Chin. Ann. of Math.
11B:1(1990), 33—44.

ON THE DISTRIBUTION OF VALUES OF RANDOM DIRICHLET SERIES (II)**

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(*Dedicated to the Tenth Anniversary of CAM*)

Abstract

For certain Dirichlet series almost surely (a. s.) of order (R) $\rho\in(0,\infty)$ in the right-half plane, a. s. every point of the imaginary axis is a Borel point of order $\rho+1$ and with no finite exceptional value.

In [9, 10] we studied the distribution of values of random Dirichlet series a. s. of infinite order(R) in the right-half plane or in the whole plane and introduced the N-sequence $\{Z_n(\omega)\}$ $(n\in N_+)$ of random variables, a sequence of independent, symmetric and equally distributed real or complex variables of finite variance in the probability space $(\Omega, \mathscr{A}, \mathscr{P})$ $(\omega\in\Omega)$ for which $\exists k_0\in N_+$ such that

$$\int_{|Z|<1}|Z_n|^{-1/k_0}\mu(dZ_n)<\infty,$$

where μ is the common measure defined by $Z_n(\omega)$. The classical Rademacher, Steinhaus and Gauss sequences are special cases of the N-sequence.

In this paper corresponding to the N-sequence we study the distribution of values of random Dirichlet series almost surely (a. s.) of finite order(R) in the right-half plane and improve some results in [6] and [7]. Here we have Borel points of an accurate order (R) and with no finite exceptional value as in the case of Borel directions in [3], the method adopted being different. We indicate corresponding results for some random Dirichlet series a. s. of finite order (R) in the whole plane. The results in this paper can be extended to the case of (p, q)-order(R) as in [9], [10].

§1. Some Lemmas on Meromorphic Functions

We shall give some lemmas on meromorphic functions and we generalize first the modified second fundamental theorem in the unit disc in [5], p. 291 by replacing

Manuscript received January 16, 1989.
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** Projects Supported by the National Natural Science Foundation of China.

constants a_i by functions of slower growth. Let $f(z)$ be meromosphic in $|z|<1$ and put

$$S(r, f)=\frac{1}{\pi}\iint\limits_{|z|<r}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 r\,dr\,d\theta \quad (z=re^{i\theta},\ 0\leqslant r<1).$$

Consider the Ahlfors–Shimizu characteristic function

$$T_0(r, f)=\int_0^r \frac{S(t, f)}{t}\,dt$$

for which the relation

$$|T(r, f)-T_0(r, f)-\log^+|f(0)||\leqslant 2\log 2 \tag{1}$$

holds, where $T(r, f)$ is the Nevanlinna characteristic function[1]. We have

Lemma 1. *Suppose that $f(z)$, $\varphi_1(z)$, $\varphi_2(z)$ and $\varphi_3(z)$ are four different functions meromorphic in $|z|<1$. Then $\forall m>0$,*

$$T_0(r, f)\leqslant\left(1+\frac{2}{m}\right)\sum_{j=1}^{3}N\left(\frac{mr+1}{m+1},\ f=\varphi_j\right)+6\sum_{j=1}^{3}T\left(\frac{mr+1}{m+1},\ \varphi_j\right)$$
$$+A(m+2)\log\frac{1}{1-r}+B,$$

where $N(r,\ f=\varphi_j)$ is the usual notation in the theory of meromorphic functions, A is an absolute constaint and B is a constant depending only upon $\varphi_1(z)$, $\varphi_2(z)$, $\varphi_3(z)$ and $f(0)$.

Proof Put

$$g(z)=\frac{f(z)-\varphi_1(z)}{f(z)-\varphi_3(z)}\cdot\frac{\varphi_2(z)-\varphi_3(z)}{\varphi_2(z)-\varphi_1(z)}.$$

By Theorem VI. 21 in [5] (p. 260), $\forall m>0$,

$$S(r,\ g(z))\leqslant\sum_{j=1}^{3}n(R,\ g(z)=b_j)+\frac{AR}{R-r}, \tag{2}$$

where $n(R,\ g(z)=b_j)$ is a usual notation, $b_1=0$, $b_2=1$, $b_3=\infty$ and $0\leqslant r<R<1$. $\forall m>0$, let $R=\frac{mr+1}{r+1}$. From (2) we obtain

$$\int_{1/2}^{r}\frac{mS(t,\ g)}{m+1/t}\,\frac{dt}{t}\leqslant\sum_{j=1}^{3}\int_0^R\frac{n(t,\ g=b_j)}{t}\,dt+\int_{1/2}^{r}\frac{Am(mt+1)}{(1-t)(m+1/t)}\,\frac{dt}{t},$$

and consequently

$$\left(\frac{m}{m+2}\,T_0(r,\ g)-T_0\left(\frac{1}{2},\ g\right)\right)\leqslant\sum_{j=1}^{3}N(R,\ g=b_j)+2Am\log\left(\frac{1}{1-r}\right).$$

and

$$T_0(r,\ g)\leqslant\left(1+\frac{2}{m}\right)\sum_{j=1}^{3}N\left(\frac{mr+1}{m+1},\ g=b_j\right)+2A(m+2)\log\frac{1}{1-r}+T_0\left(\frac{1}{2},\ g\right).$$

Since

$$T(r,\ f)\leqslant T(r,\ g)+4\sum_{j=1}^{3}T(r,\ \varphi_j)+B,$$

where B is a constant depending only on φ_1, φ_2, φ_3, and $f(0)$, we have

$$\sum_{j=1}^{3}N\left(\frac{mr+1}{m+1},\ g=b_j\right)\leqslant\sum_{j=1}^{3}N\left(\frac{mr+1}{m+1},\ f=\varphi_j\right)+2\sum_{j=1}^{3}T\left(\frac{mr+1}{1+m}\ \varphi_3,\right)$$

and obtain the conclusion of Lemma 1 by (1).

If we take $\varphi_j=a_j$ and $m=1/3$, we obtain immediately Theorem VII. 15 in [5] (p.291).

Definition. *If the non-decreasing positive function $h(r)$ $(0\leqslant r<1)$ satisfies*

$$\varlimsup_{r\to 1-0}\left(\log h(r)/\log\frac{1}{1-r}\right)=\rho,$$

then we say that $h(r)$ is of order ρ.

Let $f(z)$ be meromorphic in the sector $A(t, b)=\{z\,|\,|z|<1\}\cup\{z\,|\,|\arg z-t|<b\}$. If $a\in \boldsymbol{C}\cup\{\infty\}$ (with at most two exceptional values) and $\forall\delta\in(0, b)$, $n(r, A(t, \delta), f=a)$ is of order ρ, then e^{it} is called a Borel point of order ρ of $f(z)$, where $n(r, A(t, \delta), f=a)$ is the number of roots of $f(z)=a$ in $A(t, \delta)\cap\{z\,|\,|z|<r\}$.

Put

$$S(r, A(t, b), f)=\frac{1}{\pi}\iint_{A(t,b)\cap\{|z|<r\}}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 r\,dr\,d\theta,$$

$$H(\rho, t, b)$$
$$=\{\{\varphi\}\cup\boldsymbol{C}\cup\{\infty\}\,|\,\varphi \text{ meromorphic in } |z|<1 \text{ and } S(r, A(t, b), \varphi) \text{ of order}<\rho\}.$$

If $\forall\varphi\in H(\rho, t, b)$ (with at most two exceptional functions) and $\forall\delta\in(0, b)$, $n(r, A(t, \delta), f=\varphi)$ is of order ρ, then e^{it} is called a strong Borel point of order ρ of $f(z)$.

We consider a conformal mapping in the following lemma.

Lemma 2. $\forall\varepsilon>0$, $\exists b\in(0, 1)$ *such that*

$$w(\{z|b<|z|<r\}\cap\{z\,|\,|\arg z|<\varepsilon/2\}\subset\{\omega\,|\,|\omega|<1-(1-r)\pi/20\varepsilon\}$$
$$\subset w\{z\,|\,|z|<(319+r)/320\}\cap\{z\,|\,|\arg z|<\varepsilon\},$$

where

$$w(z)=\frac{z^{\pi/2}+2z^{\pi/2\varepsilon}-1}{z^{\pi/2}-2z^{\pi/2\varepsilon}-1} \tag{3}$$

and b is a constant depending on ε.

Proof Since $\lim\limits_{p\to 1-0}((1-p^{\pi/\varepsilon})/(1-p))=\pi/\varepsilon$, $\exists b\in((1/2)^{2\varepsilon/\pi}, 1)$ such that $1/2<p^{\pi/2\varepsilon}<1$ and $(1-p)\pi/2\varepsilon<1-p^{\pi/\varepsilon}<(1-p)^{2\pi/\varepsilon}$ when $b<p<1$.

It is easy to verify that $\omega(z)$ maps conformally the sector $A(0, \varepsilon)$ into the unit disc $|\omega|<1$ and $\omega((\sqrt{2}-1)^{2\varepsilon/\pi})=0$.

Let $z_0=pe^{i\varphi}\in A(0, \varepsilon)$. By (3) we have

$$1-|w(z_0)|=1-\sqrt{\frac{A^2+B^2}{C^2+D^2}}=\frac{C^2+D^2-A^2-B^2}{C^2+D^2+\sqrt{(A^2+B^2)(C^2+D^2)}}$$
$$=\frac{8p^{\pi/2\varepsilon}(1-p^{\pi/\varepsilon})\cos(\varphi\pi/2\varepsilon)}{C^2+D^2+\sqrt{(A^2+B^2)(C^2+D^2)}}$$

where

$$A=p^{\pi/\varepsilon}\cos(\varphi\pi/\varepsilon)+2p^{\pi/2\varepsilon}\cos(\varphi\pi/2\varepsilon)-1,$$
$$B=p^{\pi/\varepsilon}\sin(\varphi\pi/\varepsilon)+2p^{\pi/2\varepsilon}\sin(\varphi\pi/2\varepsilon),$$

$$C=p^{\pi/s}\cos(\varphi\pi/s)-2p^{\pi/2s}\cos(\varphi\pi/2s)-1,$$
$$D=p^{\pi/s}\sin(\varphi\pi/s)-2p^{\pi/2s}\sin(\varphi\pi/2s).$$

Since

$$1\leqslant p^{2\pi/s}+2p^{\pi/s}+1+4p^{\pi/2s}(1-p^{\pi/s})\cos(\varphi\pi/2s)+2p^{\pi/s}(1-\cos(\varphi\pi/s))$$
$$=C^2+D^2\leqslant C^2+D^2+\sqrt{(A^2+B^2)(C^2+D^2)}$$
$$\leqslant 2(C^2+D^2)\leqslant 20,$$

we have

$$\max\{1-|w(pe^{i\varphi})|\,|pe^{i\varphi}\in A(0,\ s),\ p=r \text{ or } \varphi=\pm s\}\leqslant 8(1-r)(2\pi/s)=16(1-r)\pi/s$$

and

$$\min\{1-|w(pe^{i\varphi})|\,|\,r>p>b,\ |\varphi|>s/2\}>8\cdot(1/2)(\pi/2s)(1-p)(\sqrt{2}/2)/20$$
$$>(1/20)(\pi/s)(1-p)>(\pi/20s)(1-r).$$

Hence

$$w(\{z|b<|z|<r\}\cap\{z|\,|\arg z|<s/2\})\subset\{w|1-|w|>(\pi/20s)(1-r)\}$$
$$=\{w|\,|w|<1-(\pi/20s)(1-r)\}$$
$$=\{w|\,|w|<1-(16\pi/s)(1-(319+r)/320)\}$$
$$\subset w(\{z|\,|z|<(319+r)/320\}\cap\{z|\,|\arg z|<s\}.$$

We establish now some relation between Borel points and strong Borel points.

Lemma 3. *Suppose that e^{it_0} is a Borel point of at least order $\rho(>1)$ of $f(z)$ meromorphic in the sector $A(t_0,\ b)$. Then e^{it_0} is also a storng Borel point of at least order ρ of $f(z)$.*

Proof Take $a\in \boldsymbol{C}\cup\{\infty\}$ such that for sufficiently small $\delta>0$, we have

$$\varlimsup_{r\to 1-0}\log n(r,\ A(t_0,\delta/2),\ f=a)/\log(1/(1-r))\geqslant\rho>1. \tag{4}$$

Without loss of generality we can assume that $t_0=0$ and that $f((\sqrt{2}-1)^{2s/\pi})\neq\infty$, a. By (4), $\exists(0<)r_n\uparrow 1$ such that $\forall s>0$,

$$n(r_n,\ A(0,\ \delta/2),\ f=a)\geqslant(1/(1-r_n))^{\rho-s}.$$

Consider the mapping (3) and let $R_n=1-(\pi/20\delta)(1-r_n)$. Then

$$1/(1-r_n)=(20\delta/\pi)(1/(1-R_n)).$$

Hence by Lemma 2, we have

$$n(R_n,\ f(z(w))=a)+C\geqslant n(r_nA(0,\ \delta/2),\ f=a)\geqslant(1/(1-r_n))^{\rho/s}\geqslant C(1/(1-R_n))^{\rho-s}$$

where $z=z(w)$ is the inverse mapping of (3) and C is a suitable constant. Hence for the function $f(z(w))$ meromorphic in the unit circle, $n(R,\ f(z(w))=a)$ is of order at least ρ. By a lemma in [4], $N(R,\ f(z(w))=a)$ is of order at least $\rho-1(>0)$. By Nevanlinna first fundemental theorem, $T(R,\ f(z(w)))$ is of order at least $\rho-1$.

$\forall\varphi(z)\in H(\rho,\ 0,\ b)$, i.e, $S(R,\ A(0,\ b),\ \varphi)$ is of order less than ρ and consequently $S(R,\ A(0,\ \delta/2),\ \varphi)(0<\delta<b)$ is of order less than ρ. By Lemma 2, $S(R',\ \varphi(z(w)))$ is of order less than ρ and by a lemma in [4], $T_0(R',\ \varphi(z(w)))$ is of order at least $\rho-1$. By Lemma 1, $N(R,\ f(z(w))=\varphi(z(w)))$ is of order at least $\rho-1$(with two possible

exceptional values in $H(\rho, 0, b)$). By a lemma in [4], $n(R, f(z(w))=\varphi(z(w)))$ is of order at least ρ. By Lemma 2, $n(r, A(0, \delta), f(z)=\varphi(z))$ is of order at least ρ. The proof is completed.

§2. Random Dirichlet Series of Finite Order (R) in the Right-Half Plane

Let $(C_n, \mathscr{B}_n)$ $(n\in N_+)$ be a sequence of complex Borel measurable spaces and let $\{Z_n(w)\}$ be a sequence of independent random variables defined in the probability space $(\Omega, \mathscr{A}, \mathscr{P})$ and taking their valves respectively in $\{(C_n, \mathscr{B}_n)\}$. Evidently) $Z(w)=\{Z_1(w), Z_2(w), \cdots\}$ is a mapping of Ω into $\prod_{n=1}^{\infty} C_n$. Let

$$\mu_n(B_n)=\mathscr{P}(Z_n^{-1}(B_n)) \quad (\forall n\in N_+, \forall B_n\in\mathscr{B}_n).$$

Then $\{(C_n, \mathscr{B}_n, \mu_n)\}$ forms a sequence of probability spaces. Suppose that they are complete, i. e., if $A\in\mathscr{B}_n(\forall n\in N_+)$, $\mu_n(A)=0$ and $B\subset A$, then $B\in\mathscr{B}_n$. Let $\prod_{n=1}^{\infty}\mathscr{B}_n$ be the least σ-algebra of all sets of the form

$$B=\left\{Z_1, Z_2, \cdots) \mid (Z_1, Z_2, \cdots)\in\prod_{1=u}^{\infty} C_n;\ Z_j\in B_j\in\mathscr{B}_j,\ \forall j\in\{1, 2, \cdots, n_0\},\ n_0\in N_+.\right\} \tag{5}$$

Since for any set B of form (5)

$$Z^{-1}(B)=\{w \mid Z_j(w)\in B_j,\ j\in\{1, 2, \cdots, n_0\},\ n_0\in N_+\}=\bigcap_{j=1}^{n_0}\{Z^{-1}(B_j)\}\in\mathscr{A},$$

$\forall B\in\prod_{n=1}^{\infty}\mathscr{B}_n$ we have $Z^{-1}(B)\in\mathscr{A}$. Hence $Z(w)$ is a random variable defined in $(\Omega, \mathscr{A}, \mathscr{P})$ and taking values in the product space $\left(\prod_{n=1}^{\infty} C_n, \prod_{n=1}^{\infty}\mathscr{B}_n\right)$. Let

$$\mu(B)=\mathscr{P}(Z^{-1}(B)),\quad \forall B\in\prod_{n=1}^{\infty}\mathscr{B}_n.$$

For any set B in $\prod_{n=1}^{\infty}\mathscr{B}_n$ of form (5),

$$\mu(B)=\mathscr{P}(\{w \mid z(w)\in B\})=\mathscr{P}(\{w \mid Z_j(w)\in B_j,\ \forall j\in\{1, 2, \cdots, n_0\},\ n_0\in N_+\}$$
$$=\prod_{j=1}^{n_0}\mathscr{P}(\{w \mid Z_j(w)\in B_j\}=\prod_{j=1}^{n_0}\mu_j(B_j).$$

Consequently $\left(\prod_{n=1}^{\infty} C_n, \prod_{n=1}^{\infty}\mathscr{B}_n, \mu\right)$ is the product probability space of the sequence of probability spaces $\{(C_n, \mathscr{B}_n, \mu_n)\}$ (see [2]).

Now we study Borel points of some random function meromorphic in a sector.

Lemma 4. *Suppose that $\{Z_n'(w)\}$ $(n\in N_+)$ is a sequence of the independent complex random variables defined in probability space $(\Omega, \mathscr{A}, \mathscr{P})$ and satisfying*

$$K=\sup\{\mathscr{P}(Z_n(w)=C) \mid C\in \boldsymbol{C},\ n\in N_+\}<1. \tag{6}$$

Suppose that the sequence of functions $\{\varphi_n(z)\}\subset H(\rho, \theta, b)-\{\infty\}$ $(\rho>1,\ \theta\in R,\ 0<b<$

π) *and thbt the random series*

$$f_w(Z)=\sum_{n=1}^{\infty}Z_n(w)\varphi_n(z)$$

defines a random meromorphic function in the sector $\{z\mid |\arg Z-\theta|<b\}\cap\{z\mid |z|<1\}$. *If* $e^{i\theta}$ *is a. s. a Borel point of order* $\rho(>1)$ *of* $f_w(z)$, *then it is a. s. a Borel point of order* ρ *and with no finite exceptional value.*

Proof We can assume that $e^{i\theta}$ is surely a Borel point of order $\rho(>1)$ of $f_w(z)$. Otherwise we exclude an event of probability zero.

$\forall s\in(0,1)$, take $N>\log s/\log K$. $\forall\{c_n\}_{N+1}^{\infty}$ if $\{w\mid Z_n(w)=c_n,$
$n=N+1, N+2, \cdots\}=\phi$, put $\mathscr{D}(\{c_n\}_{N+1}^{\infty})=\phi$.

If $\exists w_0\in\Omega$ such that $Z_n(w_0)=c_n$, $n=N+1, N+1, \cdots$, put

$$\mathscr{D}(\{b_n\}_{N+1}^{\infty})=\left\{\begin{array}{l}(X_1, X_2, \cdots, X_n)\mid \text{For } \sum_{n=1}^{N}X_n\varphi_n(z)+\sum_{n=N+1}^{\infty}c_n\varphi_n(z),\\ e^{i\theta}\text{ is a Borel point of order }\rho\text{ with a finite exceptional value.}\end{array}\right\}$$

We now prove that there are at most two different elements in $\mathscr{D}(\{c_n\}_{N+1}^{\infty})$. Otherwise there would exist three different

$$(X_{j1}, X_{j2}, \cdots, X_{jN})\in\mathscr{D}(\{c_n\}_{N+1}^{\infty})\quad(j=1, 2, 3),$$

three corresponding constants $a_j\in C$ and $\eta>0$ such that the order of

$$n\Big(r, \theta, \eta, \sum_{n=1}^{N}X_{jn}\varphi_n(z)+\sum_{n=N+1}^{\infty}c_n\varphi_n(z)=a_j\Big)$$
$$=n\Big(r, \theta, \eta, \sum_{n=1}^{N}X_{jn}\varphi_n(z)+\sum_{n=N+1}^{\infty}Z_n(w_0)\varphi_n(z)=a_j\Big)$$

would be less than ρ. Put

$$\varphi_j(z)=\sum_{n=1}^{N}(Z_n(w_0)-X_{jn})\varphi_n(z)+a_j\quad(j=1, 2, 3).$$

Then the order of $n(r, \theta, \eta, f_{w_0}(z)=\varphi_j(z))$ would be less than ρ. Hence $e^{i\theta}$ would not be a strong Borel point of $f_{w_0}(z)$, which is in contradiction to Lemma 3.

In order to complete the proof we need only to prove that $\mathscr{P}(H)=0$, where

$H=\{w\mid$ For $f_w(z)$, $e^{i\theta}$ is a Borel point of order ρ with a finite exceptional valne$\}$.

Put

$$Y=\{(X_1, X_2, \cdots)\mid X_j\in \boldsymbol{C}(j\in N_+),\ (X_1, X_2, \cdots, X_N)\in\mathscr{D}(\{X_n\}_{N+1}^{\infty})\},$$
$$Q=\{w\mid Z_1(w), Z_2(w), \cdots)\in Y\}.$$

Since $\forall w\in H$, $(Z_1(w), Z_2(w), \cdots, Z_n(w))\in\mathscr{D}(\{Z_n(w)\}_{N+1}^{\infty})$, we have $(Z_1(w), Z_2(w), \cdots)\in Y$ and consequently $H\subset Q$. Hence

$$\mathscr{P}(H)\leqslant\mathscr{P}(Q)=\mu(Y)=\mathscr{E}(\mathbf{1}_Y)$$
$$=\lim_{m\to\infty}\int\cdots\int\mathbf{1}_Y\mu_1(dZ_1)\mu_2(dZ_2)\cdots\mu_m(dZ_m)\quad(m>N)$$
$$=\lim_{m\to\infty}\int\cdots\int\mu_{N+1}(dZ_{N+1})\cdots\mu_m(dZ_m)\int\cdots\int\mathbf{1}_Y\mu_1(dZ_1)\cdots\mu_N(dZ_N)$$
$$=\lim_{m\to\infty}\int\cdots\int\mu_{N+1}(dZ_{N+1})\cdots\mu_m(dZ_m)\int\cdots\int\mathbf{1}_{\mathscr{D}(\{Z_n\}_{N+1}^{\infty})}\mu_1(dZ_1)\cdots\mu_N(dZ_n)$$

$$\leqslant \lim_{m\to\infty} \int\cdots\int \mu_{N+1}(dZ_{N+1})\cdots\mu_m(dZ_m)\int\cdots\int 1_{\bigcup_j \bigcap_{n=1}^{N}(Z_n=X_{jn})}\mu_1(dZ_1)\cdots\mu_n(dZ_n)$$

$$\leqslant \lim_{m\to\infty} \int\cdots\int \sum_j \prod_{n=1}^{N} \mu_n(Z_n=X_{jn})\mu_{N+1}^{\infty}(dZ_{N+1})\cdots\mu_m(dZ_m)\leqslant 2K^N\leqslant 2\varepsilon,$$

where μ, μ_1, μ_2, $\cdots$ are measures defined by corresponding random variables and $(X_{j1}, \cdots, X_{jn})$ are elements in $\mathscr{D}(\{z_n\}_{N+1}^{\infty})$, the number of which is at most 2. Since $\varepsilon>0$ is arbitrary, the proof is completed.

We improve here a theorem in [7] and [11] on the order(R) of a Dirichlet series whose abscissa of convergence is zero.

Theorem 1. *Suppose that for the Dirichlet series*

$$f(s)=\sum_{n=0}^{\infty} b_n e^{-\lambda_n s} \quad (s=\sigma+it)$$

the abscissa of convergence is zero, where

$$0=\lambda_0<\lambda_1<\lambda_2<\cdots<\lambda_n\uparrow+\infty$$

and

$$\overline{\lim_{n\to\infty}}\, \frac{\log\log n}{\log\lambda_n}<\frac{\rho}{1+\rho} \quad (0<\rho<\infty), \tag{7}$$

then

$$\overline{\lim_{\sigma\to 0}}\, \frac{\log^+\log^+ M(\sigma, f)}{\log(1/\sigma)}=\rho \Leftrightarrow \overline{\lim_{n\to\infty}}\, \frac{\log^+\log^+|b_n|}{\log\lambda_n}=\frac{\rho}{\rho+1}, \tag{8}$$

where

$$M(\sigma, f)=\sup_{-\infty<t<\infty} |f(\sigma+it)| \quad (\sigma>0).$$

The left-side of (8) is the definition of the order(R) ρ of $f(s)$ in $\operatorname{Re}s>0$.

Proof In this theorem the conditions $\overline{\lim}_{n\to\infty}(n/\lambda_n)<\infty$ in [7] and $\overline{\lim}_{n\to\infty}(\log n/\log\lambda_n)<\infty$ in [11] are replaced by a weaker condition (7). Hence we need only to modify the determination of an upper bound of $\sum_{n\to 0}^{\infty} e^{-\lambda_n\varepsilon\sigma}$.

By (7), $\exists\rho''\in(0, \rho)$ such that

$$\overline{\lim_{n\to\infty}}\, \frac{\log\log n}{\log\lambda_n}<\frac{\rho''}{1+\rho''}.$$

Hence, $\exists N$, $\forall n>N$, $\lambda_n>(\log n)^{(1+\rho'')/\rho''}>1$ and consequently

$$\sum_{n=0}^{\infty} e^{-\lambda_n\varepsilon\sigma}\leqslant N+1+\sum_{n=N+0}^{\infty} e^{-\varepsilon\sigma(\log n)^{(1+\rho'')/\rho''}}=N+1+\sum_{n=N+1}^{\infty} n^{-\varepsilon\sigma(\log n)^{1/\rho''}}$$

$$=N+1+\sum_{n=N+1}^{T} n^{-\varepsilon\sigma}+\sum_{n=T+1}^{\infty} n^{-2}\leqslant C+\int_N^T \frac{dt}{t^{2\sigma}}$$

$$=C+\frac{1}{1-\varepsilon\sigma}T^{1-\varepsilon\sigma},$$

where $T=$ the integral part of $\exp[(2/\varepsilon\sigma)\rho'']$ and C is a constant. The rest part of the previous proofs remains unchanged.

Remark. In a certain sense condition (7) is best possible. If $\lambda_n=(\log n)^{(1+\rho)/\rho}$, then

$$\lim_{n\to\infty}\frac{\log\log n}{\log\lambda_n}=\frac{\rho}{1+\rho}$$

and for the Dirichlet series $f(s)=\sum_{n=1}^{\infty}e^{-\lambda_n s}$,

$$M(\sigma, f)=\sum_{n=1}^{\infty}e^{-\lambda_n\sigma}=\sum_{n=1}^{\infty}(1/n^{\sigma(\log n)^{1/\rho}})$$

$$\geqslant\sum_{n=1}^{T}(1/n^{1/2})\geqslant T^{1/2}\geqslant e^{(1/3\sigma)^{\rho}/2},$$

where T is the integral part of $1+\exp\left(\left(\frac{1}{3\sigma}\right)^{\rho}\right)$. Hence

$$\lim_{\sigma\to+0}\frac{\log^+\log^+ M(\sigma, f)}{\log(1/\sigma)}\geqslant\rho,$$

while

$$\lim_{n\to\infty}\frac{\log^+\log^+ 1}{\log\lambda_n}=0,$$

contrary to (8).

Given a Dirichlet series satisfying the conditions in Theorem 1, we consider the Dirichlet–N series

$$f_w(s)=\sum_{n=1}^{\infty}b_n Z_n(w)e^{-\lambda_n s}\tag{9}$$

where $\{Z_n(w)\}$ is an N-sequence defined in $(\Omega, \mathscr{A}, \mathscr{P})$. We have, $\forall w\in\Omega$, $\exists N(w)>0$ a. s. such that $\forall n>N(w)$

$$n^{-k_0}\leqslant|Z_n(w)|\leqslant n^{k_0}\ \text{(see [8, 9])}.$$

By this result and Theorem 1 we find easily that $f_w(s)$ is of order(R) ρ in $\operatorname{Re}s>0$ a. s. We shall prove that on certain sequences of points whose abscissas converge to zero $f_w(s)$ is a. s. of growth rapid enough.

Theorem 2. *If the Dirichlet–N series (9) satisfies the conditions in Theorem 1 and if the sequence of complex numbers $\{s_m\}$ satisfies*

$$\frac{B}{Q^m}<\operatorname{Re}s_m<\frac{A}{Q^m},\tag{10}$$

where A, B and $Q(>1)$ are positive constandts, then

$$\overline{\lim_{m\to\infty}}\frac{\log^+\log^+|f_w(s_m)|}{-\log\sigma_m}=\rho\ \text{a. s.}\qquad(\operatorname{Re}s_m=\sigma_m).\tag{11}$$

Proof By Theorem 1 we need to prove $\mathscr{P}(H)=0$, where

$$H=\left\{w\,\middle|\,\overline{\lim_{m\to\infty}}\frac{\log^+\log^+|f_w(s_m)|}{-\log\sigma_m}<\rho\right\}.$$

Take $(\rho>)\varepsilon_n\downarrow0$ and put

$$H_n=\left\{w\,\middle|\,\overline{\lim_{m\to 8}}\frac{\log^+\log^+|f_w(s_m)|}{-\log\sigma_m}<\rho-\varepsilon_n\right\}.$$

We see that $H=\bigcup_n H_n$.

Suppose that $\mathscr{P}(H)>0$. Then $\exists n_0\in N_+$ such that $\forall w\in H_{n_0}$

$$\overline{\lim_{m\to\infty}} \frac{\log^+ \log^+ |f_\omega(s)|}{-\log \sigma_m} < \rho - \varepsilon_{n_0} \text{ and } \mathscr{P}(H_{n_0}) > 0.$$

Hence, $\forall \omega \in H_{n_0}$, $\exists M_\omega \in N_+$ such that $\forall m > M_\omega$

$$\left| \sum_{n=0}^{\infty} b_n Z_n(\omega) e^{-\lambda_n s_m} \right| < \exp\exists((1/\sigma_m)^{\rho-\varepsilon_{n_0}}). \tag{12}$$

Evidently $H_{n_0} = \bigcup_m \{\omega | \omega \in H_{n_0}, M_\omega > m\}$ and consequently $\exists m_0 \in N_+$ such that $\rho(H')$ > 0, where

$$H' = \{\omega | \omega \in H_{n_0}, M_\omega < m_0\}.$$

Then $\forall m > m_0$, $\forall \omega \in H'$, (12) holds.

By an extension of a lemma of Paley–Zygmund to N-sequences $\exists N = N(H')$, $e = e(H) > 0$ such that

$$\sum_{n=N}^{\infty} |b_n|^2 e^{-2\lambda_n \sigma_m} \leqslant (1/e) \int_{H'} \left| \sum_{n=N}^{\infty} b_n Z_n(\omega) e^{-\lambda_n s_m} \right|^2 \varphi(d\omega) \quad \text{(see [9, 10])}.$$

Since $\forall m > m_0$, $\forall \omega \in H'$,

$$\begin{aligned} \left| \sum_{n=N}^{\infty} b_n Z_n(\omega) e^{-\lambda_n s_m} \right| &\leqslant \left| \sum_{n=1}^{N-1} b_n Z_n(\omega) e^{-\lambda_n s_m} \right| + \exp((1/\sigma_m)^{\rho-\varepsilon_{n_0}}) \\ &\leqslant K\Big(1 + \sum_{n=1}^{N-1} |Z_n(\omega)|\Big) \exp((1/\sigma_m)^{\rho-\varepsilon_{n_0}}), \end{aligned}$$

where $K = \max\{1, |b_1|, |b_2|, \cdots, |b_{N-1}|\}$, we have $\forall m > m_0$

$$\begin{aligned} \sum_{u=N}^{\infty} |b_n|^2 e^{-2\lambda_n \sigma_m} &\leqslant (1/e) \int_{H'} K^2 \Big(1 + \sum_{n=1}^{N-1} |Z_n(\omega)|\Big)^2 \exp(2(1/\sigma_m)^{\rho-\varepsilon_{n_0}}) \mathscr{P}(d\omega) \\ &\leqslant C \exp(2(1/\sigma_m)^{\rho-\varepsilon_{n_0}}), \end{aligned}$$

where C is a constant. Hence $\forall n > N$, $\forall m > m_0$

$$|b_n| e^{-\lambda_n \sigma_m} \leqslant \sqrt{c} \exp((1/\sigma_m)^{\rho-\varepsilon_{n_0}})$$

and

$$\log^+ |b_n| \leqslant \log \sqrt{c} + \lambda_n \sigma_m + (1/\sigma_m)^{\rho-\varepsilon_{n_0}}. \tag{13}$$

Take n sufficiently large, $\exists\ m = m(n)$ such that

$$Q^m \leqslant \lambda_n^{1/(\rho+1-\varepsilon_0)} \leqslant Q^{m+1} \quad (\varepsilon_0 = \varepsilon_{n_0}).$$

By (10) and the above inequalities

$$\lambda_n^{1/(\rho+1-\varepsilon_0)}/B \geqslant \frac{Q^m}{B} \geqslant \frac{1}{\sigma_m} \geqslant \frac{Q^m}{A} \geqslant \lambda_n^{1/(\rho+1-\varepsilon_0)}/QA,$$

and by (13)

$$\log^+ |b_n| \leqslant \log \sqrt{c} + QA\lambda_n^{(\rho-\varepsilon_0)/(\rho+2-\varepsilon_0)} + B^{-\rho+\varepsilon_0} \lambda_n^{(\rho-\varepsilon_0)/(\rho+1-\varepsilon_0)}.$$

Hence

$$\overline{\lim_{n\to\infty}} \frac{\log^+ \log^+ |b_n|}{\log \lambda_n} \leqslant \frac{\rho - \varepsilon_0}{\rho + 1 - \varepsilon_0} < \frac{\rho}{\rho + 1},$$

contrary to Theorem 1. The proof of Theorem 2 is completed.

Now we can prove the main theorem in this paper.

Theorem 3. *If the random Dirichlet-N series*

$$f_\omega(s) = \sum_{n=1}^{\infty} b_n Z_n(\omega) e^{-\lambda_n s} \tag{14}$$

satisfies (7) *and the right side of* (8), *where* $\{Z_n(\omega)\}$ *is an* N*-sequence of random variables, then for* $f_\omega(s)$, *a. s. every point it* $(t\in R)$ *is a Borel point of order*(R) $\rho+1$ *and with no finite exceptional value, i. e.,* $\forall\omega\in\Omega-E$, $\forall t\in R$, $\forall\delta>0$ *and* $\forall a\in \mathbf{C}$,

$$\overline{\lim_{\sigma\to+0}}\frac{\log n(\sigma,\ t,\ \delta,\ f_\omega=a)}{\log(1/\sigma)}=\rho+1,$$

where $\mathscr{P}(E)=0$ *and* $n(\sigma,\ t,\ \delta,\ f_\omega=a)$ *denotes the number of roots of*

$$f_\omega(s)=a \text{ in } \{s\,|\,\mathrm{Re}\ s>\sigma>0,\ |\mathrm{Im}s-t|\leqslant\delta\}.$$

Proof Fix $t_0\in R$ and take $\varepsilon\in(0,\ \pi/2)$. It is easy to verify that

$$Z(s,\ t_0,\ \varepsilon)=\frac{e^{\pi(it_0-s)/\varepsilon}-1+2e^{\pi(it_0-s)/2\varepsilon}}{-e^{\pi(it_0-s)/\varepsilon}+1+2e^{\pi(it_0-s)/2\varepsilon}}$$

maps conformally the half-strip $\Delta(t_0,\ \varepsilon)=\{s\,|\,\mathrm{Re}s>0,\ |\mathrm{Im}\ s-t_0|<\varepsilon\}$ into the unit disc $\{z\,|\,|z|<1\}$. The inverse transformation

$$s(z,\ t_0,\ \varepsilon)=it_0-\frac{2\varepsilon}{\pi}\log\frac{-1+z+\sqrt{2+2z^2}}{1+z}$$

maps conformally the unit circle into $\Delta(t_0,\ \varepsilon)$, where for the logarithmic functions concerned we take the branch which takes real values on $(0,\ 1)$.

We shall prove in the following that the random analytic function in the unit disc

$$g_\omega(z)=f_\omega(s(z,\ t_0,\ \varepsilon))$$

satisfies a. s.

$$\overline{\lim_{r\to1-0}}\frac{\log T(r,\ g_\omega)}{-\log(1-r)}=\rho. \tag{15}$$

$\forall\omega$ for which (15) holds, by Nevanlinna Second Fundamental Theorem, $\forall a\subset C$ possibly with one exception, the order of $N(r,\ g_\omega=a)$ is ρ and by a lemma in [5] the order of $n(r,\ g_\omega=a)$ is $\rho+1$. By Lemma 4 and a conformal mapping we see that it_0 is a. s. a Borel point of order(R) $\rho+1$ and without finite exceptional value of $f_\omega(s)$. Arrange all the rational numbers in a sequence $\{t_n\}$. Then $it_n(n\in N_+)$ is a. s. a Borel point as it_0. Hence for $f_\omega(s)$, a. s. every it_n, and consequently every it is a Borel point of order(R) $\rho+1$ and with no finite exceptional value.

In order to complete the proof we need only to prove that (15) holds a. s. Otherwise as before we see that $\exists\delta\in(0,\ \rho/2)$, $\exists E\in\mathscr{A}(P(E)>0)$, $\exists b\in(2^{-1/4},\ 1)$ such that $\forall\omega\in E$, $\forall r\in(b,\ 1)$,

$$\int_0^{2\pi}\log^+|g_\omega(re^{i\varphi})|\,d\varphi\leqslant(1/(1-r))^{\rho-2\delta}. \tag{16}$$

We shall prove that this is impossible by Theorem 2. Take $\{r_n\}$ such that $b<r_n\uparrow1$ and that

$$(1-r_n)^\delta=(\pi/32)(1/2^n). \tag{17}$$

Put

$$A_n(\varphi,\ \omega)=\left\{\{(\varphi,\ \omega)\,|\,|g_\omega(r_ne^{i\varphi})|>\exp\left(\frac{1}{1-r_n}\right)^{\rho-\rho}\right\}.$$

If (16) were verified, for arbitrary $n_0 \in N_+$ and $\omega_0 \in E$ we would have

$$mA_{n_0}(\varphi, \omega_0) \leqslant (1-r_n)^\delta, \tag{18}$$

where m is the Lebesgue measure on $[0, 2\pi]$. Put

$$B(\varphi, \omega) = [0, \pi/8] \times E - \bigcup_{n=1}^{\infty} A_n(\varphi, \omega). \tag{19}$$

There would exist $\varphi_0 \in (0, \pi/8)$ such that

$$\mathscr{P}(B(\varphi_0, \varphi)) \geqslant (1/2)\mathscr{P}(E), \tag{20}$$

since otherwise by (20), Fubini Theorem, (19), (18) and (17), we would have

$$\begin{aligned}(1/2)(\pi/8)\mathscr{P}(E) &= \int_0^{\pi/8} (1/2)\mathscr{P}(E)\,d\varphi > \int_0^{\pi/8} \int_E \mathbf{1}_{B(\varphi,\omega)} \mathscr{P}(d\omega)\,d\varphi \\ &= \int_E \int_0^{\pi/8} \mathbf{1}_{B(\varphi,\omega)} d\varphi\, \mathscr{P}(d\omega) \geqslant \int_E \Big(\pi/8 - \sum_{n=1}^{\infty} mA_n(\varphi,\omega)\Big)\mathscr{P}(d\omega) \\ &\geqslant \int_E (\pi/8 - \pi/32)\mathscr{P}(d\omega) = (3/32)\pi\mathscr{P}(E).\end{aligned}$$

Hence, by (20), $\mathscr{P}(B(\varphi_0, \omega)) > 0$ and $\forall r_n$, $\forall \omega \in B(\varphi_0, \omega) \subset E$,

$$|f_\omega(s \in r_n e^{i\varphi_0}, t_0, 2\varepsilon))| = |g_\omega(r_n e^{i\varphi_0})| \leqslant \exp((1/(1-r_n))^{\rho-\delta}. \tag{21}$$

Let $s(r_n e^{i\varphi_0}, t_0, 2\varepsilon) = \sigma_n + it_0$, where $\sigma_n = -(4\varepsilon/\pi)\log u_n$,

$$u_n = \frac{-1 + r_n e^{i\varphi_0} + \sqrt{2 + 2r_n^2 e^{2i\varphi_0}}}{1 + r_n e^{i\varphi_0}}.$$

The calculations concerning u_n, r_n and σ_n are as follows:

$$\begin{aligned}1 - |u_n| &= 1 - \sqrt{(A^2+B^2)/(C^2+D^2)} \\ &= \frac{4r_n \cos\varphi_0 - 2\sqrt{(1-r_n^2)^2 + 4r_n^2\cos^2\varphi_0 - 4xr\cos\varphi_0 + 2x - 2r_n y \sin\varphi_0}}{C^2 + D^2 + \sqrt{(A^2+B^2)(C^2+D^2)}} \\ &= \frac{1 - r_0^2 + 2r_n^2\cos^2\varphi_0 - ar_n\cos\varphi_0 + a - r_n\cos\varphi_0 - r_n^3\cos\varphi_0 + 2r_n x\cos\varphi_0 - ax}{(1/2)(C^2+D^2) + \sqrt{(A^2+B^2)(C^2+D^2)}\,x} \\ &= \frac{(1-r_n)(1+r_n)(H + r_n H\cos\varphi_0 - r_n(1-r_n^2)\cos\varphi_0 + 1 - r_n^2 - x(1-r_n^2))}{(1/2)xH(C^2+D^2+\sqrt{(A^2+B^2)(C^2+D^2)})},\end{aligned}$$

where

$$\begin{aligned}\sqrt{2} &\leqslant a = \sqrt{(1-r^2)^2 + 4r_n^2\cos^2\varphi_0} \leqslant 2, \\ \sqrt{2} &\leqslant x = \sqrt{1 + r_n^2\cos 2\varphi_0 + a} \leqslant 2, \\ 0 &\leqslant y = \sqrt{-1 - r_n^2\cos 2\varphi_0 + a} \leqslant 1, \\ \sqrt{2} - 1 &\leqslant A = -1 + r_n\cos\varphi_0 + x \leqslant 2, \\ 0 &\leqslant B = r_n\sin\varphi_0 + y \leqslant 2, \\ 1 &\leqslant C = 1 + r_n\cos\varphi_0 \leqslant 2, \\ 0 &\leqslant D = r_n\sin\varphi_0 \leqslant 1, \\ 1 + \sqrt{2} &\leqslant H = 2r_n\cos\varphi_0 + a \leqslant 4.\end{aligned}$$

Hence we obtain

$$(1/20)(1-r_n) \leqslant 1 - |u_n| \leqslant 32(1-r_n),$$

$$\begin{aligned}(\varepsilon/5\pi)(1-r_n) &\leqslant -(\varphi\varepsilon/\pi)\log(1-(1-r_n)/20) \leqslant -(4\varepsilon/\pi)\log|u_n| \\ &\leqslant -(4\varepsilon/\pi)\log(1-32(1-r_n)) \leqslant (320\varepsilon/\pi)(1-r_n),\end{aligned}$$

and consequently by (17),

$$(s/5\pi)(\pi/32)^{1/\delta}2^{-n/\delta}\leqslant\sigma_n\leqslant(320s/\pi)(\pi/32)^{1/\delta}2^{-n/\delta}.$$

By Theorem 2, we would obtain a contradiction to (16). The proof of the theorem is completed.

In the case of infinite orders(R), it is not difficult to obtain lemmas analogous to Lemmas 3 and 4. Hence the Borel points concerned can be strengthened as with no finite exceptional value.

§3. Random Dirichlet Series of Finite Order (R) in the Whole Plane

Suppose that $\{Z_n(\omega)\}$ is the Rademacher or Steinhaus sequence and suppose that the random Dirichlet series

$$f_\omega(s)=\sum_{n=1}^{\infty}b_nZ_n(\omega)e^{-\lambda_n s}$$

verifies

$$\overline{\lim_{n\to\infty}}(\log|b_n|/\lambda_n\log\lambda_n)=-1/\rho\quad(0<\rho<\infty)$$

and

$$\overline{\lim_{n\to\infty}}(\log n/\lambda)<\infty.$$

In [6] it was proved that the series is a. s. an entire function of order(R) ρ and that for $f_\omega(s)$, a. s. a Borel line of order(R) ρ in every horizonthal strip of width π/ρ. According to ideas in this paper we can take $\{Z_n(\omega)\}$ as an N-sequence and the Borel line can be strengthened as with no finite exceptional value. But we cannot now diminish the width of the horizontal strip.

On the other hand, we can prove the results of J. E. Littlewood and A. C. Offord[3] on Borel directions of random Taylor series for some random variable sequence and prove them with a shorter proof.

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