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TWO QUESTIONS ON LEVESQUE'S CYCLOTOMIC UNIT INDEX

FENG KEQIN (冯克勤)*

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Abstract

C. Levesque [3] raised two questions on the index of his independent cyclotomic unit system in the whole unit group of $\mathbf{Z}[\zeta_n+\bar{\zeta}_n]$. In this paper it is shown that the first question has affirmative answer for $n=p^\alpha$ and $n=p^\alpha q^\beta$, the second question has affirmative answer for $n=p^\alpha$, but has negative answer for $n=p^\alpha q^\beta$.

§1. Intioduction

Let $n\geqslant 0$, $\zeta_n=e^{\frac{2\pi i}{n}}$, $n\not\equiv 2 \pmod 4$. For each a, $(a, n)=1$, let

$$\varepsilon_a=\frac{1-\zeta_n^a}{1-\zeta_n}\zeta^{\frac{1-a}{2}}.$$

The ε_a is a unit in real cyclotomic field $K_n^+=\mathbf{Q}(\zeta_n+\bar{\zeta}_n)$, and is called (real) cyclotomic unit. For the case of $n=p^\alpha$(p is an odd prime number), Kummer has known in the last century that the cyclotomic unit system

$$C_n=\{\varepsilon_a \mid 2\leqslant a<n/2,\ (a, n)=1\}$$

is independent, and

$$[E_n^+ : \langle\pm C_n\rangle]=h_n^+,$$

where E_n^+ and h_n^+ denote the unit group and the class number of K_n^+ respectively. Hilbert[2] has suspected that C_n may be dependent if n is not a power of a prime number. The first example of such kind n has been given explicitly by Ramachandra[5], who also constructed new cyclotomic unit system C_n' of K_n^+ which is maximal independent for any n. Exactly speaking, let $n=\prod_{i=1}^{s}p_i^{e_i}$. For each proper subset I of $\{1, 2, \cdots, s\}$, let

$$n_I=\prod_{i\in I}p_i^{e_i}.$$

For $2\leqslant a<n/2$, $(a, n)=1$, let

$$\xi_a'=\zeta_n^{d_a}\prod_I\frac{1-\zeta_n^{an_I}}{1-\zeta_n^{n_I}},\quad d_a=\frac{1}{2}(1-a)\sum_I n_I,$$

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* Department of Mathematics, China University of Science and Technology, Hefei, Anhui, China

where I takes all proper subset of $\{1, 2, \cdots, s\}$. Ramachandra[5] (also see Washington [6, Theorem 8.3]) proved that the cyclotomic unit system

$$C_n' = \{\xi_a' \mid 2 \leqslant a < n/2, \ (a, n) = 1\}$$

of K_n^+ is maximal independent for any $n \geqslant 5$, $n \not\equiv 2 \pmod 4$. Besides, the index of $\langle \pm C_n' \rangle$ in the whole unit group E_n^+ is

$$[E_n^+ : \langle \pm C_n' \rangle] = h_n^+ \prod_{\substack{\chi \neq 1 \\ \chi(-1)=1}} \prod_{\varphi_i \nmid f_\chi} (\varphi(p_i^{e_i}) + 1 - \chi^*(p_i)), \tag{1}$$

where χ takes all non-trivial even characters of mod n, f_χ is the conductor of χ, χ^* is the primitive character correspondent to χ.

Pei Dingyi and Feng Keqin[4] (also see Feng[1]) presented several necessary and sufficient conditions for independence of C_n, and then determined all values of n such that C_n is independent. Recently, Levesque[3] constructed a series of new maximal independent system of K_n^+ having smaller index in E_n^+ than Ramachandra's one. That means that the subgroup generated by the Levesque's system is closer to the whole unit group E_n^+. Exactly speaking, let $\mathscr{D}$ be a set of some proper divisors of n. For each a, $2 \leqslant a < n/2$, $(a, n) = 1$, we define the real cyclotomic unit

$$\lambda_a = \lambda_a(\mathscr{D}) = \zeta_n^{b_a} \prod_{d \in \mathscr{D}} \frac{1 - \zeta_n^{ad}}{1 - \zeta_n^d}, \quad b_a = \frac{1}{2}(1 - a) \sum_{d \in \mathscr{D}} d$$

and

$$C_n(\mathscr{D}) = \{\lambda_a \mid 2 \leqslant a < n/2, \ (a, n) = 1\}.$$

Then Levesques[3] proved that

$$[E_n^+ : \langle \pm C_n(\mathscr{D}) \rangle] = h_n^+ i(\mathscr{D}), \tag{2}$$

where

$$i(\mathscr{D}) = \prod_{\substack{\chi \neq 1 \\ \chi(-1)=1}} \Big[\sum_{\substack{d \in \mathscr{D} \\ f_\chi \mid n/d}} \frac{\varphi(n)}{\varphi(n/d)} \prod_{p \mid n/d} (1 - \chi^*(p)) \Big]. \tag{3}$$

Therefore $C_n(\mathscr{D})$ is a maximal independent unit system iff $i(\mathscr{D}) \neq 0$. Moreover, let $n = \prod_{i=1}^{s} p_i^{e_i}$, $S = \{1, 2, \cdots, s\}$, $T_0 = \{i \in S \mid$ there exists even character $\chi \neq 1$ of mod n such that $\chi^*(p_i) = 1\}$. For each subset T, $T_0 \subseteq T \subseteq S$, let $\mathscr{D} = \mathscr{D}(T) = \{n_I \mid I \subseteq T\}$. Levesque[3] proved that

$$i(\mathscr{D}) = \prod_{\substack{\chi \neq 1 \\ \chi(-1)=1}} \Big[\prod_{\substack{i \in T \\ p_i \nmid f_\chi}} (\varphi(p_i^{e_i}) + 1 - \chi^*(p_i)) \prod_{j \in S - T} (1 - \chi^*(p_j)) \Big] \neq 0. \tag{4}$$

Therefore $C_n(\mathscr{D})$ is indepependent for such $\mathscr{D} = \mathscr{D}(T)$. (Remark: is [3] the formula (4) is proved for only $T = T_0$. But it is easy to see that the proof also works well for any T such that $T_0 \subseteq T \subseteq S$.) Taking $T = S$ we get Ramachandra's system and the index formula (1). For smaller subset T, $i(\mathscr{D}(T))$ has smaller index and the subgroup generated by $C_n(\mathscr{D}(T))$ is closer to E_n^+.

Based on several computing data, Levesque[3] raised the following two questions

(A). If $\mathscr{D}^* = \mathscr{D} \cup \{d^*\}$ with $d^* \mid n$, $d^* < n$, $d^* \notin \mathscr{D}$, is $i(\mathscr{D}) < i(\mathscr{D}^*)$?

(B). If $d_1 \in \mathscr{D}_1$, $d_2 \in \mathscr{D}_2$ are such that $\mathscr{D}_1 - \{d_1\} = \mathscr{D}_2 - \{d_2\}$ and if $d_1 < d_2$, is $i(\mathscr{D}_2)$

$<i(\mathscr{D}_1)$?

In this paper I will show that

(I) Question (A) has affirmative answer foı the cases $n=p^\alpha$ and $n=p^\alpha q^\beta$;

(II) Question (B) has affirmative answer for $n=p^\alpha$, but has negative answer for $n=p^\alpha q^\beta$. The simplest counter-example is $n=3^2 5^3$, $\mathscr{D}_1=\{1, 3, 15\}$, $\mathscr{D}_2=\{1, 3, 25\}$.

§2. For the Case of $n=p^\alpha$

We assume that $1\in\mathscr{D}$ for all $\mathscr{D}$ being considered since otherwise $i(\mathscr{D})=0$. In this section I will show the following theorem which says that both Questions (A) and (B) have affirmative answer for $n=p^\alpha$ except few trivial cases.

Theorem 1. *Suppose that $n=p^\alpha$ and p is a prime number, $\alpha\geqslant 1$. Then*

1. *The answer of question (A) is affirmative except $p=2$, $d^*=2^{\alpha-2}$, $2^{\alpha-1}$; and $p=3$, $d^*=3^{\alpha-1}$ for which we have $i(\mathscr{D})=i(\mathscr{D}^*)$.*

2. *The answer of question (B) is affirmative except $p=2$, $d_1=2^{\alpha-2}$, $d_2=2^{\alpha-1}$ for which we have $i(\mathscr{D}_1)=i(\mathscr{D}_2)$.*

Proof For $n=p^\alpha$, we have $\chi(p)=0$ if $\chi\neq 1$. Formula (3) becomes

$$i(\mathscr{D})=\prod_{\substack{\chi\neq 1\\ \chi(-1)=1}}\sum_{\substack{d\in\mathscr{D}\\ f_\chi|n/d}} d=\prod_{\lambda=1}^{\alpha-1}\prod_{\substack{\chi(-1)=1\\ f_\chi=p^{\alpha-\lambda}}}\Big(\sum_{\substack{p^\beta\in\mathscr{D}\\ \beta\leqslant\lambda}} p^\beta\Big)=\prod_{\lambda=1}^{\alpha-1}\Big(\sum_{\substack{p^\beta\in\mathscr{D}\\ \beta\leqslant\lambda}} p^\beta\Big)^{F_{\alpha-\lambda}}, \tag{5}$$

where

$$F_\lambda=\#\{\text{even }\chi\,(\text{mod }n)\,|f_\chi=p^\lambda\}.$$

For odd prime number p, it is easy to see that

$$F_\lambda=\begin{cases}[\varphi(p^\lambda)-\varphi(p^{\lambda-1})]/2, & \text{if }\lambda\geqslant 2,\\ (p-1)/2-1, & \text{if }\lambda=1.\end{cases} \tag{6}$$

For $p=2$, $n=2^\alpha(\alpha\geqslant 2)$, we have

$$F_1=F_2=0,\ F_\lambda=2^{\lambda-3}\quad(\text{for }\lambda\geqslant 3). \tag{7}$$

Suppose that

$$\mathscr{D}=\{1, p^{\beta_1}, \cdots, p^{\beta_t}\},\ 1\leqslant\beta_1<\cdots<\beta_t\leqslant\alpha-1.$$

Let $x_{\beta_i}=p^{\beta_i}(1\leqslant i\leqslant t)$; $x_l=0$ for $1\leqslant l\leqslant\alpha-1$, $l\notin\{\beta_1, \cdots, \beta_t\}$. Then we have from formula (5) that

$$i(\mathscr{D})=G(x_1, \cdots, x_{\alpha-1})=\prod_{\lambda=1}^{\alpha-1}(1+x_1+\cdots+x_\lambda)^{F_{\alpha-\lambda}}. \tag{8}$$

Form formulas (6) and (7) we know that

$$F_{\alpha-\lambda}=0\Leftrightarrow\begin{cases}p=3,\ \lambda=\alpha-1;\text{ or}\\ p=2,\ \lambda=\alpha-1,\ \alpha-2,\end{cases}$$

which implies by formula (8) that $i(\mathscr{D})=i(\mathscr{D}^*)$ for $p=3$, $d^*=3^{\alpha-1}$ and $p=2$, $d^*=2^{\alpha-1}$, $2^{\alpha-2}$. And otherwise $i(\mathscr{D})<i(\mathscr{D}^*)$, so the answer of question (A) is affirmative for $n=p^\alpha$.

For question (B) we can assume that $d_1=p^l$, $d_2=p^{l+1}$ $(1\leqslant l\leqslant \alpha-2)$ without loosing generality. Then from formula (8) we get

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}=\frac{G(x,\ \cdots,\ x_{l-1},\ p^l,\ 0,\ x_{l+2},\ \cdots,\ x_{\alpha-1})}{G(x_1,\ \cdots,\ x_{l-1},\ 0,\ p^{l+1},\ x_{l+2},\ \cdots,\ x_{\alpha-1})}$$
$$=\left(\frac{\Sigma_0^{l-1}+p^l}{\Sigma_0^{l-1}}\right)^{F_{\alpha-l}}\left(\frac{\Sigma_*^{l-1}+p^l}{\Sigma_*^{l-1}+p^{l+1}}\right)^{F_{\alpha-l-1}}$$
$$\cdot\prod_{\lambda=l+2}^{\alpha-1}\left(\frac{\Sigma_0^{l-1}+p^l+\Sigma_{l+2}^{\lambda}}{\Sigma_0^{l-1}+p^{l+1}+\Sigma_{l+2}^{\lambda}}\right)^{F_{\alpha-\lambda}},$$

where $\Sigma_a^b=x_a+x_{a+1}+\cdots+x_b(x_0=1)$. Our aim is to show $\dfrac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}>1$, so we can assume that $\Sigma_{l+2}^{\lambda}=0$ $(l+2\leqslant\lambda\leqslant\alpha-1)$ which means $x_{l+2}=\cdots=x_{\alpha-1}=0$ without loosing generality. Then we have (let $\Sigma=\Sigma_0^{l-1}$)

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}=\left(\frac{\Sigma+p^l}{\Sigma}\right)^{F_{\alpha-l}}\left(\frac{\Sigma+p^l}{\Sigma+p^{l+1}}\right)^{F_1+F_2+\cdots+F_{\alpha-l-1}}. \qquad (9)$$

If p is an odd prime number, then (by formula (6))

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}=\left(\frac{\Sigma+p^l}{\Sigma}\right)^{\frac{p-1}{2}(p^{\alpha-l-1}-p^{\alpha-l-2})}\left(\frac{\Sigma+p^l}{\Sigma+p^{l+1}}\right)^{\frac{p-1}{2}p^{\alpha-l-2}-1}.$$

Let

$$f(x)=\left(\frac{p-1}{2}p^{\alpha-l-1}-1\right)\log(x+p^l)-\frac{p-1}{2}(p^{\alpha-l-1}-p^{\alpha-l-2})\log x$$
$$-\left(\frac{p-1}{2}p^{\alpha-l-2}-1\right)\log(x+p^{l+1})$$
$$=\left(\frac{p-1}{2}p^{\alpha-l-1}-1\right)\log\left(1+\frac{p^l}{x}\right)-\left(\frac{p-1}{2}p^{\alpha-l-2}-1\right)\log\left(1+\frac{p^{l+1}}{x}\right).$$

Then

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}>1\Leftrightarrow f(\Sigma)>0.$$

Since $f(+\infty)=0$, $\Sigma\geqslant1$, and

$$f'(x)=\frac{-2p^l(p-1)x-p^{\alpha+l-1}(p-1)^2}{2x(x+p^l)(x+p^{l+1})}<0$$

for $x>0$, we have $f(\Sigma)>0$, namely $\dot{i}(\mathscr{D}_2)<\dot{i}(\mathscr{D}_1)$. So the answer of question (B) is affirmative for $n=p^\alpha$ $(p\geqslant3)$. If $p=2$, $n=2^\alpha$, $d_1=2^l$, $d_2=2^{l+1}$, $1\leqslant l\leqslant\alpha-2$, from formula (7) we know that formula (8) becomes

$$\dot{i}(\mathscr{D})=\prod_{\lambda=1}^{\alpha-3}\Bigg(\sum_{\substack{2^\beta\in\mathscr{D}\\ \beta\leqslant\lambda}}2^\beta\Bigg)^{F_{\alpha-l}}=\prod_{\lambda=1}^{\alpha-3}(1+x_1+\cdots+x_\lambda)^{F_{\alpha-\lambda}}$$

and formula (9) becomes

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}=\left(\frac{\Sigma+2^l}{\Sigma}\right)^{F_{\alpha-\lambda}}\left(\frac{\Sigma+2^l}{\Sigma+2^{l+1}}\right)^{F_3+F_4+\cdots+F_{\alpha-l-1}}.$$

Thus $\dot{i}(\mathscr{D}_1)=\dot{i}(\mathscr{D}_2)$ for $l=\alpha-2$ (namely for $d_1=2^{\alpha-2}$, $d_2=2^{\alpha-1}$), and for $1\leqslant l\leqslant\alpha-3$ $(\alpha\geqslant4)$ the above formula becomes (by (7))

$$\frac{\dot{i}(\mathscr{D}_1)}{\dot{i}(\mathscr{D}_2)}=\left(\frac{\Sigma+2^l}{\Sigma}\right)^{2^{\alpha-l-3}}\left(\frac{\Sigma+2^l}{\Sigma+2^{l+1}}\right)^{2^{\alpha-l-3}-1}.$$

Let

$$f(x)=(2^{\alpha-l-2}-1)\log(x+2^l)-2^{\alpha-l-3}\log x-(2^{\alpha-l-3}-1)\log(x+2^{l+1})$$
$$=(2^{\alpha-l-2}-1)\log(1+2^l/x)-(2^{\alpha-l-3}-1)\log(1+2^{l+1}/x).$$

Then $\frac{i(\mathscr{D}_1)}{i(\mathscr{D}_2)}>1\Leftrightarrow f(\Sigma)>0$. Since $f(+\infty)=0$, $\Sigma\geqslant 1$, and

$$f'(x)=\frac{-2^l x-2^{\alpha+l-2}}{x(x+2^l)(x+2^{l+1})}<0$$

for $x>0$, we know that $f(\Sigma)>0$ which implies that the answer of question (B) is affirmative for $n=p^\alpha$ except the case $d_1=2^{\alpha-2}$, $d_2=2^{\alpha-1}$. This completes the proof of Theorem 1.

§ 3. For the Case of $n=p^\alpha q^\beta$

Theorem 2 *Suppose that* $n=p^\alpha q^\beta\not\equiv 2 \pmod 4$, *where* p *and* q *are distinct prime numbers,* α, $\beta\geqslant 1$. *Then the answer of question (A) is affirmative; but the answer of question (B) is negative in general.*

Proof we look at question (A) at first. Let $\frac{n}{d^*}=p^u q^v$, u, $v\geqslant 0$, and

$$f(d,\ \chi)=\frac{\varphi(n)}{\varphi(n/d)}\prod_{p\mid n/d}(1-\chi^*(p)),$$
$$f(\mathscr{D},\chi)=\sum_{\substack{d\in\mathscr{D}\\ f_\chi\mid n/d}} f(d,\ \chi).$$

Fromformula (3) we know that

$$\frac{i(\mathscr{D}^*)}{i(\mathscr{D})}=\prod_{\substack{\chi\neq 1\\ \chi(-1)=1\\ f_\chi\mid n/d^*}}\left(1+\frac{f(d^*,\ \chi)}{f(\mathscr{D},\ \chi)}\right)=\prod_{\substack{0\leqslant s\leqslant u\\ 0\leqslant t\leqslant v}}\ \prod_{\substack{\chi\neq 1\\ \chi(-1)=1\\ f_\chi=p^s q^t}}\left(1+\frac{f(d^*,\ \chi)}{f(\mathscr{D},\ \chi)}\right). \tag{10}$$

Since d^* is a proper divisor of n, at least one of u and v is not zero. Suppose that $v=0$ at first. Then $u\geqslant 1$ and

$$\frac{i(\mathscr{D}^*)}{i(\mathscr{D})}=\prod_{1\leqslant s\leqslant u}\ \prod_{\substack{\chi(-1)=1\\ f_\chi=p^s}}\left(1+\frac{f(d^*,\ \chi)}{f(\mathscr{D},\ \chi)}\right), \tag{11}$$

where

$$f(d,\ \chi)=\begin{cases}\dfrac{\varphi(n)}{\varphi(n/d)}(1-\chi^*(q)), & \text{if } q\mid n/d,\\[2ex] \dfrac{\varphi(n)}{\varphi(n/d)} & \text{if } q\nmid n/d.\end{cases}$$

Thus we have from (11)

$$\frac{i(\mathscr{D}^*)}{i(\mathscr{D})}=\prod_{1\leqslant s\leqslant u}\ \prod_{\substack{\chi(-1)=1\\ f_\chi=p^s}}\left(1+\frac{c(1-\chi^*(q))}{A+B(1-\chi^*(q))}\right),$$

where $C=\frac{\varphi(n)}{\varphi(n/d)}\geqslant 1$, A, $B\geqslant 0$. If $\chi^*(q)=\pm 1$, the term of the product in the right hand side of above equation is$\geqslant 1$. If $\zeta=\chi^*(q)\neq\pm 1$, ζ is a root of 1, χ and its conjugate character $\bar{\chi}$ have the same conductor. The contribution of χ and $\bar{\chi}$ to the right hand side of above equation is

$$\left|1+\frac{C(1-\zeta)}{A+B(1-\zeta)}\right|^2=\left|\frac{A+(B+C)(1-\zeta)}{A+B(1-\zeta)}\right|^2. \tag{12}$$

The following real function

$$f(x)=|r+x(1-\zeta)|^2=r^2+rx(2-\zeta-\bar{\zeta})+x^2|1-\zeta|^2$$

is an extremely increasing function since $2-\zeta-\bar{\zeta}>0$. Therefore the value of formula (12) is bigger than 1 since $C>0$. Thus $i(\mathscr{D}^*)>i(\mathscr{D})$, and the answer of question (A) is affirmative for the case $v=0$. With the same reason we can prove this conclusion for the case $u=0$. Now we assume that $u,\ v\geqslant1$. Then formula (10) becomes

$$\frac{i(\mathscr{D}^*)}{i(\mathscr{D})}=\prod_{\substack{0\leqslant s\leqslant u\\0\leqslant t\leqslant v}}\prod_{\substack{\chi\neq1\\\chi(-1)=1\\f_\chi=p^sq^t}}\left(1+\frac{f(d^*,\ \chi)}{f(\mathscr{D},\ \chi)}\right)$$

$$=\prod_{\substack{1\leqslant s\leqslant u\\1\leqslant t\leqslant v}}\prod_{\substack{\chi(-1)=1\\f_\chi=p^sq^t}}+\prod_{1\leqslant t\leqslant v}\prod_{\substack{\chi(-1)=1\\f_\chi=q^t}}+\prod_{1\leqslant s\leqslant u}\prod_{\substack{\chi(-1)=1\\f_\chi=p^s}}. \tag{13}$$

If $f_\chi=p^sq^t,\ s,\ t\geqslant1$, then $\chi^*(p)=\chi^*(q)=0$. Therefore $f(d^*,\ \chi)$ and $f(\mathscr{D},\ \chi)$ are positive integers. Thus $\left(1+\frac{f(d^*,\ \chi)}{f(\mathscr{D},\ \chi)}\right)>1$, and the first product in the right hand side of (13) is >1. The other two products are $\geqslant1$ by using the same argument as we used in the case $v=0$. Thus $i(\mathscr{D}^*)>i(\mathscr{D})$ and the answer of question (A) is affirmative for $n=p^\alpha q^\beta$.

Now we consider question (B). Let $n=p^\alpha q^2,\ q<p,\ a\geqslant3$ and take

$$\mathscr{D}=\{1,\ q,\ pq,\ p^2q,\ \cdots,\ p^{\alpha-3}q\}$$

$$\mathscr{D}_1=\mathscr{D}\cup\{d_1=p^{\alpha-2}q\},\ \mathscr{D}_2=\mathscr{D}\cup\{d_2=p^{\alpha-1}\}\quad(d_1<d_2).$$

From formula (3) we know that for $\lambda=1,\ 2,$

$$i(\mathscr{D}_\lambda)=\prod_{\substack{x\neq1\\\chi(-1)=1}}\Big(\sum_{\substack{d\in\mathscr{D}_\lambda\\f_\chi|n/d}}d\Big)\cdot\prod_{\substack{\chi\neq1\\\chi(-1)=1}}(1-\chi^*(p))(1-\chi^*(q)). \tag{13}$$

At the end of [1], I presented many $(q,\ p)$ $(=(3,\ 5),\ (3,\ 7),\ (5,\ 17),\ (7,\ 11),\ (7,\ 17),\ \cdots)$ such that

$$g(n)=\prod_{\substack{\chi\neq1\\\chi(-1)=1}}(1-\chi^*(p))(1-\chi^*(q))>0$$

for $n=p^\alpha q^\beta$ with arbitrary positive integers α and β. Let $(q,\ p)$ be in such case and $I(\mathscr{D}_\lambda)=i(\mathscr{D}_\lambda)/g(n)$. We are going to show that $I(\mathscr{D}_1)<I(\mathscr{D}_2)$ for sufficient large α, which means that $i(\mathscr{D}_1)<i(\mathscr{D}_2)$ and the answer of question (B) is negative for such $n=p^\alpha q^2$ and the above-chosen $\mathscr{D}_1$ and $\mathscr{D}_2$.

In fact, let

$$F_{sr}=\#\{\text{even }\chi\ (\mathrm{mod}\ n)\,|\,f_\chi=p^sq^r\}.$$

Then we know from formula (13) that

$$I(\mathscr{D}_\lambda)=\prod_{\substack{\chi\neq1\\\chi(-1)=1}}\sum_{\substack{p^\lambda q^\mu\in\mathscr{D}_\lambda\\p^\lambda q^\mu|p^\alpha q^2f_\chi^{-1}}}p^\lambda q^\mu$$

$$=\prod_{\substack{0\leqslant s\leqslant\alpha\\0\leqslant r\leqslant2}}\prod_{\substack{\chi\neq1\\\chi(-1)=1\\f_\chi=p^sq^r}}\sum_{\substack{p^\lambda q^\mu\in\mathscr{D}_\lambda\\\lambda\leqslant\alpha-s\\\mu\leqslant2-r}}p^\lambda q^\mu$$

$$= \prod_{\substack{0\leqslant s\leqslant\alpha \\ 0\leqslant r\leqslant 2}} \Bigg(\sum_{\substack{p^\lambda q^\mu \in \mathscr{D}_\lambda \\ \lambda\leqslant\alpha-s \\ \mu\leqslant 2-r}} p^\lambda q^\mu \Bigg)^{F_{sr}}.$$

Theaefore

$$\frac{I(\mathscr{D}_1)}{I(\mathscr{D}_2)} = \left(\frac{1+q+pq+p^2q+\cdots+p^{\alpha-2}q}{1+q+pq+p^2q+\cdots+p^{\alpha-3}q} \right)^{F_{21}+F_{20}} \left(\frac{1}{1+p^{\alpha-1}} \right)^{F_{12}+F_{02}}$$
$$\cdot \left(\frac{1+q+pq+\cdots+p^{\alpha-2}q}{1+q+pq+\cdots+p^{\alpha-3}q+p^{\alpha-1}} \right)^{F_{01}+F_{11}+F_{10}}$$
$$= \left(\frac{1+q(p^{\alpha-1}-1)/(p-1)}{1+q(p^{\alpha-2}-1)/(p-1)} \right)^{(p-1)^2(q-1)/2} \left(\frac{1}{1+p^{\alpha-1}} \right)^{(p-1)(q-1)^2/2}$$
$$\cdot \left(\frac{1+q(p^{\alpha-1}-1)/(p-1)}{1+q(p^{\alpha-2}-1)/(p-1)+p^{\alpha-1}} \right)^{(p-1)(q-1)/2-1}. \tag{14}$$

For sufficient large α we have

$$\frac{I(\mathscr{D}_1)}{I(\mathscr{D}_2)} \sim p^{[(p-1)^2(q-1)-(\alpha-1)(p-1)(q-1)^2]/2} \cdot \left(\frac{pq}{q+p^2} \right)^{(p-1)(q-1)/2-1}$$
$$\to 0 \ (\text{when } \alpha\to+\infty).$$

Therefore $I(\mathscr{D}_1)<I(\mathscr{D}_2)$. This completes the proof of Theorem 2.

Remark. The simplest counter-example for question (B) is by taken $n=5^3 3^2$ ($p=5$, $q=3$, $\alpha=3$) and $\mathscr{D}_1=\{1, 3, 15\}$, $\mathscr{D}_2=\{1, 3, 25\}$. For this we have from formula (14)

$$\frac{I(\mathscr{D}_1)}{I(\mathscr{D}_2)} = \left(\frac{1+3+15}{1+3} \right)^{16} \left(\frac{1}{1+25} \right)^{8} \left(\frac{1+3+15}{1+3+25} \right)^{3}$$
$$= \left(\frac{19}{4} \right)^{16} \left(\frac{1}{26} \right)^{8} \left(\frac{19}{29} \right)^{3}$$
$$= \left(\frac{19}{29} \right)^{3} \left(\frac{361}{32\cdot 13} \right)^{8} < 1.$$

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