existence domain of the classical solution.

In the course of derivation of these a priori estimates, since the classical solution has been supposed to exist on the existence domain, the originally unknown free boundaries can be actually regarded as the given boundaries with (4.10)—(4.13) as the conditions on them, therefore, we can establish the desired estimates in a way completely similar to the previous section.

In fact, on the existence domain of the classical solution, let

$$\hat{a}_r(t)=a_r(t,x_2(t)),\ \hat{a}_s(t)=a_s(t,\ x_1(t)),\ (r=1,\ \cdots,\ m;\ s=m+1,\ \cdots,\ n), \quad (4.16)$$

$$\begin{cases}\hat{F}_1(t)=F_1(b_1(t,\ x_1(t)),u(t,\ x_1(t))),\\ \hat{F}_2(t)=F_2(b_2(t,\ x_2(t)),u(t,\ x_2(t)));\end{cases} \quad (4.17)$$

similarly to the proof of Theorem 3.1, we can still suppose that (3.13) holds on the existence domain of the classical solution; then by (4.10)—(4.13) there exists a positive constant K_0 independent of t such that on the existence domain of the classical solution we have

$$|\hat{F}_i(t)-\hat{F}_i(0)|\leqslant K_0\varepsilon_0 \quad (i=1,\ 2), \quad (4.18)$$

$$|\hat{a}_l(t)-\hat{a}_l(0)|\leqslant\frac{\varepsilon}{1+t},\ (l=1,\ \cdots,\ n), \quad (4.19)$$

$$|\hat{a}_l'(t)|\leqslant\frac{K_0\varepsilon}{1+t},\ (l=1,\ \cdots,\ n).$$

Repeating the argument in § 3, we get that (3.10)—(3.11) hold on the existence domain of the classical solution, provided that ε_0 and $\varepsilon>0$ are suitably small. This also illustrates the validity of hypothesis (3.13). The existence of global classical solutions is then proved. Moreover, (4.14) is nothing but (4.18). This finishes the proof of Theorem 4.1.

§ 5. Typical Free Boundary Problems on A Fan-Shaped Domain

For the purpose in § 6, Theorem 4.1 will be generalized to the typical free boundary problem on a fan-shaped domain (cf. [3]) in this section.

The fan-shaped domain under consideration is

$$D=\bigcup_{i=1}^{n-1}D_i=\{(t,\ x)\,|\,t\geqslant0,\ x_1(t)\leqslant x\leqslant x_n(t)\}, \quad (5.1)$$

where

$$D_i=\{(t,\ x)\,|\,t\geqslant0,\ x_i(t)\leqslant x\leqslant x_{i+1}(t)\},\ (i=1,\ \cdots,\ n-1) \quad (5.2)$$

and $x=x_i(t)\,(i=1,\ \cdots,\ n)$ are free boundaries satisfying

$$\begin{cases}x_i(0)=0,\ (i=1,\ \cdots,\ n),\\ x_i(t)<x_{i+1}(t),\ \forall t>0,\ (i=1,\ \cdots,\ n-1).\end{cases} \quad (5.3)$$

Suppose that for any fixed $i\,(i=1,\ \cdots,\ n-1)$,

$$u^i=(u_1^i,\ \cdots, u_n^i)^T \tag{5.4}$$

satisfies the following quasilinear hyperbolic system on D_i:

$$\sum_{j=1}^{n}\zeta_{lj}(u^i)\left(\frac{\partial u_j^i}{\partial t}+\lambda_l(u^i)\frac{\partial u_j^i}{\partial x}\right)=0,\ (l=1,\cdots, n). \tag{5.5}$$

Suppose furthermore that $u^i(i=1, \cdots, n-1)$ satisfy the following boundary conditions: on $x=x_i(t)\,(i=1, \cdots, n)$,

$$g_k^i(u^{i-1}, u^i)=0,\ (k=1, \cdots, n-1), \tag{5.6}$$

$$\frac{dx_i}{dt}=F^i(u^{i-1}, u^i),\ x_i(0)=0, \tag{5.7}$$

where $u^0=u^0(t, x)$ and $u^n=u^n(t, x)$ are the given C^1 functions.

We give the following assumptions for the typical free boundary problem (5.5)—(5.7) on the fan-shaped domain (5.1):

(H1) There exists a unique state $u^{i,0}(i=1, \cdots, n-1)$ such that

$$g_k^i(u^{i-1,0}, u^{i,0})=0,\ (k=1, \cdots, n-1;\ i=1, \cdots,n), \tag{5.8}$$

where $u^{0,0}=u^0(0, 0)$, $u^{n,0}=u^n(0, 0)$.

(H2) Let

$$v^i=(v_1^i, \cdots, v_n^i)^T=\zeta(u^{i,0})u^i,\ (i=0, 1, \cdots, n), \tag{5.9}$$

where $\zeta(u)=(\zeta_{lj}(u))$ is the $n\times n$ matrix. For any fixed $i(i=1, \cdots, n)$, (5.6) can be rewritten as

$$\begin{aligned} &v_r^{i-1}=G_r^{i-1}(v_i^{i-1}, \cdots, v_n^{i-1}, v_1^i, \cdots, v_i^i),\ (r=1, \cdots, i-1),\\ &v_s^i=G_s^i(v_i^{i-1},\cdots, v_n^{i-1}, v_1^i, \cdots, v_i^i),\ (s=i+1, \cdots, n). \end{aligned} \tag{5.10}$$

(H3) For any fixed $i(i=1, \cdots, n)$

$$\lambda_r^{i,0}<F^{i,0}<F^{i+1,0}<\lambda_s^{i,0},\ (r=1, \cdots, i;\ s=i+1, \cdots, n), \tag{5.11}$$

where

$$\begin{aligned} &\lambda_l^{i,0}=\lambda_l(u^{i,0}),\ (l=1, \cdots, n;\ i=1, \cdots, n-1),\\ &F^{i,0}=F^i(u^{i-1,0}, u^{i,0}),\ (i=1, \cdots,n), \end{aligned} \tag{5.12}$$

then

$$0<\sigma_r^i\triangleq\frac{F^{i,0}-\lambda_r^{i,0}}{F^{i+1,0}-\lambda_r^{i,0}},\quad \sigma_s^i\triangleq\frac{\lambda_s^{i,0}-F^{i+1,0}}{\lambda_s^{i,0}-F^{i,0}}<1 \tag{5.13}$$

$$(r=1, \cdots, i;\ s=i+1, \cdots, n;\ i=1, \cdots, n-1).$$

The characterizing matrix of problem (5.5)—(5.7) is the following $(n-1)n\times(n-1)n$ matrix (cf. [3])

$$\Theta=\begin{pmatrix} B_1 & C_1 & & & 0\\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-2} & B_{n-2} & C_{n-2}\\ 0 & & & A_{n-1} & B_{n-1} \end{pmatrix}, \tag{5.14}$$

where

$$A_i=\begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial G_s^i}{\partial v_q^{i-1}}(v^{i-1,0}, v^{i,0}) \end{pmatrix}, \quad \begin{pmatrix} s=i+1, \cdots, n; \\ q=i, \cdots, n, \end{pmatrix}$$

$$B_i=\begin{pmatrix} 0 & \frac{\partial G_r^i}{\partial v_q^i}(v^{i,0}, v^{i+1,0}) \\ \frac{\partial G_s^i}{\partial v_p^i}(v^{i-1,0}, v^{i,0}) & 0 \end{pmatrix}, \quad \begin{pmatrix} p, r=1, \cdots, i; \\ q, s=i+1, \cdots, n, \end{pmatrix} \tag{5.15}$$

$$C_i=\begin{pmatrix} \frac{\partial G_r^i}{\partial v_p^{i+1}}(v^{i,0}, v^{i+1,0}) & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} r=1, \cdots, i; \\ p=1, \cdots, i+1 \end{pmatrix}$$

are all $n\times n$ matrices and $v^{i,0}=\zeta(u^{i,0})u^{i,0}$. Let

$$\Theta_{-1}=(\mathrm{diag}(\sigma_1^1, \cdots, \sigma_n^1, \cdots, \sigma_1^{n-1}, \cdots, \sigma_n^{n-1}))^{-1}\Theta, \tag{5.16}$$

similar to Theorem 4.1 we can prove

Theorem 5.1. *Under assumptions* (H1)—(H3), *suppose that* ζ_{ij}, λ_i, G_l^i, F^i, u^0, u^n *are all* C^1 *functions and*

$$\|\Theta_{-1}\|_{\min}<1; \tag{5.17}$$

then there exists a positive constant ε *so small that if on* $x=x_1(t)$

$$|u^0(t, x)-u^0(0, 0)|, \left|\frac{\partial u^0}{\partial x}(t, x)\right|, \left|\frac{\partial u^0}{\partial t}(t, x)\right|\leqslant\frac{\varepsilon}{1+t}, \ \forall t\geqslant 0 \tag{5.18}$$

and on $x=x_n(t)$

$$|u^n(t, x)-u^n(0, 0)|, \left|\frac{\partial u^n}{\partial x}(t, x)\right|, \left|\frac{\partial u^n}{\partial t}(t, x)\right|\leqslant\frac{\varepsilon}{1+t}, \ \forall t\geqslant 0, \tag{5.19}$$

then problem (5.5)—(5.7) *admits a unique global classical solution:* $u^i(t, x)\in C^1(D_i)$ $(i=1, \cdots, n-1)$ *and* $x_i(t)\in C^2$ $(i=1, \cdots, n)$ *on the fan-shaped domain* (5.1), *moreover,*

$$\begin{cases} |u^i(t, x)|\leqslant\frac{K\varepsilon}{1+t}, \ \forall(t, x)\in D_i (i=1, \cdots, n-1), \\ \left|\frac{\partial u^i}{\partial x}(t, x)\right|, \left|\frac{\partial u^i}{\partial t}(t, x)\right|\leqslant\frac{K\varepsilon}{1+t}, \ \forall(t, x)\in D_i \quad (i=1, \cdots, n-1), \end{cases} \tag{5.20}$$

$$|x_i'(t)-x_i'(0)|\leqslant\frac{K\varepsilon}{1+t}, \ \forall t\geqslant 0, \ (i=1, \cdots, n), \tag{5.21}$$

where K *is a positive constant independent of* t.

Remark 5.2. *Similar results to Theorem* 5.1 *still hold for the typical free boundary problem on more general fan-shaped domains*(*cf.* [3]).

§ 6. Discontinuous Initial Value Problems

In this section we turn to the discontinuous initial value problem (1.9)—(1.10). We give the following hypotheses:

(H1) (1.9) is a hyperbolic system, i. e., it can be reduced to a system of characteristic form

$$\sum_{j=1}^{n}\zeta_{lj}(u)\left(\frac{\partial u_j}{\partial t}+\lambda_l(u)\frac{\partial u_j}{\partial x}\right)=0,\ (l=1,\ \cdots,\ n) \tag{6.1}$$

with $\det|\zeta_{lj}|\neq 0$.

(H2) System (6.1) is strictly hyperbolic and all characteristics are genuinely nonlinear in the sense of P. D. Lax. Without loss of generality, we may suppose that

$$\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u) \tag{6.2}$$

and

$$\nabla\lambda_k(u)\cdot\zeta^k(u)\equiv 1,\ (k=1,\cdots,n), \tag{6.3}$$

where $\zeta^k(u)$ stands for the k-th column vector of the inverse matrix $\zeta^{-1}(u)$ of $\zeta(u)$.

(H3) The solution to the corresponding Riemann problem (1.9), (1.12) is composed of constant states and n typical shocks (cf. [4], [3]).

We shall solve the discontinuous initial value problem (1.9)—(1.10) in a class of piecewise continuous and piecewise smooth functions. For this purpose we first give the following

Definition *$u=u(t, x)$ is a classical discontinuous solution containing a k-th shock $x=x(t)$ in a class of piecewise continuous and piecewise smooth functions, if $u=u(t, x)$ satisfies (1.9) out of $x=x(t)$ in the classical sence and satisfies on $x=x(t)$ the Rankine-Hugoniot condition*

$$f(u_+)-f(u_-)=s(u_+-u_-) \tag{6.4}$$

and the entropy condition

$$\begin{aligned}&\lambda_k(u_+)<s<\lambda_k(u_-),\\&\lambda_{k+1}(u_+)>s>\lambda_{k-1}(u_-),\end{aligned} \tag{6.5}$$

where $u_\pm=u(t, x(t)\pm 0)$, and $s=\dfrac{dx(t)}{dt}$ (when $k=1$ (resp. $k=n$), the term $\lambda_{k-1}(u_-)$ (resp. $\lambda_{k+1}(u_+)$) disappears in (6.5)).

By hypothesis (H3), the solution to Riemann problem (1.9), (1.12) is composed of $n+1$ constant states $\hat{u}^i(i=0, 1, \cdots, n)$ and n typical shocks $x=\hat{F}^i t(i=1, \cdots, n)$ such that the solution takes the constant value $\hat{u}^i$ on the angular domain $\hat{D}^i(i=0, 1, \cdots,n)$, where

$$\begin{aligned}&\hat{D}^0=\{(t, x)\,|\,t\geqslant 0,\ x\leqslant\hat{F}^1 t\},\\&\hat{D}^i=\{(t, x)\,|\,t\geqslant 0,\ \hat{F}^i t\leqslant x\leqslant\hat{F}^{i+1}t\},\ (i=1, \cdots,n-1),\\&\hat{D}^n=\{(t, x)\,|\,t\geqslant 0,\ \hat{F}^n t\leqslant x\}\end{aligned} \tag{6.6}$$

and

$$\hat{u}^0=u_-\triangleq u_0^-(0),\ \hat{u}^n=u_+\triangleq u_0^+(0). \tag{6.7}$$

Moreover, for $i=1, \cdots, n$,

$$f(\hat{u}^i)-f(\hat{u}^{i-1})=\hat{F}^s(\hat{u}^i-\hat{u}^{i-1}), \tag{6.8}$$

$$\begin{cases}\lambda_i(\hat{u}^i)<\hat{F}^i<\lambda_i(\hat{u}^{i-1}),\\ \lambda_{i+1}(\hat{u}^i)>\hat{F}^i>\lambda_{i-1}(\hat{u}^{i-1})\end{cases} \tag{6.9}$$

(when $i=1$ (resp. $i=n$), the term $\lambda_{i-1}(\hat{u}^{i-1})$ (resp. $\lambda_{i+1}(\hat{u}^i)$) disappears in (6.9)). In particular, noting (6.2), for $i=1$ and $i=n$, it follows from (6.9) that

$$\begin{aligned}&\hat{F}^1<\lambda_1(u_-)<\cdots<\lambda_n(u_-),\\&\lambda_1(u_+)<\cdots<\lambda_n(u_+)<\hat{F}_n.\end{aligned}\tag{6.10}$$

Regarding the discontinuous initial value problem (1.9)—(1.10) as a perturbation of Reimann problem (1.9), (1.12), we have

Theorem 6.1. *Under assumptions* (H1)—(H3), *suppose that* $u_0^-(x)$ *and* $u_0^+(x)$ *are* C^1 *function on* $x\leq 0$ *and on* $x\geq 0$ *respectively,* $f(u)$ *is a* C^2 *vector function and*

$$\eta\triangleq|u_+-u_-|=|u_0^+(0)-u_0^-(0)|>0\tag{6.11}$$

is suitably small, then there exists a positive constant ε *so small that if*

$$\begin{aligned}&|u_0^-(x)-u_0^-(0)|,\quad |u_0^{-\prime}(x)|\leq\frac{\varepsilon}{1+|x|},\ \forall x\leq 0,\\&|u_0^+(x)-u_0^+(0)|,\quad |u_0^{+\prime}(x)|\leq\frac{\varepsilon}{1+|x|},\ \forall x\geq 0,\end{aligned}\tag{6.12}$$

then problem (1.9)—(1.10) *admits a unique global classical discontinuous solution* $u=u(t,x)$ *only containing* n *shocks* $x=x_i(t)\,(x_i(0)=0)$ $(i=1,\cdots,n)$, *such that* $u(t,x)$ *belongs to* C^1 *on each domain* $D^i(i=0,1,\cdots,n)$ *and* $x_i(t)$ $(i=1,\cdots,n)$ *to* C^2 *on* $t\geq 0$ *with*

$$\begin{aligned}&|u(t,x)-\hat{u}^i|\leq\frac{K\varepsilon}{1+t},\ \forall(t,x)\in D^i,\ (i=0,1,\cdots,n),\\&\left|\frac{\partial u}{\partial x}(t,x)\right|,\left|\frac{\partial u}{\partial t}(t,x)\right|\leq\frac{K\varepsilon}{1+t},\ \forall(t,x)\in D^i,\ (i=0,1,\cdots,n),\end{aligned}\tag{6.13}$$

$$|x_i'(t)-x_i'(0)|\leq\frac{K\varepsilon}{1+t},\ (i=1,\cdots,n),\ \forall t\geq 0,\tag{6.14}$$

where

$$\begin{aligned}&D^0=\{(t,x)\,|\,t\geq 0,\ x\leq x_1(t)\},\\&D^i=\{(t,x)\,|\,t\geq 0,\ x_i(t)\leq x\leq x_{i+1}(t)\},\ (i=1,\cdots,n-1),\\&D^n=\{(t,x)\,|\,t\geq 0,\ x_n(t)\leq x\}\end{aligned}\tag{6.15}$$

and K *is a positive constant independent of* t. *Moreover,* $u(0,0)=u^i$ *on the domain* $D^i(i=0,1,\cdots,n)$ *and* $x_i'(0)=\hat{F}^i(i=1,\cdots,n)$. *Therefore, as a global perturbation,* $u(t,x)$ *possesses a structure similar to the solution to Riemann problem* (1.9), (1.12) *on* $t\geq 0$.

Proof We first use the initial condition on $x\leq 0$ to solve system (6.1) on the domain

$$\hat{D}_-=\{(t,x)\,|\,t\geq 0,\ x\leq\xi_-t\},\tag{6.16}$$

where

$$\xi_-=\frac{\lambda_1(u_-)+\hat{F}^1}{2}.\tag{6.17}$$

It is easy to see from (6.10) that (3.27) holds, then by Corollary 3.2 we can get a unique global C^1 solution $u=u^0(t,x)$ on the domain D_- and there exists a positive constant K_0 such that

$$|u^0(t,x)-u_-|,\ \left|\frac{\partial u^0}{\partial x}(t,x)\right|,\ \left|\frac{\partial u^0}{\partial t}(t,x)\right| \leqslant \frac{K_0\varepsilon}{1+t},\ \forall (t,x)\in \hat{D}_-, \tag{6.18}$$

provided that $\varepsilon>0$ is suitably small. Similarly, by means of the initial data on $x\geqslant 0$, we can get a unique global solution $u=u^n(t,x)$ on the domain

$$\hat{D}_+=\{(t,x)\,|\,t\geqslant 0,\ x\geqslant \xi_+ t\}, \tag{6.19}$$

where

$$\xi_+=\frac{\lambda^n(u_+)+\hat{F}^n}{2}, \tag{6.20}$$

and we have

$$|u_n(t,x)-u_+|,\ \left|\frac{\partial u^n}{\partial x}(t,x)\right|,\ \left|\frac{\partial u^n}{\partial t}(t,x)\right| \leqslant \frac{K_0\varepsilon}{1+t},\ \forall (t,x)\in \hat{R}_+. \tag{6.21}$$

Accooding to the local existence theorem of discontinuous solutions (see [3]), the discontinuous initial value problem (1.9)—(1.10) admits a unique classical discontinuous solution only containing n shocks $x=x_i(t)$ $(i=1,\cdots,n)$ on a local domain $D(\delta)=\{(t,x)\,|\,0\leqslant t\leqslant\delta,\ -\infty<x<\infty\}$ $(\delta>0)$, and this solution has a structure similar to the solution to the corresponding Riemann problem in a neighborhood of the origin. Moreover, the entropy condition implies that $x=x_1(t)$ must lie to the interior of $\hat{D}_-$, $x=x_n(t)$ must lie to the interior of the domain $\hat{D}_+$ and to the right side of $x=x_1(t)$; hence the solution on the left side of $x=x_1(t)$ and on the right side of $x=x_n(t)$ should be furnished by $u^0(t,x)$ and $u^n(t,x)$ respectively. Thus, in order to prove the global existence of classical discontinuous solutions to the discontinuous initial value problem (1.9)—(1.10), it is only necessary to solve the following typical free boundary problem on the fan-shaped domain

$$D=\bigcup_{i=1}^{n-1}D^i=\{(t,x)\,|\,t\geqslant 0,\ x_1(t)\leqslant x\leqslant x_n(t)\}: \tag{6.22}$$

On the domain $D^i=\{(t,x)\,|\,t\geqslant 0,\ x_i(t)\leqslant x\leqslant x_{i+1}(t)\}$ $(i=1,\cdots,n-1)$,

$$\sum_{j=1}^{n}\zeta_{lj}(u^i)\left(\frac{\partial u_j^i}{\partial t}+\lambda_l(u^i)\frac{\partial u_j^i}{\partial x}\right)=0\quad (l=1,\cdots,n); \tag{6.23}$$

On $x=x_i(t)$ $(i=1,\cdots,n)$,

$$f(u^i)-f(u^{i-1})=\frac{dx_i}{dt}(u^i-u^{i-1}), \tag{6.24}$$

$$\begin{gathered}\lambda_i(u^i)<\frac{dx_i}{dt}<\lambda_i(u^{i-1}),\\ \lambda_{i+1}(u^i)>\frac{dx_i}{dt}>\lambda_{i-1}(u^{i-1}),\end{gathered} \tag{6.25}$$

where u^i is the unknown function on $D^i(i=1,\cdots,n-1)$, $u^0=u^0(t,x)$ and $u^n=u^n(t,x)$.

Since at the origin (6.24)—(6.25) gives (6.8)—(6.9), in a neighborhood of $u^i=\hat{u}^i(i=0,1,\cdots,n)$, (6.24) can be rewritten as (cf. [3])

$$\begin{cases} v_r^{i-1}=G_r^{i-1}(v_i^{i-1},\cdots,v_n^{i-1},v_1^i,\cdots,v_i^i)\quad (r=1,\cdots,i-1),\\ v_s^i=G_s^i(v_i^{i-1},\cdots,v_n^{i-1},v_1^i,\cdots,v_i^i)\quad (s=i+1,\cdots,n),\\ \dfrac{dx_i}{dt}=F^i(u^{i-1},u^i),\end{cases} \tag{6.26}$$

where

$$v^i=\zeta(\hat{u}^i)u^i \quad (i=0, 1, \cdots, n). \tag{6.27}$$

Besides, for the minimal characterizing number of the characterizing matrix of problem (6.23)—(6.25) we have (cf. [3])

$$\|\Theta\|_{\min}=O(\eta), \tag{6.28}$$

and then

$$\|\Theta_{-1}\|_{\min}=O(\eta), \tag{6.29}$$

where η is defined by (6.11). Therefore, if η is suitably small, then

$$\|\Theta_{-1}\|_{\min}<1. \tag{6.30}$$

It is easy to verify that all other hypotheses in Theorem 5.1 are satisfied; then by Theorem 5.1 there exists a positive constant ε so small that if (6.12) (then (6.18), (6.21)) holds, then problem (6.23)—(6.25) admits a unique global classical solution $u^i=u^i(t, x)\in C^1(D^i)$ $(i=1, \cdots, n-1)$ and $x_i(t)\in C^2$ $(i=1, \cdots, n)$ on the fan-shaped domain (6.22), and (6.13)—(6.14) hold. This also shows that $x=x_1(t)$ and $x=x_n(t)$ always lie to the interior of $\hat{D}_-$ and $\hat{D}_+$ respectively, provided that $\varepsilon>0$ is suitably small; then the previous procedure of constructing the global classical discontinuous solution is reasonable. The proof of Theorem 6.1 is finished.

References

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