Chin. Ann. of Math.
11B:1(1990), 84—92.

ON BOUNDARY VALUE PROBLEMS FOR OVERDETERMINED ELLIPTIC SYSTEMS OF TWO COMPLEX VARIABLES**

LI MINGZHONG (李明忠)* CHENG JIN (程 晋)*

(*Dedicated to the Tenth Anniversary of CAM*)

Abstract

In this paper, based on previous results, the Riemann-Hilbert boundary value problems with general forms for overdetermined elliptic equations of first order are considered. The characteristic of modified function space is given. It is proved that there exists a unique solution for modified problem of the problem which we discuss. By the way, it is pointed out that there are great differences between overdetermined elliptic systems and first order elliptci systems in the plane.

§1. Introduction

Overdetermined elliptic partial differential equations first arise in the theory of functions of several complex variables. Since the number of equations is greater than the number of unknown functions, there are great differences between them and first order partial differential equations in the plane. The formulations of the corresponding boundary value problems are different. The formulation of the boundary ualue problems depends on the shape of domain.

For the existence theorem of generalized solutions of the overdetermined elliptic systems and expressions of the solutions, R. P. Gilbert and J. L. Buchanan have done some work[1]. W. Tutschke and the authors have considered several boundary value problems with special forms [2-8]. In this paper, we discuss the general R-H problems for overdetermined elliptic systems in bicylinder. We look for w which satisfies:

$$\begin{cases} \dfrac{\partial w}{\partial \bar{z}_1}=f_1(z_1, z_2, w) \\ \dfrac{\partial w}{\partial \bar{z}_2}=f_2(z_1, z_2, w) \end{cases} \text{in } G_1\times G_2, \qquad (*)$$
$$\mathrm{Re}(\lambda w)=\gamma \quad \text{on } \partial G_1\times\partial G_2.$$

Manuscript received April 15, 1989.

* Department of Mathematics, Fudan University, Shanghai, China.

** Supported by NSF and supported in part by Science and Technology Development Foundation of Shanghai.

The difference between the work presented here and previous works is that, in this paper, we discuss above problems for general $\lambda\neq 0$, however in previous works, we only considered the problems for special λ. Here, we give out the characteristic of the modified functions space and prove that there exists a unique solution for the modified problem of above problem (*).

The plan of this paper is as follows: In section 2, some definitions and notations are given and the boundary condition in above problem is transformed into canonical form. In section 3, we discuss the properties of several boundary singular integral operators. In section 4, the main result of this paper is presented and the solvability for the modified problem of the problem (*) is discussed. In last section, we present some remarks for the problem (*).

§ 2. Definitions, Notations and Canonical Form of the Problem

Suppose $G_j=\{z_j \mid |z_j|<1\}$, $\partial G_j=\{z_j \mid |z_j|=1\}$, $j=1, 2$. $G=G_1\times G_2$ is a bicylinder in C^2 and $\partial G_1\times\partial G_2$ is the characteristic boundary of G.

We consider the problem:

$$\left.\begin{aligned}\frac{\partial w}{\partial \bar z_1}&=\mu f_1(z_1, z_2, w)\\ \frac{\partial w}{\partial \bar z_2}&=\mu f_2(z_1, z_2, w)\end{aligned}\right\}\quad \text{in } G_1\times G_2, \tag{1}$$

$$\operatorname{Re}(\lambda w)=\gamma \quad \text{on } \partial G_1\times\partial G_2, \tag{2}$$

where λ is a Hölder continuous function on $\partial G_1\times\partial G_2$ and $\lambda\neq 0$; γ is a real Hölder continuous function on $\partial G_1\times\partial G_2$; μ is a parametre and $\mu\neq 0$.

We assume that

(i) f_j is continuous with respect to $(z_1, z_2)\in G_1\times G_2$ and is holomorphic with respect to $w\in\{w \mid |w|<K+1\}$; $\frac{\partial f_1}{\partial \bar z_2}$ and $\frac{\partial f_2}{\partial \bar z_1}$ exist and are continuous, where K is a given sufficient large positive number.

(ii) Equation (1) is complete integrable, that is

$$\frac{\partial f_1}{\partial \bar z_2}+\mu\frac{\partial f_1}{\partial w}f_2=\frac{\partial f_2}{\partial \bar z_1}+\mu\frac{\partial f_2}{\partial w}f_1.$$

These conditions are called condition (C).

In the following, we transform the boundary value condition (2) into a canonical form.

Since $\lambda\neq 0$ on $\partial G_1\times\partial G_2$ and 1-dimensional homology group of $\partial G_1\times\partial G_2$ is $Z\times Z$, there exist two integers n_1, n_2 such that: $\arg(z_1^{n_1}z_2^{n_2}\lambda)=n_1\arg z_1+n_2\arg z_2+\arg\lambda$ is a real single-valued Hölder continuous function. By the lemma in [7], we know that there exist two holomorphic functions $\widetilde{\Phi}(z_1, z_2)$, $\widetilde{\Psi}(z_1, z_2)$ such that, on $\partial G_1\times\partial G_2$,

$$\mathrm{Re}[\widetilde{\Phi}(z_1, z_2)+\widetilde{\Psi}(\bar{z}_1, z_2)]=n_1\arg z_1+n_2\arg z_2+\arg\lambda.$$

By using Vekua method, we do the transformation:

$$\widetilde{w}=w\exp[i\widetilde{\Psi}(\bar{z}_1, z_2)].$$

The equations (1) can be transformed into

$$\left.\begin{aligned}\frac{\partial\widetilde{w}}{\partial\bar{z}_1}&=\mu\widetilde{f}_1(z_1, z_2, \widetilde{w})\\ \frac{\partial\widetilde{w}}{\partial\bar{z}_2}&=\mu\widetilde{f}_2(z_1, z_2, \widetilde{w})\end{aligned}\right\}\quad \text{in } G, \tag{3}$$

where $\widetilde{f}_j(z_1, z_2, \widetilde{w})=\exp[i\widetilde{\Phi}(z_1, z_2)]f_j(z_1, z_2, \widetilde{w}\exp[-i\widetilde{\Phi}(\bar{z}_1, z_2)])$, $j=1, 2$.

The boundary condition (2) is transformed into

$$\mathrm{Re}[z_1^{-n_1}z_2^{-n_2}\exp[i\widetilde{\Psi}(\bar{z}_1, z_2)]\widetilde{w}]=\widetilde{\gamma}, \tag{4}$$

where
$$\widetilde{\gamma}=\gamma\exp[-\mathrm{Im}\Phi(z_1, z_2)-\mathrm{Im}\widetilde{\Psi}(\bar{z}_1, z_2)]\,|\lambda|^{-1}.$$

It is easy to verify directly that condition (C) still holds for equations (3).

Therefore, we can assume without loss of generality that $\lambda=z_1^{-n_1}z_2^{-n_2}\exp[\Psi(\bar{z}_1, z_2)]$ in problem (1), (2), (where $\Psi(z_1, z_2)$ is holomorphic in G and can be continuously extended to boundary, and the boundary value satisfies Hölder condition.

In the following, we only consider the problems for $n_2<0$. From [7], we know that, for $n_1\geqslant 0$, $n_2\geqslant 0$ the conclusions are more complicated when $f_1=f_2=0$. The nonlinear problems for this case will be discussed in detail later.

§ 3. Several Singular Integral Operators on Characteristic Boundary

Let $C^\beta(\partial G_1\times\partial G_2)$ be the set of all Hölder continuous functions with respect to $(z_1, z_2)\in\partial G_1\times\partial G_2$ (The Hölder index is β); $C^\beta_R(\partial G_1\times\partial G_2)$ the set of all real functions in $C^\beta(\partial G_1\times\partial G_2)$.

We consider four singular integral operators defined on $\partial G_1\times\partial G_2$:

$$\begin{aligned}P_1h(z_1, z_2)=&\frac{1}{4}h(z_1, z_2)+\frac{1}{2\cdot 2\pi i}\int_{\partial G_1}\frac{h(\zeta_1, z_2)}{\zeta_1-z_1}d\zeta_1\\&+\frac{1}{2\cdot 2\pi i}\int_{\partial G_2}\frac{h(z_1, \zeta_2)}{\zeta_2-z_2}d\zeta_2\\&+\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{h(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2,\end{aligned}$$

$$\begin{aligned}P_2h(z_1, z_2)=&\frac{1}{4}h(z_1, z_2)-\frac{1}{2\cdot 2\pi i}\int_{G_1}\frac{h(\zeta_1, z_2)}{\zeta_1-z_1}d\zeta_1\\&+\frac{1}{2\cdot 2\pi i}\int_{\partial G_2}\frac{h(z_1, \zeta_2)}{\zeta_2-z_2}d\zeta_2\\&-\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{h(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2,\end{aligned}$$

$$P_3h(z_1, z_2)=\frac{1}{4}\,h(z_1, z_2)+\frac{1}{2\cdot 2\pi i}\int_{\partial G_1}\frac{h(\zeta_1, z_2)}{\zeta_1-z_1}\,d\zeta_1$$
$$-\frac{1}{2\cdot 2\pi i}\int_{\partial G_2}\frac{h(z_1, \zeta_2)}{\zeta_2-z_2}\,d\zeta_2$$
$$-\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{h(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\,d\zeta_1 d\zeta_2,$$
$$P_4h(z_1, z_2)=\frac{1}{4}\,h(z_1, z_2)-\frac{1}{2\cdot 2\pi i}\int_{\partial G_1}\frac{h(\zeta_1, z_2)}{\zeta_1-z_1}\,d\zeta_1$$
$$-\frac{1}{2\cdot 2\pi i}\int_{\partial G_2}\frac{h(z_1, \zeta_2)}{\zeta_2-z_2}\,d\zeta_2$$
$$+\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{h(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\,d\zeta_1 d\zeta_2,$$

where $h\in C^\beta(\partial G_1\times\partial G_2)$, $(z_1, z_2)\in\partial G_1\times\partial G_2$

Proposition 1. *P_j, $j=1, 2, 3, 4$, are bounded operators on $C^\beta(\partial G_1\times\partial G_2)$ and*

$$\sum_{j=1}^{4}P_jh=h, \quad \forall h\in C^\beta(\partial G_1\times\partial G_2). \tag{5}$$

Proof The boundedness of P_j can be obtained from the properties of Cauchy singular integral and the expression of P_j. It is obvious that equality (5) holds.

Proposition 2. $P_jP_k=\delta_{jk}P_j$, j, $k=1, 2, 3, 4$.

Proof We only prove that $P_1^2=P_1$, $P_2P_1=0$. Others can be proved by similar method.

From the proof of Lemma 1 in [7], we know that, for $h\in C^\beta(\partial G_1\times\partial G_2)$, P_1h is the boundary value of a holomorphic function in $G_1\times G_2$. Therefore, piecewise holomorphic function

$$\Phi_1(z_1, z_2)=\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{P_1h(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\,d\zeta_1 d\zeta_2$$

satisfies $\Phi(z_1, z_2)=0$, $z_2\in\mathbf{C}\backslash\overline{G}_2$ or $z_1\in\mathbf{C}\backslash\overline{G}$. So we have that $P_jP_1=0$, $j=2, 3, 4$. By proposition 1, we obtain $P_1^2=P_1$.

In the following we discuss four singular integral operators on $C_R^\beta(\partial G_1\times\partial G_2)$ and $C^\beta(\partial G_1\times\partial G_2)$.

$$Th=P_1\{\lambda^{-1}[P_1h+\overline{P_4h}+P_2h+\overline{P_3h}]\}, \tag{6}$$
$$Nh=\mathrm{Re}(\lambda Th), \tag{7}$$
$$T^\perp h=P_2\{\lambda^{-1}[P_1h+\overline{P_4h}+P_2h+\overline{P_3h}]\}, \tag{8}$$
$$N^\perp h=\mathrm{Re}(\lambda T^\perp h). \tag{9}$$

It is easy to verify that T, N, $T^\perp$, $N^\perp$ are bounded operators from $C^\beta(\partial G_1\times\partial G_2)$ to $C^\beta(\partial G_1\times\partial G_2)$.

Proposition 3. *For any $h\in C_R^\beta(\partial G_1\times\partial G_2)$, it holds that*

$$Nh+N^\perp h=h.$$

Proof First, it is noticed that, for $n_2<0$, $\lambda^{-1}[P_1h+\overline{P_4h}+P_2h+\overline{P_3h}]$ is the boundary value of a function on $\partial G_1\times\partial G_2$ which is holomorphic with respect to z_2.

By the results in [7], we have

$$Th-T^{\perp}h=\lambda^{-1}[P_1h+\bar{P}_4h+P_2h+\bar{P}_3h].$$

By proposition 1, we have the conclusion.

Proposition 4. *For operators N, $N^{\perp}$ on $C_R^{\beta}(\partial G_1\times\partial G_2)$, it holds that*

$$NN^{\perp}=N^{\perp}N=0,\quad N^2=N,\quad N^{\perp 2}=N^{\perp},$$

Proof First we prove $N^{\perp}N=NN^{\perp}$.

Since $N^{\perp}+N=I$ on $C_R^{\beta}(\partial G_1\times\partial G_2)$ we have

$$N^{\perp}=N^{\perp 2}+N^{\perp}N=N^{\perp 2}+NN^{\perp}.$$

Therefore, we have

$$NN^{\perp}=N^{\perp}N.$$

Next, we will prove $N^{\perp}Nh=NN^{\perp}h=0$, for any $h\in C_R^{\beta}(\partial G_1\times\partial G_2)$.

By using the integral expressions of $N^{\perp}$ and N we know

$$\operatorname{Re}\{\lambda[TN^{\perp}h-T^{\perp}Nh]\}=0,\quad \text{on } \partial G_1\times\partial G_2.$$

By the results in [7], we have

$$TN^{\perp}h-T^{\perp}Nh=\lambda^{-1}[\varphi(z_1)-\overline{\varphi(z_1)}],\tag{10}$$

where $\varphi(z_1)$ is a Hölder continuous function on ∂G_1 and is the boundary value of a holomorphic function in G_1.

Since the left part of (10) is the boundary value of a function with respect to z_2 and $n_2<0$, $z_1^{n_1}z_2^{n_2}\exp[-\overline{\Psi}(\bar{z}_1, z_2)][\varphi(z_1)-\overline{\varphi(z_1)}]$ is the boundary value of a function with respect to z_2. So we have $\varphi(z_1)-\overline{\varphi(z_1)}=0$.

By Proposition 2, we have

$$TN^{\perp}h=T^{\perp}Nh=0.$$

Therefore $\quad N^{\perp}N=NN^{\perp},\ N^{\perp 2}=N^{\perp},\ N^2=N.$

§ 4. Solvability of Modified Problems of Problems (1) (2)

By [5], we know that, for $f_1=f_2=0$, problem (1), (2) is not always solvable for abitrary γ. Here, we present the characteristic of the necessary and sufficient conditions for the solvabilityof generalized Riemann-Hilbert problems.

Lemma 1. *Suppose $\Phi(z_1, z_2)$ is a holomorphic function in G which can be extended to the characteristic boundary and satisfies*

$$\operatorname{Re}(\lambda\Phi)=\gamma\quad on\ \partial G_1\times\partial G_2,\tag{11}$$

then we have $\Phi=T\gamma$, on $\partial G_1\times\partial G_2$.

Proof By the results in [7], we know that $\Phi(z_1, z_2)$ has expression

$$\Phi(z_1, z_2)=T\gamma+\lambda^{-1}(\varphi(z_1)-\overline{\varphi(z_1)}),$$

where φ is a Hölder continuous function on ∂G_1 and is the boundary value of a holomorphic function in G_1. By using the similar ethod mused for proposition 4, we have $\varphi(z_1)-\overline{\varphi(z_1)}=0$. So we have the conclusion.

Theorem 1. *For $f_1=f_2=0$, the necessayy and sufficient condition for the solvability of problem (1), (2) is $N^\perp\gamma=0$. When this condition is satisfied, the solution can be expressed as*

$$w(z_1\ \ z_2)=\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{T\gamma(\zeta_1,\ \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2.$$

Proof Sufficiency: When $N^\perp\gamma=0$, $N\gamma=\gamma$. Therefore $T\gamma$ satisfies

$$\mathrm{Re}(\lambda T\gamma)=N\gamma=\gamma.$$

Next, we will prove the uniqueness of the solution.

If there exist two solutions w_1, w_2, we have $\mathrm{Re}[\lambda(w_1-w_2)]=0$. By the results in [7], we have $w_1-w_2=\lambda^{-1}[\varphi(z_1)-\overline{\varphi(z_1)}]$, where $i(\varphi(z_1)-\overline{\varphi(z_1)})$ is a Hölder continuous function on ∂G_1. By using the similar method used for Proposition 4, we have $w_1=w_2$. The soluotin is unique.

Necessity: By Lemma 1, we get the conclusion immediately.

From above, we know

$$w(z_1,\ z_2)=\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{T\gamma(\zeta_1,\ \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\ d\zeta_1 d\zeta_2.$$

We define modified function space $N^\perp$

$$N^\perp=\{N^\perp h\,|\,h\in C^\alpha_R(\partial G_1\times\partial G_2)\}.$$

We consider modified problem:

$$\left.\begin{aligned}\frac{\partial w}{\partial\bar z_1}&=\mu f_1(z_1,\ z_2,\ w)\\ \frac{\partial w}{\partial\bar z_2}&=\mu f_2(z_1,\ z_2,\ w)\end{aligned}\right\}\quad \text{in } G, \tag{12}$$

$$\mathrm{Re}(\lambda w)=\gamma+h\quad \text{on } \partial G_1\times\partial G_2, \tag{13}$$

where $h\in N^\perp$ is a modified function.

By using Pompieu operator[8], we know that the solutions of equation (12) must be the solutions of following integral equation

$$w=\mu T_1 f_1+\mu T_2 f_2-\mu T_1 T_2\frac{\partial f_2}{\partial\bar z_1}+\Phi(z_1,\ z_2), \tag{14}$$

where T_1, T_2 are Pompieu operators with respect to z_1, z_2 respectively (see [9]), $\Phi(z_1,\ z_2)$ is a holomorphic function in G.

We denote

$$Pw=T_1 f_1+T_2 f_2-T_1 T_2\frac{\partial f_2}{\partial\bar z_1}.$$

From (13), we have

$$\mathrm{Re}(\lambda\Phi)=\gamma+h-\mathrm{Re}(\mu Pw),\quad \text{on } \partial G_1\times\partial G_2.$$

By Lemma 1 and Theorem 1, we know that, on $\partial G_1\times\partial G_2$,

$$\Phi=T\gamma-T(\mathrm{Re}\mu Pw),\quad h=N^\perp(\mathrm{Re}\mu Pw-\gamma).$$

Equation (14) can be transformed into

$$w=\mu Pw-\tilde{T}(\operatorname{Re}\mu Pw)+\tilde{T}\gamma \quad \text{in } G, \tag{15}$$

where
$$\tilde{T}h=\frac{1}{(2\pi i)^2}\int_{\partial G_1\times\partial G_2}\frac{Th(\zeta_1, \zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2.$$

We denote equation (15) as $w=Hw$.

In the following, we will prove that there exists a unique solution of integral equation (15).

We define functions space B:

$$B=\{g(z_1, z_2)\mid g(z_1, z_2), \frac{\partial g(z_1, z_2)}{\partial \bar{z}_1}, \frac{\partial g(z_1, z_2)}{\partial \bar{z}_2} \text{ are continuous in } \bar{G}\}.$$

The norm is defined as

$$\|g\|_B=\sup_{(z_1,z_2)\in G}\left[|g|+\left|\frac{\partial g}{\partial \bar{z}_1}\right|+\left|\frac{\partial g}{\partial \bar{z}_2}\right|\right].$$

It is easy to verify that B is a Banach space.

Let B_k be the ball in B centered in O with radius K, i. e. $B_k=\{g\mid \|g\|<K\}$. Without loss of generality, we take K sufficient large such that $H0\in B_k$.

Let $(z_1, z_2, w)\in \bar{G}_1\times\bar{G}_2\times\{w\mid |w|\leqslant K\}$, it holds that

$$\max\left\{\frac{\partial^2 f_2}{\partial w^2}, \frac{\partial^2 f_2}{\partial w\partial \bar{z}_1}, \frac{\partial f_2}{\partial w}, \frac{\partial f_1}{\partial w}\right\}\leqslant L_k.$$

By using some integral estimations and some results in section 3 we have that, for w_1, $w_2\in B_k$, it holds that

$$\|w_1-w_2\|_B\leqslant \mu M(L_k, K)\|w_1-w_2\|,$$

where $M(L_k, K)$ is a positive number only depending on L_k and K. When $\mu\leqslant\frac{1}{2M(L_k, K)}$, H is a contraction mapping on B_k. So there exists a unique $w^*\in B_k$ such that

$$w^*=\mu Pw^*-\mu\tilde{T}(\operatorname{Re}Pw^*)+\tilde{T}\gamma. \tag{16}$$

Next, we prove that $w^*(z_1, z_2)$ is the solution of the modified problem (12), (13).

By using (16), it is easy to verify that w^* satisfies boundary value condition (13).

By using the expression of Pompieu operator, we have

$$\frac{\partial w^*}{\partial \bar{z}_1}=\mu f_1(z_1, z_2, w^*(z_1, z_2)), \tag{17}$$

$$\frac{\partial w^*}{\partial \bar{z}_2}=\mu f_2(z_1, z_2, w^*(z_1, z_2))+\mu T_1\left[\frac{\partial f_1(z_1, z_2 w^*(z_1, z_2))}{\partial \bar{z}_2}-\frac{\partial f_2(z_1, z_2, w^*(z_1, z_2))}{\partial \bar{z}_1}\right]. \tag{18}$$

By using complete integrable condition, equation (18) can be written as

$$\frac{\partial w^*}{\partial \bar{z}_2}-\mu f_2=\mu T_1\left[\frac{\partial f_1}{\partial w}\left(\frac{\partial w^*}{\partial \bar{z}_2}-\mu f_2\right)\right].$$

Therefore

$$\left|\frac{\partial w^*}{\partial \bar{z}_2}-\mu f_2\right| \leqslant 2\mu L_k \left|\frac{\partial w^*}{\partial \bar{z}_2}-\mu f_2\right|.$$

When $\mu \leqslant \frac{1}{4L_k}$, we have $\frac{\partial w^*}{\partial \bar{z}_2}=\mu f_2$.

Therefore, $w(z_1, z_2)$ is the unique solution of the modified problem (12), (13). So we have following theorem.

Theorem 2. *Consider problem* (12), (13). *Suppose that* $\gamma \in C_R^\beta(\partial G_1 \times \partial G_2)$ *is given. When condition* (C) *is satisfied, there exists a unique solution of the modified problem if* $\mu \leqslant \min\left(\frac{1}{4L_k}, \frac{1}{2M}\right)$.

§ 5. Some Remarks

In this section, we show the difficulties we meet when we give up the hypothesis that "f_j is holomorphic with respect to w" in condition (C). This is very important. Due to this reason, we can not treat the problem by using the method similar to Vekua method.

Consider the equation:

$$\frac{\partial w}{\partial \bar{z}_1}=\bar{w}, \tag{19}$$

$$\frac{\partial w}{\partial \bar{z}_2}=0. \tag{20}$$

The compatibility equation is

$$\frac{\partial \bar{w}}{\partial \bar{z}_2}=0. \tag{21}$$

From (20), (21), we know that w is independent of z_2. This overdetermined equations in fact is a complex elliptic equation in the plane:

$$\frac{\partial w}{\partial \bar{z}_1}=\bar{w}.$$

Obviously, it is not suitable to propose the boundary value condition similar to (2) for equations (20), (21).

References

[1] Gilbert, R. P. & Buchanan, J. L., First Order Elliptic Systems: A Function Theoretic Approach, *Math. in Science and Engineering* 163, Academic Press, New York-London (1983).

[2] Tutschke, W., Boundary value problems for generalized analytic functions of several complex variables, *Ann. Polon. Math.*, **39** (1981).

[3] Tutschke, W., Partielle komplexe Differentialgleichungen, VEB Deutscher Verlag d. Wiss., Berlin (1977).

[4] Li Mingzhong, Generalized Riemann-Hilbert problems for a system of first order quasilinear elliptic equations of several complex variables, *Complex Variables*, **7**: 4 (1987), 383—393.

[5] Li Mingzhong, On Riemann-Hilbert problems for elliptic systems of first order with two unknown functions in bicylinder, *Chin. Ann. Math.*, **8A**: 5 (1987) (in Chinese).

[6] Li Mingzhong & Cheng Jin, On Riemann-Hilbert problems for overdetermined elliptic systems with

several unknown functions of several complex variables, *Acta Math. Sinica,* **31:** 5 (1988), 671—679 (in Chinese).

[7] Chen Jin (Cheng Jin), On Riemann-Hilbert boundary value problems for complex overdetermined partial differential equations of first order, *Complex Variables,* **11** (1989).

[8] Li Mingzhong & Cheng Jin, On boundary value problems for overdetermined systems, Invited lecture on the 5-th integral equations and boundary value problems conference of China, *V. Sichun Normal University,* (1989) (in Chinese).

[9] Vekua, I. N., Generalized Analytic Functions, Addison-Wesley, Reading, Massachusetts (1962).