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ON THE SECTIONAL CURVATURE OF A RIEMANNIAN MANIFOLD**

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Abstract

In this paper the author establishes the following

1. If $M^n(n\geqslant 3)$ is a connected Riemannian manifold, then the sectional curvature $K(p)$, where p is any plane in $T_x(M)$, is a function of at most $n(n-1)/2$ variables. More precisely, $K(p)$ depends on at most $n(n-1)/2$ parameters of group SO(n).

2. Let $M^n(n\geqslant 3)$ be a connected Riemannian manifold. If there exists a point $x\in M$ such that the sectional curvature $K(p)$ is independent of the plane $p\in T_x(M)$, then M is a space of constant curvature.

This latter improves a well-known theorem of F. Schur.

Let M^n be a connected Riemannian manifold of dimension $\geqslant 3$. If the sectional curvature $K(p)$, where p is a plane in $T_x(M)$, depends only on x, for each $x\in M$ then M is a space of constant curvature. This is a well-known theorem due to F. Schur. Since the group (SO)n depends on $n(n-1)/2$ parameters, the sectional curvature $K(p)$, as a function on M, depends generally on $n(n+1)/2$ variables. In the following theorem we determine the exact number of independent variables of $K(p)$ on M and as a consequence of it we improve the above theorem of Schur.

Theorem 1. *If $M^n(n\geqslant 3)$ is a connected Riemannian manifold, then the sectional curvature $K(p)$, where p is any plane in $T_x(M)$, is a function of at most $n(n-1)/2$ variables. More precisely, $K(p)$ depends on at most $n(n-1)/2$ parameters of group $SO(n)$.*

Proof Let $L(M)$ be the bundle of linear frames over M and $O(M)$ be the bundle of orthogonal frames over M. $O(M)$ is a subbundle of $L(M)$.

Let $X_1,\ X_2\in T_x(M)$ be an orthonormal basis of a plane p in $T_x(M)$ and let u be a point of $O(M)$ such that $\pi(u)=x$. We set $\xi_1=u^{-1}(X_1)$, $\xi_2=u^{-1}(X_2)$, $B_1=B(\xi_1)$, $B_2=B(\xi_2)$, where $B(\xi_1)$ and $B(\xi_2)$ are the restrictions to $O(M)$ of the standard horizontal vector fields corresponding to ξ_1 and ξ_2 respectively. At any point

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u of $L(M)$ with $\pi(u)=x$, X^* and Y^* are vectors of $L(M)$ at u with $\pi(X^*)=X$ and $\pi(Y^*)=Y$. We have ([1], p. 133)

$$R(X, Y)Z=u(\Omega(X^*, Y^*))(u^{-1}Z), \text{ for } X, Y, Z\in T_x(M), \tag{1}$$

where Ω is the curvature form of the connection form ω, and

$$R(X, Y)Z=[\nabla_X, \nabla_Y]Z-\nabla_{[X,Y]}Z. \tag{2}$$

Moreover, the sectional curvature of plane p is given by

$$K(p)=R(X_1, X_2, X_1, X_2)=g(R(X_1, X_2)X_2, X_1), \tag{3}$$

where g is the metric tensor of M.

Observing that $u\in O(M)$ as a mapping of $\mathbf{R}^n$ onto $T_x(M)$ is isometric, we have by (1)

$$K(p)=g(u(\Omega(X_1^*, X_2^*)(u^{-1}X_2), X_1)=(\Omega(X_1^*, X_2^*)(u^{-1}X_2), u^{-1}X_1).$$

Since $\pi B(\xi_1)_u=u\xi_1=X_1$, $\pi B(\xi_2)_u=u\xi_2=X_2$, we have $X_1^*=B(\xi_1)$, $X_2^*=B(\xi_2)$, and consequently, for $n\geqslant 3$,

$$K(p)=(\Omega(B(\xi_1)_u, B(\xi_2)_u)\xi_2, \xi_1), \tag{4}$$

where (,) denotes the natural inner product in $\mathbf{R}^n$.

Let θ be the canonical form of $L(M)$; let ω and Ω be respectively the connection form and the curvature form of a Riemannian connection Γ of M. Then we have the structure equations

$$d\theta=-\omega\wedge\theta, \tag{5}$$

$$d\omega=-\omega\wedge\omega+\Omega. \tag{6}$$

Since $B(\xi_1)$ and $B(\xi_2)$ are both horizontal vectors $\omega(B(\xi_1))=\omega(B(\xi_2))=0$, we have from (5)

$$d\theta(B_1, B_2)=-\omega(B_1)\theta(B_2)+\omega(B_2)\theta(B_1)=0.$$

On the other hand, since $\theta(B(\xi_1))=\xi_1$, $\theta(B(\xi_2))=\xi_2$, $\xi_1, \xi_2\in\mathbf{R}^n$, we have

$$\begin{aligned} d\theta(B_1, B_2)&=B_1(\theta(B_2))-B_2(\theta(B_1))-\theta([B_1, B_2])\\ &=B_1(\xi_2)-B_2(\xi_1)-\theta([B_1, B_2])\\ &=-\theta([B_1, B_2]).\end{aligned}$$

Consequently, we have

$$\theta([B_1, B_2])=0. \tag{7}$$

By definition $\theta([B_1, B_2]_u)=u^{-1}(\pi[B_1, B_2])$, we have $\pi([B_1, B_2]_u)=0$, that is, $[B_1, B_2]_u$ is a vector tangent to the fibre through u.

Given a principal fibre bundle $P(M, G)$, the action of G on P induces a homomorphism σ of the Lie algebra g of G into the Lie algebra $\mathscr{X}(P)$ of vector fields on P. For each $A\in g$, $A^*=\sigma(A)$ is called the fundamental vector field corresponding to A. Since the action of G sends each fibre into itself, A_u^* is tangent to the fibre at each $u\in P$. It is known that $\{A_u^*\}$ span the tangent space G_u at u of the fibre through u, where G_u is the vertical subspace of $T_u(P)$.

When $G=SO(n)$, $\dim g=n(n-1)/2$, we can choose a basis $A_{i,j}$ of g such that $A_{i,j}$

$=-A_{j,i}(i, j=1, \cdots, n)$. If $\{\xi_i\}(i=1, \cdots, n)$ is a basis of $\mathbf{R}^n$, as a consequence of (7) we can put

$$[B(\xi_i), B(\xi_j)]=A^*_{i,j}. \tag{8}$$

Denote, for simplicity, $B(\xi_i)=B_i$, we have

$$d\omega(B_i, B_j)=B_i(\omega(B_j))-B_j(\omega(B_i))-\omega([B_i, B_j])$$
$$=-\omega([B_i, B_j])=-\omega(A^*_{i,j})=-A_{i,j},$$

and

$$\omega\wedge\omega(B_i, B_j)=\omega(B_i)\omega(B_j)-\omega(B_j)\cdot\omega(B_i)=0.$$

Hence we have by (6)

$$\Omega(B(\xi_i)_u, B(\xi_j)_u)=-A_{i,j}. \tag{9}$$

Since $A_{i,j}\in g$ depends only on ξ_i and ξ_j, the left-hand member of (9) $\Omega(B(\xi_i)_u, B(\xi_j)_u)$ is independent of u. Thus we conclude by (4) that the sectional curvature on M is a function locally of at most $n(n-1)/2$ independent variables $A_{i,j}$. Since it is a continuous function and M is connected, it must be a function of at most $n(n-1)/2$ independent variables on M.

Theorem 2. *Let $M^n(n\geqslant3)$ be a connected Riemannian manifold. If there exists a point $x\in M$ such that the sectional curvature $K(p)$ is independent of the plane $p\in T_x(M)$, then M is a space of constant curvature.*

The above theorem improves the theorem of F. Schur.

Proof This Theorem is a direct consequence of Theorem 1. We give here another proof.

Let Y_1 and Y_2 be an orthonormal basis of a plane $q\in T_x(M)$, we set $\eta_1=u^{-1}(Y_1)$, $\eta_2=u^{-1}(Y_2)$ such that, for $a\in SO(n)$, $a\xi_1=\eta_1$, $a\xi_2=\eta_2$. We have by (4)

$$K(q)=(\Omega(B(\eta_1)_u, B(\eta_2)_u)\eta_2, \eta_2). \tag{10}$$

Since

$$R_a(B(\xi))=B(a^{-1}\xi),$$

we have

$$\Omega(B(\eta_1)_u, B(\eta_2)_u)=\Omega(B(a\xi_1)_u, B(a\xi_2)_u)$$
$$=\Omega(R_{a^{-1}}(B(\xi_1)_{ua}), R_{a^{-1}}(B(\xi_2)_{ua}))$$
$$=ad(a)(\Omega(B(\xi_1)_{ua}, B(\xi_2)_{ua})$$
$$=a\cdot\Omega(B(\xi_1)_{ua}, B(\xi_2)_{ua})\cdot a^{-1}.$$

Substituting this expression into (10) and noticing that a as a mapping is isometric, we obtain

$$K(q)=(\Omega(B(\xi_1)_{ua}, B(\xi_2)_{ua})\xi_2, \xi_1). \tag{11}$$

From (4) and (11) we see that, for fixed ξ_1 and ξ_2, the sectional curvature $K(q)$ for any plane $q\in T_x(M)$ can always be obtained by choosing $a\in SO(n)$. Since we have assumed that $K(p)=K(q)$ at $x=\pi(u)$, this implies that

$$\Omega(B(\xi_1)_u, B(\xi_2)_u)=\Omega(B(\xi_1)_{ua}, B(\xi_2)_{ua})$$

for every $a \in SO(n)$. Moreover, we have proved in (9) that $\Omega(B(\xi_1)_u, B(\xi_2)_u)$ is independent of u. Hence, for fixed ξ_1 and ξ_2, the function $(\Omega(B(\xi_1)_{ua}, B(\xi_2)_{ua})\xi_2, \xi_1)$ is locally constant. This means that $K(p)$, considered as a function on M, is locally constant. Since it is continuous and M is connected, it must be a constant on M.

Reference

[1] Kobayashi, S. & Nomizu, K., Fundations of differential geometry, 1 (1963).