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ON THE RELATIVE POSITION OF LIMIT CYCLES OF A REAL QUADRATIC DIFFERENTIAL SYSTEM

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Abstract

The followunh results are proved in this paper

1) If a real quadratic differential system has two strong foci, then around them there cannot appear $(2n, 2m)$ distribution of non-semi-stable limit cycles, where n and m are natural numbers.

2) If a real quadratic differential system has two strong foci of different stability, then around them there cannot appear $(2n, 2m)$ distribution of non-semi-stable limit cycles, where n and m are natural numbers.

In the papers [1, 2] we have discussed the problem concerning the impossibility of (2, 2) distribution of limit cycles of any real quadratic differential system. But we have not solved the problem completely. Even in [2], there was still a proposition not strictly proved. Moreover, we have not described clearly the process of escaping the appearance of the limiting Case 5), in which we have two infinite separatrix cycles each passing through a pair of critical points at infinity not diametrically opposite. In this paper we continue to develop the ideas in [2] and add three new theorems strictly proved, by which we not only solve the above mentioned problem satisfactorily but also prove the impossibility of some other distributions of limit cycles for real quadratic differential systems.

As in [2], we assume that the system

$$\dot{x} = -y + \delta_0 x + lx^2 + m_0 xy + ny_2 = P(x, y),\ \dot{y} = x(1 + ax - y) = O(x, y) \tag{1}$$

(where $a \neq 0$, $\delta_0 > 0$, $0 < n < 1$, $m_0 + n\delta_0 < 0$) has a (2, 2) distribution of limit cycles as follows

$$\Gamma_2 \supset \Gamma_1 \supset O(0, 0), \quad \Gamma'_2 \supset \Gamma'_1 \supset N(0, 1/n), \tag{2}$$

where O is an unstable strong focus, N is a stable strong focus, Γ_1 and Γ'_2 are stable

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limit cycles, Γ_2 and Γ_1' are unstable limit cycles.[1)] Without loss of generality, we may assume $a<0$. In case $a>0$, only a few words of the present paper should be changed correspondingly.

In [2] we have used three different families of generalized rotated vector fields (RVF, for abbreviation):

F_1: to add a term $\delta_1 x(1+ax-y)$ to the right hand side of the first equation in system (1), where δ_1 is a parameter. Since

$$\frac{\partial\theta}{\partial\delta_1}=-x^2(1+ax-y)^2/(P^2+Q^2),\ (\theta=\mathrm{tg}^{-1}Q/P),$$

F_1 is a whole plane generalized family of RVF.

F_2: to add a term $\delta_2 x$ similar to that in F_1. Since

$$\frac{\partial\theta}{\partial\delta_2}=-x^2(1+ax-y)/(P^2+Q^2),$$

F_2 is a half plane generalized family of RNF.

F_3: to add a term $m_3x(y-1)$ similar to that in F_1. Since

$$\frac{\partial\theta}{\partial m_3}=x^2(1+ax-y)(1-y)/(P^2+Q^2),$$

F_3 defines a family of generalized RVF in each one of the four regions:

$$1+ax-y\gtrless 0,\ y-1\gtrless 0.$$

The influence of the increases and decreases of δ_1, δ_2 and m_3 to Γ_1, Γ_2, Γ_1' and Γ_2' can be seen in the following table:

	Γ_1	Γ_2	Γ_1'	Γ_2'
δ_1 increases	expands	contracts	expands	contracts
δ_1 decreases	contracts	expands	contracts	expands
δ_2 increases	expands	contracts	contracts	expands
δ_2 decreases	contracts	expands	expands	contracts
m_3 increases	contracts	expands	contracts	expands
m_3 decreases	expands	contracts	expands	contracts

By applying F_2 and F_3 to (1), we get:

$$\dot{x}=-y+(\delta_0+\delta_2-m_3)x+lx^2+(m_0+m_3)xy+ny^2;\quad \dot{y}=x(1+ax-y). \tag{3}$$

Theorem 1. *If system* (1) (*in which* $a<0$, $\delta_0>0$, $0<n<1$, $m_0+n\delta_0<0$) *has* (2, 2) *distribution of limit cycles as shown in* (2), *and we take* $m_3>0$ *in* F_3 *and* δ_2 *in* F_2 *such that*

$$\delta_0+\delta_2-m_3=m_0+m_3=0, \tag{4}$$

then the system obtained from (3):

1) In general, we can prove that, when N is a strong focus and O is a weak focus of order three, then they have the same stability.

$$\dot{x}=-y+lx^2+ny^2,\ \dot{y}=x(1+ax-y) \tag{5}$$

has no limit cycle as well as separatrix cycle, and O, N will change their stability due to the fact that:

$$\Gamma_1\to 0 \text{ and } \Gamma_1'\to \mathrm{N}.$$

Moreover, we must have $2l<1$.[1)]

Proof If $2l=1$, then 0 is a center of (5). But we cannot get a center from (3), which was assumed to have limit cycles around O and N, by applying first F_3 (in which m_3 increases from zero to $-m_0$) and then F_2 (in which δ_2 varies from zero to $-m_0-\delta_0$), except that after F_3 is applied, Γ_1 and Γ_2 both disappear. In this case we should have $\delta_0+m_0<0$ (i. e., Γ_1 disappears before m_3 attains $-m_0$), and so δ_2 increases in F_2. Moreover, when δ_2 increases but still less then $-\delta_0-m_0$, no limit cycle can appear around O. Only when $\delta_2=-\delta_0-m_0$, a family of closed orbits suddenly appears around O.

On the other hand, if $\delta_0+m_0<0$, then after (1) is applied by F_3, N is still a stable focus and Γ_1' still exists. Then under F_2, Γ_1' contracts again and attains N when $\delta_2=-\delta_0-m_0$. But this contradicts the fact that N is also a center of (5) and the non-intersection property of RVF.

Therefore, under the condition (2), $2l\neq 1$. The non-existence of limit cycle and separatrix cycle for system (5) when $2l\neq 1$ can be proved easily by using the Dulac function $(1-y)^{2l-1}$ (See [3], Theorem 15. 1).

Next, assume $1-2l<0$. Then O will be an unstable focus of (5), which cannot be the limiting position of a stable limit cycle Γ_1. Similarly, N will be a stable focus of (5), which cannot be the limiting position of Γ_1'. The only possibility for O to be unstable in (5) can occur in the following procedure:

When $m_3=-m_0>0$, we must have $\delta_0-m_3=\delta_0+m_0<0$. The second inequality means that before m_3 arrivesat $-m_0$, O hasalready changed its stability for a certain m_3', $0<m_3'<-m_0$ (due to $\Gamma_1\to 0$, or due to the sudden appearance of an unstable limit cycle Γ_3 from 0). So we have at this moment the system:

1) Notice that the transformation of coordinates:

$$y=\frac{1}{n}-w,\ x=\sqrt{\frac{n}{1-n}}u,\ \frac{dt}{d\tau}=\sqrt{\frac{n}{1-n}} \tag{6}$$

transforms (5) into

$$\begin{aligned}\frac{du}{d\tau}&=-w+\frac{nl}{1-n}u^2-nw^2,\\ \frac{dw}{d\tau}&=u\left[1+\frac{n}{n-1}\sqrt{\frac{n}{1-n}}\,au+\frac{n}{n-1}\,w\right].\end{aligned} \tag{7}$$

Since

$$\frac{n}{1-n}\sqrt{\frac{n}{1-n}}\,a\left[\frac{n}{n-1}+\frac{2nl}{1-n}\right]=\frac{n}{1-n}\sqrt{\frac{n}{1-n}}\,\frac{an(2l-1)}{1-n},$$

we see that the first focal quantity of N in (5) has the same sign as $a(2l-1)$, so the focus $N\left(0,\ \frac{1}{n}\right)$ of (5) has different stability with the focus $O(0,0)$.

$$\dot{x}=-y+(\delta_0-m_3')x+lx^2+(m_0+m_3')xy+ny^2,\ \dot{y}=x(1+ax-y).$$

Evidently, $\delta_0-m_3'=0$. As m_3 increases from m_3' to $-m_0$, O becomes a strong stable focus. On the other hand. Γ_1' contracts but still exists when $m_3=-m_0$, while Γ_2' expands but disappears before $m_3=-m_0$[1]; also Γ_3, Γ_1, Γ_2 all disappear before $m_3=-m_0$ (for the reason, see Theorem 2 below).

We then apply F_2. When δ_2 increases from zero to $-\delta_0-m_0$, the unstable limit cycle Γ_2 (or Γ_3) reappears, it contracts to O and changes the stability of O. However, Γ_1' always contracts under F_3 and F_2, so it can not contract to N. But when $\delta_2=-m_0-\delta_0$, system (3) has no limit cycle; this is a contradiction.

Therefore, we must have $1-2l>0$. Now there are three subcases:

1) $\delta_0+m_0>0$. This means that when m_3 increases from zero to $-m_0$, Γ_1 contracts but still exists, N changes its stability before $m_3=-m_0$. So N is a strong unstable focus when $m_3=-m_0$. Then as δ_2 decreases from zero to $-\delta_0-m_0$, Γ_1 contracts to O and changes the stability of O, while N changes from unstable strong focus into unstable weak focus. Meanwhile, Γ_2 expands and disappears under F_3 before m_3 attains $-m_0$, or disappears under F_2 before δ_2 attains $-\delta_0-m_0$.

2) $\delta_0+m_0=0$. Then $\Gamma_1(\Gamma_1')$ contracts to O (N) when m_3 increases from zero to $-m_0$, $\Gamma_2(\Gamma_2')$ expands and disappears before m_3 attains $-m_0$. And we may take $\delta_2=0$.

3) $\delta_0+m_0<0$. Then as δ_2 increases from zero to $-\delta_0-m_0$, Γ_1 expands, Γ_2^0 contracts (they may coincide and disappear before)$_2$ attains $-\delta_0-m_0$), Γ_1' contracts but still exists. As m_3 increases from zero to $-m_0$, Γ_1 contracts (or reappears then contracts) to O and changes O into a weak stable focus. Γ_2 expands (or reappears then expands) to $\overline{\Gamma}$ and disappears again (or $\overline{\Gamma}$ appears first, it breaks and generates an unstable cycle $\Gamma_3\supset\Gamma_2$, then they close to each other, coincide and disappear) before m_3 attains $-m_0$. On the other hand, Γ_1' contracts again and attains N when $m_3=-m_0$.

From Theorem 1 we see that, in order to use this theorem and Theorem 2 below to prove the impossibility of (2, 2) distribution of limit cycles for system (1), to use only F_2 and F_3 is insufficient. So in the following we will use three RVF's F_1, F_2 and F_3 altogether. We take suitable values m_3 $(-m_0>m_3>0)$, ($\delta_1<0$ and $\delta_2>0$ such that I).

$$m_0+m_3-\delta_1=0 \quad\text{and}\quad \delta_0-m_3+\delta_2+\delta_1>0. \tag{8}$$

Then after adding $\delta_1x(1+ax-y)$, δ_2x and $m_3x(y-1)$ to system (1) we will get a system

$$\dot{x}=-y+(\delta_0-m_3+\delta_2+\delta_1)x+(l+a\delta_1)x^2+ny^2,\ \dot{y}=x(1+ax-y). \tag{9}$$

1) It may expand, becomes a separatrix cycle $\overline{\Gamma}'$, then disappears; or $\overline{\Gamma}'$ may appear first, which then breaks and generates an unstable cycle Γ_3', it contracts and coincides with Γ_2' and then disapyears.

II). The applications of F_1, F_2 and F_3 to (1) should be divided into many sub-steps and performed alternatively. This means: We set

$$\delta_1=\sum_{j=1}^{N}\delta_{1j},\ m_3=\sum_{j=1}^{M}m_{3i},\ \delta_2=\sum_{k=1}^{M+N}\delta_{2k},$$

where N and M are sufficient large natural numbers, and $\delta_{1j}<0$, $\delta_{2k}>0$, $m_{3i}>0$. We then add $\delta_{1j}x(1+ax-y)$, $\delta_{2k}x$ and $m_{3i}x(y-1)$ to (1) alternatively for $2(N+M)$ times altogether[1], such that:

Γ_1 and Γ_2 can at most coincide and become a semi-stable cycle Γ_k^*, but Γ_k^* does not disappear under F_{2k}.

When conditions I and II are satisfied, system (9), in general, will have even number of non-semi-stable cycles (at least two cycles Γ_1 and Γ_2) around a strong unstable focus O.

The above purpose can be achieved, because:

a) Under condition II, although Γ_1 may attain O and disappear, or O may change its stability first and generates a third cycle Γ_3, which expands, coincides with Γ_1 and then disappears when Γ_1 decreases or m_3 increases. But as δ_2 increases to a considerable amount, O will become unstable and regenerates Γ_1, which expands again.

b) Similarly, although Γ_2 may expand and become a separatrix cycle $\overline{\Gamma}$, then disappear; or $\overline{\Gamma}$ may appear first, breaks and generates a third cycle Γ_4, which contracts, coincides with Γ_2 and then disappoars when $\overline{\Gamma}_1$ decreases or m_3 increases. But as δ_2 increases to a considerable amount, a new $\overline{\Gamma}$ will reappear again, it breaks and regenerates a new Γ_2, which contracts again.

c) We will now give a more detailed explaination of the last sentence in b). As we know, when F_{1j} is applied to system (1), the number and position of critical points on $1+ax-y=0$ do not change. But the position of critical points on $1+ax-y=0$ will move under the application of F_{3i} (with $m_{3i}>0$), and maybe one or two critical points disappear at infinity, may be two critical points coincide and then disappear. Also, under the application of F_{2k} (with $\delta_{2k}>0$), critical points on $1+ax-y=0$ cannot disappear at infinity, but the position where two critical points coincide will move.

Suppose we apply now F_{3i} and F_{2k} to an intermediate system:

$$\dot{x}=-y+\delta'x+l'x^2+m'xy+ny^2,\ \dot{y}=x(1+ax-y), \tag{E}$$

where $\delta'>0$ and $m'<0$, and get a system:

$$\dot{x}=-y+(\delta'-m_{3i}+\delta_{2k})x+l'x^2+(m'+m_{3i})xy+ny^2,\ \dot{y}=x(1+ax-y). \tag{*}$$

The x-coordinates of the two critical points on $1+ax-y=0$ are determined by the quadratic equation:

1) We denote such applications of F_1, F_2 and F_3 by F_{1j}, F_{2k} and F_{3i}, respectively.

$$[l'+(m'+m_{3i})a+na^2]x^2+(2an+m'+\delta'+\delta_{2k}-a)x+n-1=0. \qquad (\triangle)$$

There are two possibilities for the disappearance of these critical points:

1) When

$$m_{3i}=\frac{l'+m'a+na^2}{-a}=m_{3i},$$

the coefficient of x^2 in (Δ) equals zero, and hence a critical point disappears at infinity[1].

Without loss of generality, we may assume:[2]

$$2an+m'+\delta'-a>0,$$

so the other critical point S (saddle) on $1+ax-y=0$ lies at the right hand side of the y-axis. As m_{3i} increases from m_{3i}^*, a new critical point R (node) appears from infinity at the right hand side of S. R and S move close to each other as m_{3i} increases.

2) When m_{3i} attains the value (which makes the discriminant of $(\Delta)_{\delta_{2k}}=0$ equal to zero)

$$m_{3i}=\frac{(2an+m'-a+\delta')^2+4(1-n)(l'+m'a+na^2)}{4a(n-1)}>m_{3i}^*$$

and continues to increase, R and S coincide and then disappear.[3]

Now, if $\overline{m}_{3i}\leqslant\delta'$, or equivalently,

$$(a+m'+\delta')^2+4l'(1-n)\leqslant 0 \qquad (\#)$$

and we take $\delta_{2k}=0$, $m_{3i}>\delta'$ in (*), then after the application of F_{3i}, Γ_2 expands and may become a separatrix cycle $\overline{\Gamma}$ passing through S, $\overline{\Gamma}$ disappears together with the disappearance of S, [4] while Γ_1 contracts to O and becomes a stable focus.

We then apply F_{2k}. From (*) We see that Γ_1 will reappear if only $\delta_{2k}+\delta'-m_{3i}>0$. Moreover, from $(\triangle)$ and $(\#)$ we know that for δ_{2k} sufficiently large

1) If $l'+m'a+na^2<0$, this possibility does not exist, because we have already assumed $m_{3i}>0$. If $l'+m'a+na^2=0$, then after the application of F_{3i} (with $m_{3i}>0$), one new critical point appears from infinity.

2) Otherwise, if $2an+m'+\delta'-a<0$, we may investigate the behavior of N, Γ_1' and Γ_2'. For when $a<0$, separatrices surounding $O(N)$ come from the saddle point on $1+ax-y=0$ lying at the right (left) hand side of the y-axis, if the later exists. If $2an+m'+\delta-a=0$, then the application of F_{3i}^* makes two critical points disappear at infinity in different directions.

Notice that under the transformatiou: $x=-\sqrt{\frac{1-n}{n}}\,u$, $y=\frac{1}{n}-\frac{1-n}{n}w$, $\frac{dt}{d\tau}=-\sqrt{\frac{n}{1-n}}$ system (E) is transformed into:

$$\frac{du}{d\tau}=-w-\sqrt{\frac{n}{1-n}}\left(\delta'+\frac{m'}{n}\right)u+l'u^2+\sqrt{\frac{1-n}{n}}m'uw+(1-n)w^2,\ \frac{dw}{d\tau}=u\left[1+\sqrt{\frac{n}{1-n}}\,au-w\right]$$

and for this system the quantity cooresponding to $2an+m'+\delta'-a$ is $\sqrt{\frac{n}{1-n}}\,(a-2an-m'-\delta')$.

3) If $\overline{m}_{3i}<0$, then there is no critical point on $1+ax-y=0$ for $\delta_{2k}=0$ and $m_{3i}>0$; separatrices surrounding O come from critical points at infinity

4) In case $\overline{m}_{3i}<0$, Γ_2 may expand and become an infinite separatrix cycle passing through two non-diametrical opposite critical points at infinity and then disappear.

$$-(a+m'+\delta'+\delta_{2k})^2+4l'(1-n)>0,$$

or equivalently

$$\frac{(2an+m'+\delta'+\delta_{2k}-a)^2+4(1-n)(l'+m'a+na^2)}{4a(n-1)}>\delta'+\delta_{2k}.$$

Then for $m_{3i}=m_{3i}$ and the above δ_{2k}, ($\triangle$) will have two different positive roots, i. e., the critical points S and R on $1+ax-y=0$ reappear. If the two separatrices (L_1 goes from S to the left and L_2 goes in S from the left) do not coincide and make a cycle[1)], then we may increase δ_{2k} again so that $L_1=L_2$, and then change their relative position and generate a new cycle.[2)]

d) Notice that the values of $|\delta_1|$ and m_3 are bounded, while δ_2 can increase indefinitely. When $\delta_0-m_3+\delta_2+\delta_0>0$, 0 will become an unstable node, Γ_1 and Γ_2 will both disappear heretofore.

Therefore, there will exist suitable values of δ_1, m_3 and δ_2, suitable subdivisions of F_1, F_2 and F_3, and suitable order of applications of the F_{1j}'s, F_{2k}'s and F_{3i}'s to system (1), such that system (9) has a strong unstable focus O, as well as two non-semi-stable limit cycles Γ_1 and Γ_2, which were assumed to exist at the very beginning of this paper (of coulse, (9) may have another new appeared even number of non-semi-stable cycles as well)[3)]. But the impossibility of this situation will be proved by the following Theorem 2.

Now, rewrite (9) into:

$$\dot{x}=-y+\delta x+l_1x^2+ny^2, \quad \dot{y}=x(1+ax-y), \tag{10}$$

where $a<0$ and $0<n<1$.

Theorem 2. *If in system (10) $\delta>0$ and $l_1<1/2$ ($>1/2$), then it has no limit cycle around N (O), and can have only an odd number of non-semi-stable limit cycles around O (N)*[4)]. *It has no limit cycle in the whole plane for $l_1=1/2$ and any δ.*

Proof The second part is clear from the theory of RVF, since when $l_1=1/2$, $\delta=0$, (10) has two centers. As to the first part, we prove it only for the case $l_1<1/2$. We have seen in Theorem 1, when $\delta=0$, O is a weak stable focus, $N(0, 1/n)$ is a weak unstable focus. When δ increases from zero, both O and N become strong unstable foci, and a stable limit cycle Γ_1 bifurcates from O, which expands with the increase of δ, but no limit cycle can appear around N, by the theory of RVF. It is easily seen

1) According to the results obtained by the computer for a special quadratic system in [5], the appearance of a new saddle-node is always accompanied by the appearance of a separatrix cycle.

2) In case Γ_2 disappears at infinity, the increase of δ_{2k} will also make the reappearance of the infinite separatrix cycle just mentioned in footnote 4), it then breaks and generates a new Γ_2 again.

3) The only exceptional case is Γ_k^* (the semi-stable cycle obtained from Γ_1 and Γ_2 under F_{2k}) approaches $\bar{\Gamma}$ (the separatrix cycle passing through the finite saddle poiut S_1 or the two critical points at infinity) as $k\to\infty$. In this case we will have for system (9): $\mathrm{div}|_s=0$ or $\alpha_1\alpha_2=1$ for the infinite separatrix cycle $\bar{\Gamma}$ as in Case II(ii) of [1]. But this is also impossible by the following Theorem 2.

4) We conjecture the limit cycle is unique in this case.

that if $1+ax-y=0$ intersects $-y+\delta x+l_1x^2+ny^2=0$, whether at one or at two points, and whether they locate both on the same side of the y-axis or they are separated by the y-axis, S_1 must lie in the right half plane, provided a separatrix cycle $\bar{\Gamma}$ exists around O and passes through a saddle point S_1 on $1+ax-y=0$.

Now the line

$$P_x+Q_y=\delta+(2l_1-1)x=0$$

is a vertical line to the right of the y-axis for all $\delta>0$. When $0<\delta\ll 1$, it separates O and S_1. On eliminating x and y from

$$\delta+(2l_1-1)x=0, \quad l+ax-y=0, \quad -y+\delta x+l_1x^2+ny^2=0$$

we see at once thot there exists a unique $\delta_*>0$ such that $(10)_{\delta_*}$ passes through S_1. For this δ_*, limit cycles or separatrix cycles of (1) already disappear, since they cannot situate in a half plane in which $P_x+Q_y\geqslant 0$. Therefore, when a separatrix cycle $\bar{\Gamma}$ passing through S_1 appears, it must be inner stable, because $P_x+Q_y<0$ at S_1. This shows that when δ increases from zero, $\bar{\Gamma}$ cannot appear before some stable limit cycle Γ_* expands and passes through S_1. Γ_* may be Γ_1, at this time limit cycle around O is unique. But as δ increases, a semi-stable limit cycle Γ_3 may also appear suddenly outside Γ_1, it then spits into $\Gamma_*\supset\Gamma_3(\supset\Gamma_1)$, Γ_4 contracts, coincide with Γ_1 and then disappears, while Γ_* plays the role of Γ_1, expands and finally becomes the separatrix cycle $\bar{\Gamma}$.

When $\bar{\Gamma}$ (the limiting position of Γ_i) is an infinite separatrix cycle passing through two critical points at infinity not diametrically opposite, we can use the criterion of Case II in [1] to show that $\bar{\Gamma}$ must also be inner stable, since at this time the position and the characteristic roots of the critical points at infinity and hence the inner stability of $\bar{\Gamma}$ are not affected by the appearance of the term δ_x in the right hand side of the first equation in (1).

Similarly, when $\delta>0$, $l_1<\frac{1}{2}$ no limit cycle appears around O, and around N there can appear only an odd number of non-semi-stable limit cycles, if exist.[1)]

In a similar manner we can prove:

Theorem 3 *The system:*

$$\dot{x}=-y+\delta x(y-1)+lx^2+ny^2, \quad \dot{y}=x(1+ax-y) \tag{2}$$

(in which $l<\frac{1}{2}$ and $\delta<0$) *has an odd number of non-semi-stable cycles around O, if exist. The same conclusion also holds if O is replaced by N $(0, 1/n)$.*

Proof When $\delta=0$, O (N) is a weak stable (unstadle) focus, no limit cycles exist. When δ decreases from O to negative, O (N) becomes a strong unstable (stable)

1) In case $\delta<0$ we need only replace "$l_1<\frac{1}{2}\left(>\frac{1}{2}\right)$" by "$l_1>\frac{1}{2}\left(<\frac{1}{2}\right)$" in the first line of Theorem 1; then the conclusion of this theorem still hold.

focus, and a stable (unstable limit cycle $\Gamma_1(\Gamma_1')$ appears.

Notice that

$$P_x+Q_y=\delta(y-1)+(2l-1)x=0$$

is a straight line intersecting $1+ax-y=0$ at $(0, 1)$ when $|\delta|\ll 1$, and it coincides with $1+ax-y=0$ when $\delta=\delta_*=\frac{1-2l}{a}<0$. For this value $\delta=\delta_*$, limit cycles around O (N) already disappear. So if separatrix cycle passing through a saddle point S_1 (S_2) appears around O (N), S_1 (S_2) and O (N) must lie in different sides of the line $P_x+Q_y=0$. Since $P_x+Q_y>0$ (<0) at O (N), we have $P_x+Q_y<0$ (>0) at S_1 (S_2). Therefore, the separatrix cycle passing through S_1 (S_2) is inner stable (unstable), which has the same stability as Γ_1 (Γ_1'); the theorem follows at once.

Remark 1. The conclusion of the theorem also holds if $l>1/2$ and $\delta>0$[1].

Theorems 1 and 2 show that the assumption of the existence of Γ_2 for system (1) is incorrect. Hence we have proved:

Theorem 4. *For system* (1) *it is impossible to have a* (2, 2) *distribution of limit cycles satisfying condition* (2).

Remark 2. Notice the reason that we can transform (1) into (9) (in which we have no term mxy) without affecting the number of limit cycles around O lies in tho fact that $\delta_0 m_0<0$, or the same, that O and N have different stability.

Remark 3. From the whole procedure of the proof of Theorem 4, we see that the following theorem also holds:

Theorem 5. *If in system* (1), $O(0, 0)$ *and* $N\left(0, \frac{1}{n}\right)$ *are strong foci of different stability, then around them there cannot appear* $(2n, m)$ *non-semi-stable limit cycles, where* n *and* m *are positive integers.*

Especially, under the condition of Theorem 5, (2, 1), (2, 2), (2, 3), $\cdots$ *and* (4, 1), (4, 2) (4, 3), $\cdots$ *distributions of limit cycles are all impossible.*

Remark 4. Although we can find in [4] an example of (2, 1) distribution, there the stability of the two foci are the same, and one of them is a weak focus.

Remark 5. In [4] it was proved that the system

$$\dot{x}=P_2\cos\theta-Q_2\sin\theta,\quad \dot{y}=P_2\sin\theta+Q_2\cos\theta,$$

where $0<\theta\ll 1$, $P_2=xy$,

$$Q_2=-\frac{1}{3}(x-1)(x+2)+\frac{1}{2}y^2+\frac{1}{3}xy-\frac{1}{3}y,$$

has 2 limit cycles around $N_1(1, 0)$, and one limit cycle around $N_2(-2, 0)$. But here both N_1 and N_2 are strong stable foci.

Remark 6. If $\delta_0>0$, $m_0>0$ in (1), and the system is actually known to have two limit cycles Γ_1 (stable) and Γ_2 (unstable) such that $\Gamma_2\supset\Gamma_1\supset 0$, and a third

1) We conjecture that the limit cycle is unique around O or N.

one Γ_3 (stable) around N, such as in [5]. In order that system (9) satisfies conditions

$$\delta_0 - m_3' + \delta_2'' + \delta_1 = 0, \quad m_0 + m_3' - \delta_1 = 0,$$

we can take

$$-m_0 < m_3' < 0, \quad \delta_1 = m_0 + m_3' > 0, \quad \delta_2'' = -\delta_0 - m_0 < 0.$$

This means that Γ_3 always contracts; Γ_1 expands and Γ_2 contracts in F_1 and F_3, but Γ_1 contracts and Γ_2 expands in F_2. If we want that Γ_1 and Γ_2 do not disappear under the applications of F_1, and F_3, then $|m_3'|$ and $|\delta_1|$ must have certain upper bounds; but then there is a possibility that $m_0 + m_3' - \delta_1 = 0$ may never be satisfied. On the other hand, in order that $m_0 + m_3' - \delta_1 = 0$ be satisfied, there is a possibility that Γ_1 coincides with Γ_2 and then disappears under the application of F_1 or F_3.

Now, if we notice that even if in condition (2) N is a stable weak focus (i. e., $m_0 + n\delta_0 = 0$, hence $m_0 = -n\delta_0 < 0$), we can still prove that (2, 1), (4, 1)$\cdots$, (2, 3), (4, 3), $\cdots$ distributions of limit cycles for system (1) are impossible, using the same procedure as before. Hence we can strengthen a part of Theorem 5 as follows:

Theorem 6. *If in system* (1) $O(0, 0)$ *and* N $(0, 1/n)$ *are strong foci, then around them there cannot appear* $(2n, 2m)$ *distribution of non-semi-stable limi cycles, where* n *and* m *are positive integers.*

Proof Take (2, 2) distribution as an example. Suppose that at this time condition (2) is replaced by:

$$\Gamma_2 \supset \Gamma_1 \supset O\ (0, 0), \quad \Gamma_2' \supset \Gamma_1' \supset N\ (0, 1/n),$$

where O and N are both strong unstable foci, Γ_1 and Γ_1' arestable limit cycles, Γ_2 and Γ_2' are unstable limit cycles. Then we can apply F_1, F_2 and F_3 to system (1) suitably, so that $\Gamma_1' \to N$ and N is changed into a stable weak focus, while Γ_1, Γ_2, Γ_2' still exist. But such a (2, 1) distribution of limit cycles is still impossible for system (1), as we have just mentioned.

The author conjectures that, by using the method of this paper, maybe other problem relating to the distribution and number of limit cycles of real quadratic differential systems can be solved later on.

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