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AN EQUIVALENCE BETWEEN RING F AND INFINITE MATRIX SUBRING OVER F**

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Abstract

In 1974 Fuller[1] characterized the equivalences between certain subcategories of the category of modules over a ring and the category of unital modules over a ring with identity. In this paperthe author uses this equivalence to extend the well-known theorem: $F\approx M_n(F)$, where F is a ring with identity and $M_n(F)$ is the ring of matrices over F.

Introduction

The prototype of Morita equivalence is provided by a ring F and the ring M_n(F) of $n\times n$–matrices over F. Indeed, the Wedderburn's theorem on simple Artin rings may be viewed as one of the earliest treatments of the theory of equivalence of rings. In 1974 Fuller characterized the equivalences between complete additive subcategories of the category of modules over a ring and the category of unital modules over a ring with identity. In this paper following the Fuller's theory we shall give an extension of the well-known theorem: $F\approx M_n(F)$, where F is a ring with identity and $M_n(F)$ is the ring of matrices over F.

Throughout this paper modules over ring F with identity always mean unital modules, and rings always mean associative rings (not necessarily with identity). We always denote by $\mathscr{M}_F$ (or ${}_F\mathscr{M}$) the category of right (or left) modules over F. Besides, we also introduce some terminologies as follows.

Let $\mathscr{M}_{\mathfrak{S}}$ be the category over ring $\mathfrak{S}$. A full subcategory $O_{\mathfrak{S}}$ of $\mathscr{M}_{\mathfrak{S}}$ is called a complete additive subcategory if it is closed under submodules, epimophic images and direct sums (See [1], P.503).

Definition 1. *Let K be a ring with identity and let $\mathfrak{S}$ be a ring. We say that $K\sim\mathfrak{S}$ if and only if there exists a complete additive subcategory $O_{\mathfrak{S}}$ of the category $\mathscr{M}_{\mathfrak{S}}$ such that $\mathscr{M}_k$ and $O_{\mathfrak{S}}$ are equivalent, denoted by $\mathscr{M}_k\approx O_{\mathfrak{S}}$.*

The following Lemma plays a crucial role in this paper, its proof appeared in [2].

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Lamma. *Let ${}_F\mathfrak{M}$ be a free module over a ring F with identity, and denote by Ω the endomorphism ring of $\mathfrak{M}$, i.e. $\Omega=\mathrm{End}_F\,\mathfrak{M}$. Let $\{u_i\}_{i\in\Gamma}$ be a basis of $\mathfrak{M}$ and $\{E_i\}_{i\in\Gamma}$ be the subset of Ω such that $u_iE_i=u_i$, $u_jE_i=0$, $i\neq j$ for all i, $j\in\Gamma$. Write $\mathfrak{S}=\sum_{i\in\Gamma}\Omega E_i$. Then*

(i) *for any element $E_1\in\{E_i\}_{i\in\Gamma}$ we have a free module $E_1\Omega=E_1\mathfrak{S}=\sum_{i\in\Gamma}\oplus K\alpha_i$ over $K=E_1\Omega E_1$ with a basis $\{\alpha_i\}_{i\in\Gamma}$ such that $u_i=u_1\alpha_i$, $\alpha_i=E_1\,\alpha_i=\alpha_iE_i$ for all $i\in\Gamma$, $\alpha_1=E_1$, and $F\cong K$.*

(ii) *$\mathfrak{S}=\Omega E_i\Omega$ for all $E_i\in\{E\}_{j\in\Gamma}$; if $\mathfrak{S}\omega=0$ or $\omega\,\mathfrak{S}=0$ for $\omega\in\Omega$, then $\omega=0$.*

(iii) *denote $A=\sum_{i\in\Gamma}\oplus K\alpha_i$, then $\mathrm{End}_K A=\Omega$.*

(iv) *there exists a semilinear isomorphisms S or ${}_F\mathfrak{M}$ onto ${}_KA$ such that $\omega=S^{-1}\omega S$ for each $\omega\in\Omega$.*

Now we can prove the following

Theorem 1 *Let $\mathfrak{M}=\sum_{j\in\Gamma}\oplus Fu_i$ be a free module over a ring with identity, where Γ is any infinite set. Let $\Omega=\mathrm{End}_F\mathfrak{M}$ and $\{u_i\}_{i\in\Gamma}$ be a basis of $\mathfrak{M}$. Let $\{E_i\}_{i\in\Gamma}\subset\Omega$ satisfy $u_iE_i=u_i$, $u_jE_i=0$, $i\neq j$, i, $j\in\Gamma$. Write $\mathfrak{S}=\sum_{i\in\Gamma}\Omega E_i$, then $F\sim\mathfrak{S}$.*

Proof: Denote $K=E_1\Omega E_1$, then K and F are ring isomorphism by Lemma. Hence $\mathscr{M}_F\approx\mathscr{M}_K$. Now we want to show that $K\sim\mathfrak{S}$. Let $C_\mathfrak{S}$ be a full subcategory of $\mathscr{M}_\mathfrak{S}$ whose objects consist of the following set

$$\{N\in 0b\mathscr{M}_\mathfrak{S}\,|\,N\mathfrak{S}=N\}. \tag{1}$$

First we have to show that $C_\mathfrak{S}$ is a complete additive subcategory. In fact, let $N\in 0bC_\mathfrak{S}$, N' is a submodule of N; then for every element n' of N' we have $n'=\sum_{j=1}^{m}n_is_i$, $n_i\in N$, $S_i\in\mathfrak{S}_3$ From $\mathfrak{S}=\sum_{i\in\Gamma}\Omega E_i$ it follows that there exists an element $s^2=s\in\mathfrak{S}$ such that $s_is=s_i$, $j=1$, $\cdots$, n. Clearly $n'=n's\in N'\mathfrak{S}$, $N'\mathfrak{S}=N'$. This shows that $C_\mathfrak{S}$ is also closed under submodules. It is easy to see that $C_\mathfrak{S}$ is also closed under epimorphic images and direct sums. Now we are going to show that $\mathscr{M}_K\approx C_\mathfrak{S}$.

We continue to use the symbols of Lemma. We have $A=E_1\Omega=E_1\mathfrak{S}=\sum_{i\in\Gamma}\oplus K\alpha_i$. Write $A^*=\mathrm{Hom}_K\,(A,\ K)$. It will be shown that $A^*=\Omega E_1$. Indeed, $K\subset A$ by Lemma. Hence every element a^* of $A^*=\mathrm{Hom}_K\,(A,\ K)$ is a linear transformation of space A and $a^*E_1=a^*\in\Omega E_1$ since $\alpha_ia^*=k\in K$. This proves that $A^*\sqsubseteq\Omega E_1$. On the other hand, $\Omega E_1\sqsubseteq A^*$, since $A\Omega E_1=E_1\Omega E_1=K$. Hence $A^*=\Omega E_1$.

By Lemma $A^*=\Omega E_1=\mathfrak{S}E_1$.

We come back to prove the theorem. Let N be an object of $\mathscr{M}_K$. Then clearly $N\otimes_KA\in Ob\,C_\mathfrak{S}$. On the other hand, $(N\otimes_KA)\otimes_\mathfrak{S}A^*$ is an object of $\mathscr{M}_K$. It is well known that $(N\otimes_KA)\otimes_\mathfrak{S}A^*$ and $N\otimes_K(A\otimes_\mathfrak{S}A^*)$are naturally isomorphic. Consider

$$A\otimes_\mathfrak{S}A^*=E_1\mathfrak{S}\otimes_\mathfrak{S}\mathfrak{S}E_1=E_1\mathfrak{S}E_1\otimes_\mathfrak{S}E_1. \tag{2}$$

It is clear that $k\otimes E_1\to kE_1=k$ is an isomorphism of $A\otimes_{\mathfrak{S}}A^*$ onto $K=E_1\Omega E_1$. Applying the canonical isomorphism of $N\otimes_K K$ to N and combining all of these, we obtain a right K-module isomorphism

$$(N\otimes_K A)\otimes_{\mathfrak{S}} A^*\to N. \quad (3)$$

Since all of the intermediate isomorphisms that we defined are natural in N, $\otimes_K A\otimes_{\mathfrak{S}} A^*$ is naturally isomorphic to the identity functor $1_{\mathfrak{M}_K}$.

On the other hand, let N^* be an object of $C_{\mathfrak{S}}$. Then $N^*\otimes_{\mathfrak{S}}A^*$ is an object of $\mathscr{M}_K$ and $(N^*\otimes_{\mathfrak{S}}A^*)\otimes_K A$ is an object of $C_{\mathfrak{S}}$. Now we want to show that $A^*\otimes_K A$ and $\mathfrak{S}$ are isomorphic. To do this, first we write

$$A^*\otimes_K A=\mathfrak{S}E_1\otimes_K\sum_{i\in\Gamma}K\alpha_i.$$

Let $x\in A^*\otimes_K A$. Then

$$x=\sum_i s_iE_1\otimes x_i=\sum_j(\sum_i s_iE_1\otimes k_{ij})\alpha_j,$$

where $s_i\in\mathfrak{S}$ and $x_i\in\sum_{i\in\Gamma}K\alpha_i$. We make a correspondence

$$\eta:\sum_j(\sum_i s_iE_1\otimes k_{ij})\alpha_j\to\sum_j(\sum_i s_iE_1k_{ij})\alpha_j.$$

If $\sum_j(\sum_i s_ik_{ij})\alpha_j=0$, then $\sum_i s_ik_{ij}\alpha_j=0$. By Lemma there exists an element $\omega\in\Omega$ such that $\alpha_j\omega=E_1=\alpha_1$. Therefore $\sum_i s_ik_{ij}=0$ and $x=\sum_j(\sum_i s_iE_1\otimes k_{ij})\alpha_j=0$. This shows that $A^*\otimes_K A$ and $\mathfrak{S}$ are isomorphic. We want to show that $N\otimes_{\mathfrak{S}}\mathfrak{S}$ is natural isomorphic to N^*. In fact, let

$$\sum_i n_i^*\otimes s_i\to\sum_i n_i^*s_i,\ n_i^*\in N^*,\ s_i\in\mathfrak{S}. \quad (4)$$

If $\sum_i n_i^*s_i=0$, then it is clear that there exists an element $e^2=e\in\mathfrak{S}$ such that $s_ie=s_i$. Thus $\sum_i n_i^*\otimes s_i=\sum_i n_i^*s_i\otimes e=0$. This shows that $N^*\otimes_{\mathfrak{S}}\mathfrak{S}$ is isomorphic to N^*, since $N^*\mathfrak{S}=N^*$.

Combining all of these we obtain a right $\mathfrak{S}$-module isomorphism

$$(N^*\otimes_{\mathfrak{S}}A^*)\otimes_K A\to N^*.$$

Since all of the intermediate isomorphisms that we defined are natural in N^*, ${}_{\mathfrak{S}}\otimes A^*\otimes_K A$ is naturally isomorphic to the identity functor 1_{C_σ}. Therefore we have proved that category $\mathscr{M}_K$ and $C_{\mathfrak{S}}$ are equivalent. The pair of functors $\otimes_K A$ and ${}_{\mathfrak{S}}\otimes A^*$ define an equivalence of the category $\mathscr{M}_K$ and the category $C_{\mathfrak{S}}$. This complets the proof.

Definition 2. *Let $M_\Gamma(F)$ be the ring of row-finite $\Gamma\times\Gamma$-matrices over ring F with any index set Γ. An element $(f_{ij})_{\Gamma\times\Gamma}$ of $M_\Gamma(F)$ is called a matrix with a finite number of non-zero columns if $(f_{ij})_{\Gamma\times\Gamma}$ has at most a finite number of non-zero columns, and the subring of all matrices of $M_\Gamma(F)$ with finite numbers of non-zero columns is called the matrix ring with almost non-zero columns. This submatrix ring will be denoted by a. z. c. $M_\Gamma(F)$.*

Theorem 2. *Let F be a ring with identity, and let Γ be an arbitrary set; then F*

$\sim$a. z. c. $M_\Gamma(F)$. If $|\Gamma|=n<\infty$, then the above equivalence coincides with Morita equivalence $F\approx M_n(F)$.

Proof By Theorem 1 we need only to show that the ring stated in Theorem 1 is ring isomorphic to the ring a. z. c. $M_\Gamma(K)$. But it is easy to do. In fact, let $s\in\mathfrak{S}$, then $s=\sum_{i=1}^{m}\omega_i E_i$. Now we come to the vector space $A=\sum_{s\in\Gamma}K\alpha_i$ again, stated in Theorem 1. Then s corresponds to a matrix with a finite number of non-zero columns under a basis $\{\alpha_i\}_{i\in\Gamma}$, since $\alpha_j s=\sum_{t=1}^{m}k_{jt}\,\alpha_t$ for all $j\in\Gamma$. Conversely, if $\alpha_j\omega=\sum_{t=1}^{m}k_{jt}\alpha_t$ for $\omega\in\Omega$, then there exists $s=E_1+\cdots+E_m$ such that $\alpha_j\omega s=\alpha_j\omega$ for all $j\in\Gamma$. Hence $\omega s=\omega\in\sum_{i=1}^{m}\Omega E_i\subset\mathfrak{S}$. Thus $\mathfrak{S}\cong$a. z. c. $M_\Gamma(K)$. From Theorem 1 it follows that $F\sim$a. z. c. $M_\Gamma(F)$, since $F\cong K$. Now we want to shew the last asseetion. In fact, if $|\Gamma|=n<\infty$, we have $\mathfrak{S}=\Omega=M_n(K)$. Thus a. z. c. $M_n(K)=M_n(K)$. This proves that $F\sim M_n(F)$ coincides with $F\approx M_n(F)$. The proof is completed.

Referenees

[1] Fluller, K. R., Density and equivalence, *J. of Algeb.*, **29** (1974), 528—550.
[2] Xu Yonghua, Isomorphisms between endomorphism rings of free modules (to appear).